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Chvátal-Erdős condition for pancyclicity

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Abstract: An n-vertex graph is *Hamiltonian* if it contains a cycle that covers all of its vertices and it is *pancyclic* if it contains cycles of all lengths from 3 up to n. A celebrated meta-conjecture of Bondy states that every non-trivial condition implying Hamiltonicity also implies pancyclicity (up to possibly a few exceptional graphs). We show that every graph G with $\kappa(G) > (1+o(1))\alpha(G)$ is pancyclic. This extends the famous Chvátal-Erdős condition for Hamiltonicity and proves asymptotically a 30-year old conjecture of Jackson and Ordaz.

Key words and phrases: Hamiltonicity, pancyclicity, Chvatal-Erdos theorem

1 Introduction

The notion of Hamiltonicity is one of most central and extensively studied topics in Combinatorics. Since the problem of determining whether a graph is Hamiltonian is NP-complete, a central theme in Combinatorics is to derive sufficient conditions for this property. A classic example is Dirac's theorem [14] which dates back to 1952 and states that every n-vertex graph, for $n > 2$, with minimum degree at least $n/2$ is Hamiltonian. Since then, a plethora of interesting and important results about various aspects of Hamiltonicity have been obtained, see e.g. [1, 11, 12, 13, 19, 25, 26, 27, 33], and the surveys [21, 28].

Besides finding sufficient conditions for containing a Hamilton cycle, significant attention has been given to conditions which force a graph to have cycles of other lengths. Indeed, *the cycle spectrum of a graph*, which is the set of lengths of cycles contained in that graph, has been the focus of study of numerous papers and in particular gained a lot of attention in recent years [2, 3, 15, 20, 24, 30, 31, 32, 36]. Among other graph parameters, the relation of the cycle spectrum to the minimum degree, number of edges, independence number, chromatic number and expansion of the graph have been studied.

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We say that an n-vertex graph is *pancyclic* if the cycle spectrum contains all integers from 3 up to n. In the cycle spectrum of an n-vertex graph, it is usually hardest to guarantee the existence of the longest cycle, i.e. a Hamilton cycle. This intuition was captured in Bondy's famous meta-conjecture [6] from 1973, which asserts that any non-trivial condition which implies Hamiltonicity, also implies pancyclicity (up to a small class of exceptional graphs). As a first example, he proved in [7] an extension of Dirac's theorem, showing that minimum degree at least $n/2$ implies that the graph is either pancyclic or that it is the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. Further, Bauer and Schmeichel [5], relying on previous results of Schmeichel and Hakimi [35], showed that the sufficient conditions for Hamiltonicity given by Bondy [8], Chvátal [10] and Fan [18] all imply pancyclicity, up to a certain small family of exceptional graphs.

Another classic condition which implies Hamiltonicity is given by the famous theorem of Chvátal and Erdős [11]. It states that if the vertex connectivity of a graph G is at least as large as its independence number, that is, $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian. The pancyclicity counterpart of this result has also been investigated - see, e.g., [4] and the surveys [22, 34]. In fact, in 1990, Jackson and Ordaz [22] conjectured that G must be pancyclic if $\kappa(G) > \alpha(G)$, which if true would confirm Bondy's meta-conjecture for this classical instance. One can use an old result of Erdős [16] to show pancyclicity if $\kappa(G)$ is large enough function of $\alpha(G)$. Indeed, Erdős showed that if the number of vertices in G is larger than $4\alpha^4(G)$ (and thus, also if $\kappa(G) > 4\alpha^4(G)$), then it is pancyclic. A first linear bound on $\kappa(G)$ was given only in 2010 by Keevash and Sudakov [24], who showed that $\kappa(G) \geq 600\alpha(G)$ is enough. In this paper, we resolve the conjecture of Jackson and Ordaz asymptotically, by showing that $\kappa(G) > (1+o(1))\alpha(G)$ is already enough to guarantee pancyclicity.

Theorem 1.1. *Let $\varepsilon > 0$ and let n be sufficiently large. Then, every n-vertex graph G for which we have $\kappa(G) \geq (1+\varepsilon)\alpha(G)$ is pancylic.*

We remark that the only assumption of the above theorem is that n is sufficiently large in terms of ε. In turn, the first step of the proof will be to use the old result of Erdős mentioned before that if $n \geq 4(\alpha+1)^4$, then G is pancyclic, and therefore, we can assume that $n < 4(\alpha+1)^4$, which implies that α is also sufficiently large in terms of ε.

Next we briefly discuss some of the key steps in the proof of this theorem. It will be convenient for us to consider different ranges of cycle lengths whose existence we want to show, and for each range we have a separate subsection which deals with it. This is done in Section 3. In order to find these different cycle lengths we will combine various tools on shortening/augmenting paths and finding consecutive path lengths between two fixed vertices.

For example, for finding consecutive path lengths we crucially use that since $\kappa(G) > \alpha(G)$, it must be that G contains triangles - moreover, it contains a *path with triangles attached to many of its edges* (see Definition 2.3), which trivially implies the existence of many consecutive path lengths between the endpoints of such a path. For shortening/augmenting paths, we also introduce new tools. One of them is used to shorten paths using only the minimum degree of the graph (Lemma 2.8), while another one augments paths using both the independence and connectivity number (Lemma 2.10). Furthermore, we will also use a novel result proven in [15] using the Gallai-Milgram theorem, in order to shorten paths using the independence number of the graph (Lemma 2.9). In Section 2 we present these tools, together with some other useful results of a similar flavour. After that, in Section 3, we prove Theorem 1.1. The general proof strategy is to find a cycle of appropriate length which consists of two paths, one of which has many edges to which triangles are attached. Then we apply our shortening/augmenting results to the other path. Combining the consecutive path lengths from the first path with the path lengths obtained from the second path (see Observation 2.2), we will get all possible cycle lengths. Finally, in Section 4 we make some concluding remarks.

2 Preliminaries

2.1 Notation and definitions

We mostly use standard graph theoretic notation. Let G be a finite graph. Denote by $V(G)$ its vertex set, and let $S_1, S_2 \subseteq V(G)$. We denote by $G[S_1]$ the subgraph of G induced by S_1, and by $E[S_1,S_2]$ the set of edges with one endpoint in S_1 and the other in S_2. Let H be a subgraph of G. We denote by $G[H]$ the graph $G[V(H)]$. A path $P = (x_0, x_1, \dots, x_l)$ of length l is a graph on vertex set $\{x_0, x_1, \dots, x_l\}$ with an edge between x_{i-1} and x_i for all $i \in [l]$. We say that x_0 and x_l are the endpoints of P, and we call P an x_0x_l-path. Given disjoint sets of vertices A, B, we say that P is a path *going from A to B* if $x_0 \in A, x_l \in B$ and $x_i \notin A \cup B$ for all $0 < i < l$. We denote by $\alpha(G)$ the independence number of G. The *connectivity* $\kappa(G)$ of a connected graph G is the minimum number of vertices whose removal makes G disconnected or reduces it to a trivial graph (i.e., consisting of a single vertex).

Given sets $A_1, A_2 \subset \mathbb{N}$, we denote by $A_1 + A_2$ the set of integers c such that $c = a_1 + a_2$ for some $a_1 \in A_1$ and $a_2 \in A_2$. Throughout the paper we omit floor and ceiling signs for clarity of presentation, whenever it does not impact the argument.

Definition 2.1. Let a, b, p be positive real numbers. Given a graph G, and two vertices x and y, we say that the pair xy is p-dense in the interval $[a,b]$ if for every subinterval $[a',b']$ with $b' - a' \geq p$ such that there is an integer in $[a',b']$, we can find an integer $l \in [a',b']$ and an xy-path in G of length l. Note that, in particular, xy is 0-dense in $[a,b]$ if there are paths of all lengths in $[a,b]$ between x and y.

We now give a trivial observation which will be used in the proof of Theorem 1.1. It states that appropriate combinations of internally vertex-disjoint paths of different lengths imply the existence of cycles of many different lengths.

Observation 2.2. *Let G be a graph whose vertex set contains t disjoint sets $S_1, \dots, S_t$ and another set of t vertices $v_1, \dots, v_t$ outside of $\bigcup_{i=1}^t S_i$. For each $i \in [t]$, let $A_i \subset \mathbb{N}$ and suppose that for every i the induced subgraph $G[v_i \cup S_i \cup v_{i+1}]$ is such that it contains a v_iv_{i+1}-path of length l for each $l \in A_i$ (with $v_{t+1} = v_1$). Then for every $l \in A_1 + \dots + A_t$, the graph G contains a cycle of length l.*

2.2 Cycles and paths with triangles

One of the crucial objects which are used in our proof will be cycles which have triangles attached to some of their edges. Evidently, one can increase the length of such a cycle by precisely one, by using the two edges of a triangle, instead of the edge which lies on the cycle.

Definition 2.3. Define the graph C_l^r to be the graph formed by a cycle $v_1v_2 \dots v_lv_1$ of length l with the additional edges $v_1v_3, v_3v_5, \dots, v_{2r-1}v_{2r+1}$ (if $r = 0$, then it is just a cycle of length l). We will refer to this as a *cycle of length l with r triangles*. Similarly define P_l^r and refer to it as a *path of length l with r triangles*, where P_0^0 is just a vertex.

The following is an easy starting point for the existence of the graphs C_l^r with appropriate parameters, as subgraphs in graphs G with $\kappa(G) \geq \alpha(G)$.

Lemma 2.4. *Every n-vertex graph G with $\kappa(G) \geq \alpha(G)$ contains a C_l^r for all r such that $0 \leq r \leq \lfloor \frac{\kappa(G)-\alpha(G)}{2} \rfloor$ and some l with $l - 2(r+1) \leq \max\left(\frac{n}{\kappa(G)-2r+1}, \frac{n}{\kappa(G)-1}\right)$. In particular, it contains a P_{2r}^r for all such r.*

Proof. We will first show that G must always contain a $P_{2r'}^{r'}$ for $r' := \left\lfloor \frac{\kappa(G)-\alpha(G)}{2} \right\rfloor$, which can assume to have $r' \geq 1$, since otherwise, P_0^0 is a single vertex and clearly exists. We construct such a path greedily. Suppose that we have the vertices $v_1 v_2 v_3 \dots v_{2i+1}$ which form a P_{2i}^i, so that the edges $v_1 v_3, \dots, v_{2i-1} v_{2i+1}$ are also present. Provided that $i < r'$, we can augment this path as follows. Consider the set $S := N(v_{2i+1}) \setminus \{v_1, \dots, v_{2i}\}$. By assumption, this has size at least $\delta(G) - 2i > \kappa(G) - 2r' \geq \alpha(G)$. Therefore, it must contain an edge $v_{2i+2} v_{2i+3}$. Clearly, $v_{2i+1} v_{2i+2} v_{2i+3}$ forms a triangle and thus, $v_1 v_2 v_3 \dots v_{2i+1} v_{2i+2} v_{2i+3}$ is a P_{2i+2}^{i+1}. Continuing with this procedure until $i = r'$, gives the desired $P_{2r'}^{r'}$.

Now, fix r with the given condition. If $r = 0$, then take an edge xy in G. By Menger's theorem, there exist at least $\kappa(G)$ internally vertex-disjoint xy-paths in G and thus, at least $\kappa(G) - 1$ of these are not the edge xy. Therefore, there is such a path with at most $\frac{n}{\kappa(G)-1} + 2$ vertices, which together with the edge xy, then creates a cycle of length at most $\frac{n}{\kappa(G)-1} + 2$. If $r \geq 1$, by the previous paragraph, G contains a P_{2r}^r - let x, y be its endpoints. By Menger's theorem, there exist at least $\kappa(G)$ internally vertex-disjoint xy-paths in G. Since at most $2r - 1$ of these intersect $P_{2r}^r \setminus \{x, y\}$, there exists one which is disjoint to $P_{2r}^r \setminus \{x, y\}$ and contains at most $\frac{n}{\kappa(G)-2r+1}$ internal vertices. This produces the desired C_l^r. ☐

We can also use this type of cycle to extend the celebrated Chvátal-Erdős theorem [11].

Theorem 2.5 (Chvátal-Erdős [11])**.** *If for a graph G we have that $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.*

Our result states that if the Chvátal-Erdős condition is satisfied, then we can find a Hamilton cycle with a certain number of triangles, depending on the discrepancy between the connectivity and the independence number.

Theorem 2.6. *Every n-vertex graph G such that $\kappa(G) \geq \alpha(G)$ contains a C_n^r with $r = \left\lfloor \frac{\kappa(G)-\alpha(G)}{2} \right\rfloor$.*

Proof. Suppose for contradiction that some $l < n$ is maximal such that there exists a copy of C_l^r in G. Note that l exists by Lemma 2.4. Order the cycle as $v_1 v_2 \dots v_l v_1$ so that the edges $v_1 v_3, v_3 v_5, \dots, v_{2r-1} v_{2r+1}$ are also present. Since $l < n$, there is a vertex v not in C_l^r. Moreover, as $\kappa(G) \geq \alpha(G) + 2r$, there exist $\alpha(G)$ paths contained in $V(G) \setminus \{v_1, \dots, v_{2r}\}$, all of which go from v to C_l^r and are vertex-disjoint apart from the initial vertex v. Let us denote these paths as $P_{i_1}, P_{i_2}, \dots$ so that $v_j = P_j \cap C_l^r$ for $j \in \{i_1, i_2, \dots\}$. Consider the set $S := \{v_{i_1+1}, v_{i_2+1}, \dots\}$ with indices taken modulo l, so that $|S| \geq \alpha(G)$. Observe (as illustrated in Figure 1) that then there must be an edge contained in $S \cup \{v\}$ and that any such edge can be used to augment C_l^r to a $C_{l'}^r$ with $l' > l$, contradicting the maximality of l. ☐

We finish this section with the following partitioning lemma - it will allow us to transform even cycles found by standard density considerations into odd cycles.

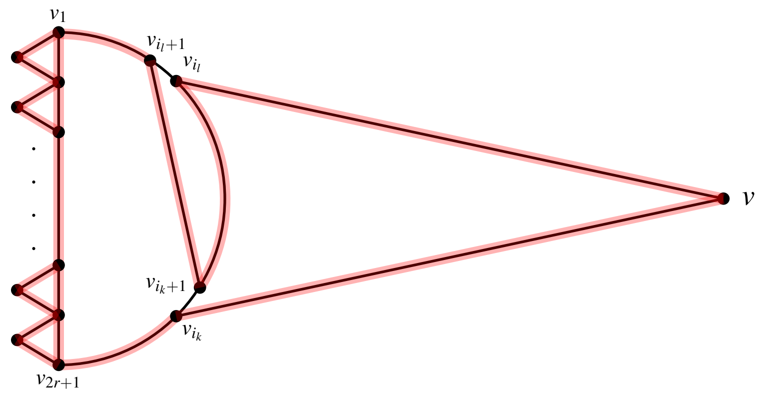

Figure 1: An illustration of how an edge between two elements v_{i_k+1}, v_{i_l+1} of S can be used to construct a new $C^r_{l'}$.

Lemma 2.7. *Let G be an n-vertex graph with $\kappa(G) > \alpha(G)$. Then, there exists $X \subseteq V(G)$ and a set of edges E contained in $G[X]$ such that the following hold.*

- $|E| \geq \left(\frac{\kappa(G)-\alpha(G)}{16}\right) n.$
- *For every edge $e = xy \in E$ there is a vertex $z \in V(G) \setminus X$ such that xzy is a triangle in G.*

Proof. First, since every vertex set in G of size at least $\alpha(G)+1$ contains an edge, every vertex v in G is such that its neighbourhood $N(v)$ contains a matching of size at least $\frac{\delta(G)-\alpha(G)}{2} \geq \frac{\kappa(G)-\alpha(G)}{2}$. Let $r := \frac{\kappa(G)-\alpha(G)}{2}$. For each v, fix such a matching M_v.

Now, let X be a random subset of $V(G)$ where each vertex is chosen independently at random with probability $1/2$. Let E denote the set of edges $e = xy$ with the following property: $x, y \in X$ and there is some $z \notin X$ such that $yz \in M_x$ or $xz \in M_y$. Clearly, E satisfies the second condition of the lemma. We need only to estimate the expected value of $|E|$ in order to prove than the first condition is satisfied for some X. Indeed, note that for an edge $e = xy$ to be present in E we must have that there is some z such that $yz \in M_x$ or $xz \in M_y$. Further, if at least one of these options holds, it is clear that then $\mathbb{P}(e \in E) \geq \frac{1}{8}$; since that is the probability that $x, y \in X$ and $z \notin X$. To finish, note that the number of such edges is at least $\frac{1}{2}\sum_v 2|M_v| = \sum_v |M_v| \geq nr$. Indeed, for each vertex $x \in G$, every vertex y in the matching M_x, gives such an edge xy, but since we possibly double counted (x might be in the matching M_y), the total number of such edges is at least $\frac{1}{2}\sum_v 2|M_v|$. Hence, $\mathbb{E}[|E|] \geq nr/8$, so there must exist such an E with $|E| \geq nr/8$ as desired. $\square$

2.3 Path shortening/augmenting tools

In this section, we describe some tools for shortening paths. First, we show the following lemma which uses only the minimum degree of the graph.

Lemma 2.8. *Let G be an n-vertex graph, $\delta := \delta(G)$ and let P be a path in G with endpoints x, y such that $|P| > 20n/\delta$. Then there is an xy-path P' such that $|P| - 20n/\delta \leq |P'| < |P|$.*

Proof. Suppose for sake of contradiction that no such path P' exists. Let $P := v_1v_2\dots v_{l-1}v_l$ with $v_1 = x, v_l = y$ and let $<_P$ denotes the given ordering of the path P as $v_1 <_P v_2 <_P \dots <_P v_l$. Since $|P| > 10n/\delta$, we can partition P into sub-paths $Q_1, Q_2, \dots, Q_k$ such that $|Q_k| \le 10n/\delta$ and $|Q_i| = 10n/\delta$ for all $i < k$. Moreover, we have $k = \left\lceil \frac{|P|}{10n/\delta} \right\rceil$. Now, take a subset $Q_1' \subseteq Q_1$ of size $\lfloor |Q_1|/3 \rfloor \ge 3n/\delta$ such that no two vertices in Q_1' are at distance at most 2 in P. Consider then the set of edges incident to Q_1', that is, $E[Q_1', V(G)]$; by the minimum degree condition, there are at least $|Q_1'| \cdot \delta \ge 3n$ such edges.

Now, clearly there cannot exist an edge spanned by Q_1 other than edges of P since this edge could be used to shorten P by at most $|Q_1| \le 10n/\delta$. Hence, $e(Q_1', Q_1) \le 2|Q_1'|$. Similarly, the following must hold.

Claim. $e(Q_1', V(G) \setminus P) \le n - |P|$.

Proof. Suppose otherwise. Then there is a vertex $v \in V(G) \setminus P$ with at least 2 neighbours in Q_1' - denote these by u, w. Note that since by construction u, w are at distance at least 2 and at most $|Q_1| \le 10n/\delta$ in P, this is a contradiction, since it produces the desired P' by substituting the sub-path of P between u and w by the path uvw. □

To give an upper bound on the total number of edges incident to Q_1' which are contained in $V(P)$, we also use the following claim.

Claim. *For all* $i > 1$*, we have* $e(Q_1', Q_i) < |Q_1'| + |Q_i|$.

Proof. Suppose otherwise. This implies that there is a cycle in $G[Q_1', Q_i]$ and hence, there must exist two crossing edges in this bipartite graph, that is, edges a_1b_1 and a_2b_2, with $a_1 <_P a_2$ and both in Q_1', and $b_1 <_P b_2$ both in Q_i. These can clearly be used to shorten P (see Figure 2) by at most $|Q_1| + |Q_i| \le 20n/\delta$, which is a contradiction as it produces the desired P'. □

Figure 2: Shortening of the path P using the crossing edges a_1b_1 and a_2b_2. The resulting path is P' and is drawn in red.

The above claim implies that

$$\sum_{i>1} e(Q_1', Q_i) < \sum_{i>1} \left(|Q_1'| + |Q_i|\right) \le (k-1)|Q_1'| + (|P| - |Q_1|) < 2|P| - 2|Q_1'|.$$

To conclude, we now must have the following

$$e(Q_1', V(G)) = e(Q_1', Q_1) + e(Q_1', V(G) \setminus P) + \sum_{i>1} e(Q_1', Q_i) < 2|Q_1'| + (n - |P|) + (2|P| - 2|Q_1'|) < 2n.$$

which contradicts the previous observation that $e(Q_1', V(G)) \ge 3n$. □

Conversely, the following lemma gives a way to shorten a path using only its independence number. It was proven in [15] and was used to solve an old conjecture of Erdős [16] - see Proposition 2.9 in [15] and let $U=\emptyset$ and $c=\frac{\lceil 20\alpha^2/|P|\rceil+3}{4}$.

Lemma 2.9. *Let G be an n-vertex graph with independence number α, let P be a path in G with endpoints x,y such that $|P|>4\alpha$. Then there is an xy-path P' such that $|P|-\lceil 20\alpha^2/|P|\rceil \le |P'| < |P|$.*

We finish this section with a lemma which contrarily to the previous lemmas, will allow us to slightly augment a path between two vertices. Furthermore, it will use both the connectivity and the independence number of the graph, and it will be used when the size of the path P we are considering is not suitable to apply the first two lemmas of this subsection.

Lemma 2.10. *Let G be an n-vertex graph with connectivity κ and independence number α, and let $r\in\mathbb{N}$ be such that $2r<\kappa-\alpha$. Let P be a path in G with endpoints x,y and with $|P|<n$. Then, there is an xy-path P' such that $|P|<|P'|\le|P|+r$ provided that $|P|>\frac{80\alpha}{r}$, and $\alpha>r>\frac{80\alpha}{r}\cdot\max\left(1,\frac{|P|}{\kappa-\alpha}\right)$.*

Proof. Consider a vertex u not contained in P and write P as $v_1v_2\dots v_l$ with $x=v_1, y=v_l$. By Menger's theorem, there exist $\min(\kappa,|P|)$ paths going from u to $V(P)$ which are vertex-disjoint apart from the vertex u. Let $S\subseteq V(P)$ be the endpoints of these paths, and for each $v_i\in S$ let P_i denote the corresponding path from u to v_i.

We first consider the case when $S=V(P)$. Note that for all i, since v_i,v_{i+1} are consecutive in P, we can substitute the edge v_iv_{i+1} by the paths P_i,P_{i+1} to form an xy-path of length $|P|+|P_i|+|P_{i+1}|-1$. Hence, if $|P_i|+|P_{i+1}|<r$ for some i, then we have constructed the desired P'. Otherwise, at least half of the paths P_i with $i\le\frac{20\alpha}{r}$ have $|P_i|\ge r/2$. Moreover, we can assume that the P_i are induced paths since if not, their length can be shortened. Let S' be the set of vertices v_i which are the endpoints of those paths, and note that $|S'|\ge\frac{10\alpha}{r}$. For each such P_i, let Q_i denote the subpath of P_i formed by its $r/4$ vertices in positions $r/4+1,\dots,r/2$, viewed in the direction $v_i\to u$. Since Q_i is an induced path, it contains an independent set I_i of size $|Q_i|/2\ge r/8$. Then we have

$$\left|\bigcup_{v_i\in S'} I_i\right| \ge |S'|\frac{r}{8} > \alpha,$$

hence there is an edge (u_a,u_b) between I_a and I_b for some $v_a,v_b\in S'$. This now completes the proof, as we can replace the part of the path in P between v_a and v_b by the path obtained by concatenating the v_au_a-path in P_a, the edge u_au_b and the u_bv_b-path in P_b, thus obtaining a path of length at least $|P|+2\cdot r/4-\frac{20\alpha}{r}>|P|$ and at most $|P|+2\cdot\frac{r}{2}$ which completes this case.

Let us now consider the case when $|S|=\kappa$. First we show the following simple claim.

Claim. *If at least $\alpha+1$ paths P_i are such that $|P_i|<r/2$, then such a P' exists.*

Proof. For each one of the endpoints $v_i\in V(P)-\{y\}$ of the paths P_i, let v_i' denote its neighbour on P which is closer to y. Let X be the set of those at least α vertices, together with the vertex u. Then there is an edge between two vertices in X. This gives an xy-path which is strictly longer than P, but by at most r (see Fig. 3 for an illustration of this operation). $\square$

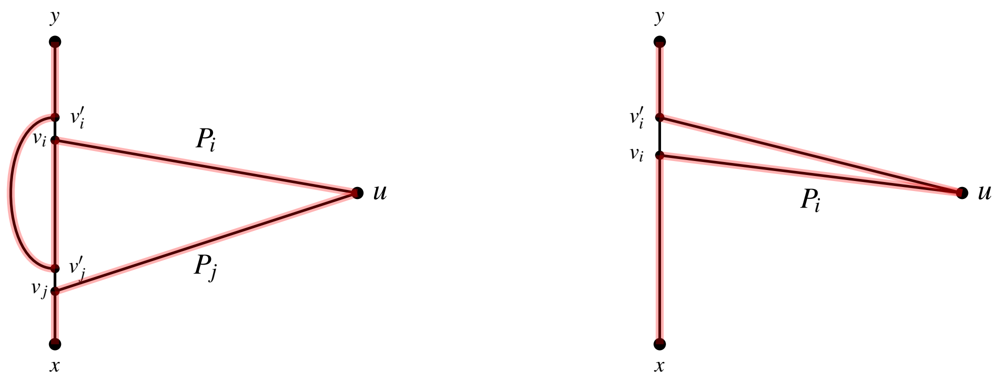

Figure 3: The first figure is for the case that the edge is in $X \setminus \{u\}$ (an edge $v_i'v_j'$) and the second figure is for when the edge contains u (an edge $v_i'u$).

By the above claim, we can assume that at least $\kappa - \alpha$ vertices $v_{i_j} \in S$ are such that $|P_{i_j}| \geq r/2$ - and moreover, we can assume that they are induced paths (since otherwise they can be shortened). Let S' be the set of those vertices in S, so that $|S'| \geq \kappa - \alpha$. Now, by letting $t = \frac{20\alpha|P|}{r(\kappa-\alpha)}$ we conclude by averaging that P contains an interval Q of length t with at least $\frac{t}{2|P|}(\kappa - \alpha) = \frac{10\alpha}{r}$ vertices in S'. By repeating the argument above – finding the independent sets $I_i \subset P_i$ for each of the $\frac{10\alpha}{r}$ paths P_i which end in Q, and then finding an edge between a pair I_i and I_j – we get a path P' of length at least $|P| - |Q| + 2 \cdot \frac{r}{4} \geq |P| - t + \frac{r}{2} > |P|$ by our assumption on r, and length at most $|P| + 2 \cdot \frac{r}{2}$, which completes the last case of the proof.

□

3 Proof of Theorem 1.1

Let $\varepsilon > 0$ and for convenience we may assume that ε is sufficiently small so that all our calculations go through. Suppose that n is sufficiently large in terms of ε and that $\kappa \geq (1+\varepsilon)\alpha$. Let G be a graph on n vertices, let α denote its independence number and κ its connectivity number. This immediately implies that α is also sufficiently large in terms of ε since otherwise, we would have $n \geq 4(\alpha+1)^4$ which by an old result of Erdős [16] would already imply pancyclicity.

Upper range: min $\left(\frac{10^5 n}{\varepsilon^2 \kappa}, \frac{100\alpha}{\varepsilon}\right)$ to n

We will first construct cycles of all lengths from $m := \min\left(\frac{10^5 n}{\varepsilon^2 \kappa}, \frac{100\alpha}{\varepsilon}\right)$ to n. First, apply Theorem 2.6 to G, which implies that it contains a $C_n^{r_1}$ with $r_1 = \varepsilon\alpha/2$. Note that if $m = \frac{10^5 n}{\varepsilon^2 \kappa}$, then we also have $r_1 \geq \frac{100n}{\kappa} =: r_2$, since in that case $\frac{10^5 n}{\varepsilon^2 \kappa} \leq \frac{100\alpha}{\varepsilon}$. Hence, in that case G trivially contains $C_l^{r_2}$.

Now, let us denote the Hamilton cycle in C_n^r by $v_1 v_2 \ldots v_n v_1$, with the edges $v_1 v_3, v_3 v_5, \ldots, v_{2r-1} v_{2r+1}$ present, where $r = r_1$ if $m = \frac{100\alpha}{\varepsilon}$, and $r = r_2$ if $m = \frac{10^5 n}{\varepsilon^2 \kappa}$. Let Q denote the path $v_1 v_2 \ldots v_{2r+1}$, and let P denote the path $v_{2r+1} v_{2r+2} \ldots v_n v_1$. Note that in the subgraph $G[Q]$, the pair $v_1 v_{2r+1}$ is 0-dense in the interval $[r,2r]$. We will now show that the same pair is $r/2$-dense in the interval $[m-2r,n]$ in the graph $G[P]$. Observation 2.2 then implies that G contains cycles of all lengths from m to n.

In order to show that v_1v_{2r+1} is $r/2$-dense in the interval $[m-2r,n]$ in the graph $G[P]$, a simple application of either Lemma 2.8 or Lemma 2.9 suffices, depending on where the minimum m is attained. Indeed, let $G' := G[P]$ and note that it has minimum degree at least $\delta' \geq \delta(G)-(2r-1) \geq \kappa - \varepsilon\alpha > (1-\varepsilon)\kappa$ and $\alpha(G') \leq \alpha$. Assume first that $m = \frac{10^5 n}{\varepsilon^2\kappa} \leq \frac{100\alpha}{\varepsilon}$, which implies that $20n/\delta' \leq 20n/(1-\varepsilon)\kappa < r/2$. Therefore, we can apply Lemma 2.8 to find a $v_{2r+1}v_1$-path P' in G' such that $|P|-r/2 \leq |P|-20n/\delta' \leq |P'| < |P|$. Further, we can repeat this on P' and continue applying Lemma 2.8 in such a manner, until we are left with a path on at most $\frac{10^5 n}{\varepsilon^2\kappa} - 2r$ vertices. Note that we can do this, since for every application of the lemma, we will have that the path will be of size at least $\frac{10^5 n}{\varepsilon^2\kappa} - 2r \geq \frac{10^5 n}{\varepsilon^2\kappa} - \frac{200n}{\kappa} \geq 20n/\delta'$. This implies that v_1v_{2r+1} is $r/2$-dense in the interval $\left[\frac{10^5 n}{\varepsilon^2\kappa} - 2r, n\right]$ as desired.

Assume now that $\frac{10^5 n}{\varepsilon^2\kappa} \geq \frac{100\alpha}{\varepsilon}$. Then, we can apply Lemma 2.9 to find a $v_{2r+1}v_1$-path P' in G' such that $|P|-r/2 \leq |P| - \lceil 20\alpha^2/|P| \rceil \leq |P'| < |P|$. We can repeat this on P' and iteratively apply the same lemma in such a way, until we are left with a path P_0 with at most $\frac{100\alpha}{\varepsilon} - 2r = \frac{100\alpha}{\varepsilon} - \varepsilon\alpha > \frac{99\alpha}{\varepsilon}$ vertices, so that for all previous paths P in this iteration we have $\lceil 20\alpha^2/|P| \rceil < r/2$. This shows that v_1v_{2r+1} is $r/2$-dense in the interval $\left[\frac{100\alpha}{\varepsilon} - 2r, n\right]$ as desired.

Middle range: $\max(\varepsilon\alpha/4000, n/\alpha)$ to $\min\left(\frac{10^5 n}{\varepsilon^2\kappa}, \frac{100\alpha}{\varepsilon}\right)$

We will now consider the middle range of cycle lengths. First, observe that we may assume that $\max(\varepsilon\alpha/4000, n/\alpha) < \min\left(\frac{10^5 n}{\varepsilon^2\kappa}, \frac{100\alpha}{\varepsilon}\right)$, as otherwise this range is empty. Hence we have that $n/\alpha < 100\alpha/\varepsilon$, which is equivalent to $\alpha > \frac{1}{10}\sqrt{\varepsilon n}$. Further, we have $\varepsilon\alpha/4000 < \frac{10^5 n}{\varepsilon^2\kappa}$, and since we have $\kappa > \alpha$, this gives $\alpha < 10^5\sqrt{n/\varepsilon^3}$. Observe that this implies that $\alpha = \Theta_\varepsilon(\sqrt{n})$.

Now, first observe that by Lemma 2.4, G contains a C_l^{2r} with $r = \varepsilon^{10}\alpha = \Theta_\varepsilon(\sqrt{n})$ and with l such that

$$4r+1 \leq l \leq \frac{n}{\kappa(G)-4r+1} + 4r + 2 \leq \frac{n}{(1+\varepsilon/2)\alpha} + 10\varepsilon^{10}\alpha \leq \frac{n}{\alpha},$$

where we used that $10^5\sqrt{n/\varepsilon^3} > \alpha > \frac{1}{10}\sqrt{\varepsilon n}$.

Note that this cycle C_l^{2r} can also be viewed as a C_l^r by omitting some triangles, which we do so that we have at least $l-2r \geq r$ vertices not among the triangles. Let P then be the path consisting of the first $2r+1$ vertices of this C_l^r (recall that P forms a P_{2r}^r), and let P' be the other path inside of the cycle with the same endpoints, denoted by x,y - so that $|P'| = l-2r \geq r$.

We will iteratively apply Lemma 2.10 to the path P' inside of the graph $G' = G-(V(P)-\{x,y\})$, with parameter r defined as above, and connectivity $\kappa' \geq \kappa - |V(P)-\{x,y\}| \geq \kappa - 2r$. Indeed, note that P' satisfies the conditions of Lemma 2.10. Indeed, since n is sufficiently large in terms of ε, we have $|P'| \geq r > \frac{80\alpha}{r}$, while $\alpha > r > \frac{80\alpha}{r} \cdot \frac{|P'|}{\kappa'-\alpha}$ (since $|P'| \leq l \leq n/\alpha$) and $\kappa' > \alpha + 2r$. Thus, there is an xy-path P'' in G' with $|P'| < |P''| \leq |P'| + r$.

We can continue applying Lemma 2.10 to the newly obtained path (now P'') inside of the same graph G', each time getting a path which is longer by at most r than the previous one. Note that the conditions of the lemma are still satisfied as long as the current path is of length at most $\frac{100\alpha}{\varepsilon}$ (again, since n is sufficiently large in terms of ε). This implies that the pair xy is r-dense in $[l-2r, 100\alpha/\varepsilon]$ in the graph G'. Now, since xy is also 0-dense in $[r, 2r]$ in $G[P]$, this gives all cycle lengths in $[l, 100\alpha/\varepsilon] \supseteq [n/\alpha, 100\alpha/\varepsilon]$ by Observation 2.2, as desired.

Lower range: 3 to $\max(\varepsilon\alpha/4000, n/\alpha)$

To finish the proof of Theorem 1.1, we now deal with the lower range. Let us first show that G contains the three smallest cycles.

Claim. *G contains a C_3, a C_4 and a C_5.*

Proof. Note that G contains C_3 since $\delta(G) \geq \kappa \geq \alpha + 1$, so the neighbourhood of a vertex necessarily spans an edge. Suppose now for sake of contradiction that G does not contain a C_4. Then, it must be that for every vertex v, the graph induced by its neighbourhood $G[N(v)]$ has maximum degree 1 - indeed, otherwise it contains a path on three vertices, which together with v forms a C_4. Moreover, this implies that $N(v)$ contains an independent set I_v of size at least $|N(v)|/2 \geq \kappa/2 \geq (1+\varepsilon)\alpha/2$. Now, take two adjacent vertices u, v in G. Since G contains no C_4, it must be that $|I_u \cap I_v| \leq 1$ and thus, $(I_u \Delta I_v) \setminus \{u, v\}$ has at least $(1+\varepsilon)\alpha - 3 > \alpha$ vertices. To finish, note that there can be no edge between $I_u \setminus \{v\}$ and $I_v \setminus \{u\}$ since together with uv it would form a C_4. Hence, the set $(I_u \Delta I_v) \setminus \{u, v\}$ is an independent set of size larger than α, which contradicts the assumption on G.

Finally, suppose for sake of contradiction that G contains no C_5. Much like before, note that it must be that for every vertex v, $G[N(v)]$ has no path on four vertices since this together with v forms a C_5. Therefore, $N(v)$ contains an independent set I_v of size at least $|N(v)|/3 \geq \kappa/3 \geq (1+\varepsilon)\alpha/3$. Now, take a vertex v, and let x_1y_1, x_2y_2, x_3y_3 be disjoint edges contained in $N(v)$ - note these exist since $|N(v)| \geq \kappa \geq \alpha + 7$. Consider also the neighbourhoods $N(x_1), N(x_2), N(x_3)$ and note that they must be disjoint (except for v) – indeed, if e.g., $z \in N(x_1) \cap N(x_2)$ then $vy_1x_1zx_2v$ is a C_5 (see Figure 4 for an illustration). Note also that there cannot exist an edge zz' with $z \in N(x_i), z' \in N(x_j)$ for some $i \neq j$ - indeed, then $vx_izz'x_jv$ is a C_5. Concluding, note that it must be that $I_{x_1} \cup I_{x_2} \cup I_{x_3}$ is an independent set and has size at least $|I_{x_1}| + |I_{x_2}| + |I_{x_3}| > \alpha$, which is a contradiction. $\square$

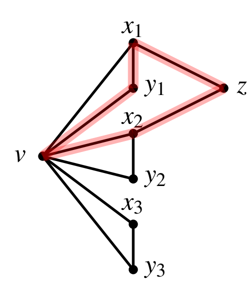

Figure 4: An illustration of the cycle $vy_1x_1zx_2v$.

For the remaining cycle lengths, it is necessary to consider two cases, depending on whether n/α is larger than $\varepsilon\alpha/4000$ or not.

Case 1: $n/\alpha \geq \varepsilon\alpha/4000$

This implies that $n \geq \varepsilon\alpha^2/4000$. Showing that G contains all cycles of lengths between 6 and n/α boils down to the study of cycle-complete Ramsey numbers. Namely, the cycle-complete Ramsey number $r(C_l, K_s)$ is the smallest number N such that every graph on N vertices either contains a copy of C_l or an independent set of size s. The following result of Erdős, Faudree, Rousseau and Schelp [17], along with a more recent result by Keevash, Long and Skokan [23] cover the mentioned range of cycle lengths we need.

Theorem 3.1 ([17]). *Let $l \geq 3$ and $s \geq 2$. Then $r(C_l, K_s) \leq \big((l-2)(s^{1/x}+2)+1\big)(s-1)$, where $x = \lfloor \frac{l-1}{2} \rfloor$.*

The next result by Keevash, Long and Skokan gives the precise behaviour of cycle-complete Ramsey numbers in a wide range of parameters, and proves a conjecture from [17].

Theorem 3.2 ([23]). *There exists $C \geq 1$ so that $r(C_l, K_s) = (l-1)(s-1)+1$ for $s \geq 3$ and $l \geq C\frac{\log s}{\log\log s}$.*

Note that since G contains no independent set of size larger than α and $n \geq \varepsilon\alpha^2/4000$, and by assumption α is sufficiently large in terms of ε, Theorem 3.1 implies the existence of a cycle of length l for every $l \in [6, \log\alpha]$, while Theorem 3.2 covers the range of $[\log\alpha, n/\alpha]$.

Case 2: $n/\alpha < \varepsilon\alpha/4000$

This implies that $\alpha > 40\sqrt{n/\varepsilon}$. We need to find all cycles from 6 to $\varepsilon\alpha/4000$. For this, we use the following classic result by Bondy and Simonovits.

Theorem 3.3 ([9]). *Let G be an n-vertex graph with $e(G) \geq \max(20ln^{1+1/l}, 200nl)$. Then, G contains a cycle of length $2l$.*

We can now use this together with Lemma 2.7 to get the desired cycle. Indeed, apply this lemma to G to obtain a set X and edge-set E of edges contained in X, such that $|E| \geq \frac{\kappa-\alpha}{16}\cdot n \geq \varepsilon\alpha n/8$, and for every edge $xy \in E$ there exists $z \in V(G)-X$ such that x, y and z form a triangle. Let $G' := (X, E)$ be the graph consisting of these edges. Observe that it is sufficient for us to show that for all $3 \leq l \leq \varepsilon\alpha/4000$, there is a cycle of length $2l$ in G' - indeed, such a cycle can then be transformed into a cycle of length $2l+1$ in G by substituting an edge xy of the cycle by the path xzy which is guaranteed to exist by Lemma 2.7. Finally, we find these even cycles in G' by applying Theorem 3.3, which gives cycles of lengths $2l$, for any l such that $\max(200nl, 20ln^{1+1/l}) \leq \varepsilon\alpha n/16$. Since $\alpha > 40\sqrt{n/\varepsilon}$ and n is sufficiently large in terms of ε, this holds for all $l \in [3, \varepsilon\alpha/4000]$. Indeed, for the first inequality note that for each such l we have $200nl \leq \varepsilon\alpha n/20$. For the second one, note that if $l < \log^2 n$ then the inequality $20ln^{1+1/l} \leq 20ln^{4/3} \leq \varepsilon\alpha n/16$ trivially holds since $\alpha > 40\sqrt{n/\varepsilon}$ and n is sufficiently large in terms of ε; on the other hand if $l > \log^2 n$, then $20ln^{1+1/l} < 40l$ which is clearly less than $\alpha n/16$ for $l < \varepsilon\alpha/4000$. □

4 Concluding remarks

In this paper we showed that if a graph G satisfies $\kappa(G) \geq (1+o(1))\alpha(G)$ then G is pancyclic. Moreover, the $o(1)$ error term can be made to be $\alpha(G)^{-c}$ for some small constant $c > 0$. This extends the classic theorem of Chvátal and Erdős, which states that $\kappa(G) \geq \alpha(G)$ implies that G is Hamiltonian, confirming asymptotically Bondy's meta-conjecture for this celebrated result. Nevertheless, it would be very interesting to prove the Jackson-Ordaz conjecture in full generality, or at least to show that it holds when $\kappa(G) \geq \alpha(G) + C$ for some constant $C > 0$.

Note added. Eight months after we posted our paper, Shoham Letzer [29] proved the conjecture of Jackson and Ordaz for large graphs.

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Effective Counting in Sphere Packings

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Abstract: Given a Zariski-dense, discrete group, Γ, of isometries acting on $(n+1)$-dimensional hyperbolic space, we use spectral methods to obtain a sharp asymptotic formula for the growth rate of certain Γ-orbits. In particular, this allows us to obtain a best-known effective error rate for the Apollonian and (more generally) Kleinian sphere packing counting problems, that is, counting the number of spheres in such with radius bounded by a growing parameter. Our method extends the method of Kontorovich [Kon09], which was itself an extension of the orbit counting method of Lax-Phillips [LP82], in two ways. First, we remove a compactness condition on the discrete subgroups considered via a technical cut-off and smoothing operation. Second, we develop a coordinate system which naturally corresponds to the inversive geometry underlying the sphere counting problem, and give structure theorems on the arising Casimir operator and Haar measure in these coordinates.

Key words and phrases: Orbital counting, Spectral theory, Automorphic representations

1 Introduction

The purpose of this paper is to give improved error estimates on the counting problem for Kleinian sphere packings (and discrete counting methods more broadly). A packing $\mathcal{P}$ of $\mathbb{S}^n$ (thought of as the boundary of hyperbolic $(n+1)$-space, $\mathbb{H}^{n+1}$) is an infinite collection of round, disjoint balls whose union is dense in $\mathbb{S}^n$. Following [KK23], such is called *Kleinian* if its residual set (left over when the interiors of the balls are removed) agrees with the limit set of some discrete, geometrically finite subgroup, Γ, of isometries of $\mathbb{H}^{n+1}$. A familiar example is the classical Apollonian circle packing in $n=2$, see e.g [Kon13] for more background and see Figure 1 for an example.

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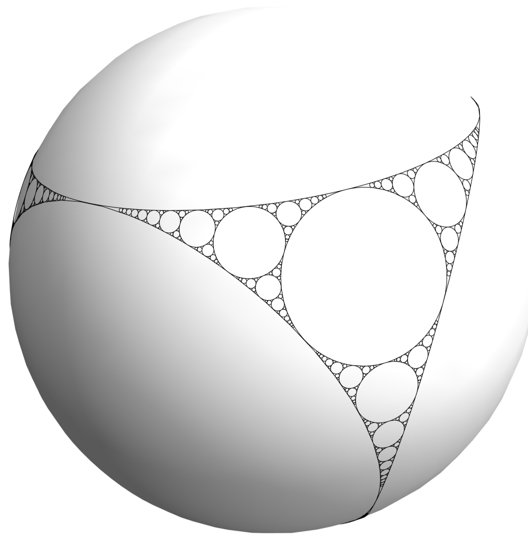

Figure 1: A classical Apollonian circle packing in the sphere $\mathbb{S}^2$. Note that the union of balls in the packing is dense in $\mathbb{S}^2$. (Image by Iván Rasskin.)

Let $\mathcal{P}$ be a given Kleinian sphere packing. For a sphere $S \in \mathcal{P}$, let $b(S)$ denote its bend, that is, its (signed) inverse-radius; this is determined after a choice of coordinates on $\mathbb{S}^n$, and in particular a choice of a point at ∞ (see §4). The Counting Problem is to estimate the number

$$N_{\mathcal{P}}(T) := \#\{S \in \mathcal{P} : b(S) < T\}$$

of spheres in $\mathcal{P}$ with bend bounded by a parameter $T \to \infty$. If the packing $\mathcal{P}$ is bounded, that is, the chosen point at infinity is contained in the interior of some ball, then $N_{\mathcal{P}}(T)$ is finite for all T; otherwise one can count spheres restricted to a bounded region (such as a period, if the packing is periodic). Let $\delta = \dim(\mathcal{P})$ be the Hausdorff dimension of the residual set of $\mathcal{P}$.

For the classical Apollonian packing, it is known that $\delta \approx 1.3$ (in general one has $n-1 < \delta < n$). In this setting, Boyd [Boy73] showed that $N_{\mathcal{P}}(T) = T^{\delta+o(1)}$, which was improved in Kontorovich-Oh [KO11] to an asymptotic formula, $N_{\mathcal{P}}(T) \sim c_{\mathcal{P}} T^{\delta}$, where $c_{\mathcal{P}}$ is a constant depending on the packing $\mathcal{P}$. An effective power savings error rate was shown in [Vin12] and [LO13] independently. These tools and results have been generalized by many authors (see, e.g., [Kim15, MO15, Pan17, EO21]). In this paper, we introduce a new method, modifying the approach in [Kon09], to produce a best known error exponent in the Counting Problem (including the classical Apollonian case).

The error exponent involves the spectrum of the hyperbolic Laplacian Δ acting on $L^2(\Gamma\backslash\mathbb{H}^{n+1})$ where Γ is the symmetry group of the packing. From work of Lax-Phillips [LP82], Patterson [Pat76], and Sullivan [Sul84], we have that the Laplace spectrum consists of a discrete isolated bottom eigenvalue $\lambda_0 = \delta(n-\delta)$, then possibly a finite number of further discrete eigenvalues in the "exceptional" range,

$$\lambda_0 < \lambda_1 \leq \cdots \leq \lambda_k < n^2/4, \tag{1}$$

and purely continuous spectrum above $n^2/4$. (It was suggested by Sarnak [Sar07] that, in the case of the classical Apollonian packing, $k=0$, that is, there are no other discrete eigenvalues below 1 except the bottom.[1]) Write $\lambda_j = s_j(n-s_j)$ with $n/2 \le s_j < \delta$.

These eigenvalues also correspond to the parameters of spherical complementary series representations occurring in the decomposition into irreducibles of $L^2(\Gamma\backslash G)$, where $G = \mathrm{SO}(n+1,1)$, see §2.2. We assume throughout that no nonspherical complementary series representations arise in this decomposition. This holds automatically for circle packings, that is, when $n=2$. Minor modifications are needed to handle the general case, leading to potentially worse error terms, see Remark 7.

Theorem 2. *Given a Kleinian packing $\mathcal{P}$ as above, there exists a constant $c_{\mathcal{P}} > 0$ such that:*

$$N_{\mathcal{P}}(T) = c_{\mathcal{P}} T^{\delta}\left(1 + O\left(T^{-\eta}\right)\right) \tag{3}$$

as $T \to \infty$, where

$$\eta = \frac{2}{n+3}(\delta - s_1). \tag{4}$$

If there is no discrete spectrum other than the base eigenvalue, then we have that

$$N_{\mathcal{P}}(T) = c_{\mathcal{P}} T^{\delta}\left(1 + O\left(T^{-\eta}(\log T)^{2/(n+3)}\right)\right),$$

where $\eta = \frac{2}{n+3}(\delta - n/2)$.

For the classical Apollonian packing $(n=2)$, our error exponent is $\eta = \frac{2}{5}(\delta - s_1)$, and if there are no discrete eigenvalues above the base then $\eta = \frac{2}{5}(\delta - 1) \approx 0.12\ldots$, whereas the best previously known exponent [LO13, Theorem 1.1] was:

$$\eta = \frac{2}{63}(\delta - s_1), \tag{5}$$

and if $s_1 = n/2$, this exponent is $\eta = \frac{2}{63}(\delta - 1) \approx 0.0097$. Hence (4) is a significant improvement over (5).

Remark 6. *Counting with a smooth cutoff and extracting all of the lower order terms corresponding to eigenvalues other than the base, we obtain the best possible error exponent $\eta = \delta - n/2$ (see Theorem 61), which improves over the smooth error exponent in the Apollonian case $\eta = \frac{2}{7}(\delta - 1)$ in [LO13, Theorem 8.2].*

Remark 7. *If we remove the assumption that there are no nonspherical complementary series representations occurring in $L^2(\Gamma\backslash G)$, then our method gives the weaker error term in (3) with $\eta = \frac{2}{n+3}(\delta - (n-1))$, that is, replacing s_1 in (4) with $n-1$; see Remark 79. With more effort, this error term could be improved, see [EO21].*

The proof of Theorem 2 introduces two new technical ideas: first, of independent interest, we overcome difficulties in the "abstract spectral theory" method of [Kon09] arising from a non-compactness condition (see the discussion below), and second, we introduce a new decomposition tailored to sphere counting problems and derive structure theorems for the arising Casimir operator and Haar measure.

[1] Added in print: Sarnak's question has now been answered in the affirmative [KKL23].

1.1 Ideas in the Proof

Previous Approach: To explain the main ideas, we first recall the method introduced in [Kon09], which itself is modeled on [LP82]. Consider the following related but simpler to exposit problem. Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a discrete, Zariski dense, geometrically finite subgroup, with critical exponent $\delta_\Gamma > 1/2$, acting on the upper half plane $\mathbb{H}^2$, and assume that ∞ is not a *point of approximation* for Γ. This last condition implies that either ∞ is a cusp of Γ with stabilizer Γ_∞ (in which case the limit set is periodic), or ∞ is not in the limit set of Γ (and hence the limit set is bounded). Either way, consider the problem of counting

$$N_\Gamma(T) := \#\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma : c^2 + d^2 < T\}.$$

(The Lax-Phillips work counts $a^2 + b^2 + c^2 + d^2 < T$, and new ideas are needed to handle the potential stabilizer when counting $c^2 + d^2$.) The assumption that ∞ is not a point of approximation assures that the set of such $c^2 + d^2$ is discrete, and hence this count is finite for any T. The main idea, as sketched below, is to use a particular function of the Laplacian to grow balls of size T from balls of bounded size.

In this subsection, write $G = \mathrm{SL}_2(\mathbb{R})$, $N = \{\left(\begin{smallmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{smallmatrix}\right)\}$, (so that $\Gamma_\infty = \Gamma \cap N$) and $K = \mathrm{SO}(2)$, and let

$$\chi_T : G \to \mathbb{R} : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \mathbb{1}_{\{c^2+d^2<T\}} \tag{8}$$

be the indicator function of the region in question in G. Note that χ_T is left-N-invariant and right-K-invariant, and let

$$F_T : \Gamma \backslash G/K \to \mathbb{R} : g \mapsto \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \chi_T(\gamma g),$$

so that $N_\Gamma(T) = F_T(e)$. As observed in [Kon09], F_T is in $L^2(\Gamma \backslash G/K) = L^2(\Gamma \backslash \mathbb{H})$ if and only if ∞ is a cusp of Γ. To access the value of F_T at the origin, we let ψ be a smooth bump function about the origin in G, and automorphize it to $\Psi(g) := \sum_{\gamma \in \Gamma} \psi(\gamma g)$. Then, we can write a smooth approximation of $N_\Gamma(T)$ as

$$\widetilde{N_\Gamma}(T) := \langle F_T, \Psi \rangle_\Gamma$$

where the inner-product is with respect to $L^2(\Gamma \backslash \mathbb{H})$. Now suppose we take the inner-product of F_T with an eigenfunction of the Laplacian ϕ, with eigenvalue $\lambda = s(1-s)$. Then, solving a second order ODE, we would have that

$$\langle F_T, \phi \rangle_\Gamma = \alpha T^s + \beta T^{1-s}, \tag{9}$$

for some α and β depending on ϕ. The key idea of [Kon09] is then to rewrite (9) in a way that involves the eigenvalue only, and not the coefficients α, β of the eigenfunction. This is achieved by setting $T = 1$ and $T = b$ (for some $b < 10$, see §3.1) in (9) and solving for α, β, to give an expression of the form:

$$\langle F_T, \phi \rangle = K_T(s) \langle F_1, \phi \rangle + L_T(s) \langle F_b, \phi \rangle,$$

for some functions K_T, L_T. This then allows one to prove the "main identity", which states that, in the sense of L^2, we have:

$$F_T = K_T(\Delta) F_1 + L_T(\Delta) F_b.$$

This is exactly what we mean by "growing" the ball of radius T from the Laplacian and bounded norm balls. It is this identity that can be proved rigorously even in the absence of explicit Whittaker expansions and spectral decompositions.

New Ideas: Much of this approach fails if Γ does not have a cusp at ∞, and the main purpose of [KO12] was to bypass this "PDE" approach and replace it with homogeneous dynamics, at the cost of worse error terms. The main new ideas of this paper allow us to recover the PDE approach (and then extend it to the Kleinian setting).

The first issue is that F_T is not in L^2. The observation which eliminates this issue (made already in [KO12]) is to add a second cut off in the N-direction without losing any of the orbit, since the limit set is anyway bounded; this adds compactness in the x-variable, restricted to some sufficiently large interval $[-X,X]$. Now we again unfold the inner product $\langle F_T,\Psi\rangle_\Gamma$ leading to the following integral:

$$\int_0^\infty \int_{-X}^{X} \chi_T(z)\widetilde{\chi}_X(z)\Psi(z)dx\frac{dy}{y^2}.$$

We proceed with harmonic analysis on $[-X,X]\times(0,\infty)$. However this truncation requires a delicate smoothing procedure to avoid introducing boundary terms in the analysis of the Laplacian. Once this is accomplished, the proof follows in a similar fashion. See §3.1 for the details.

In addition to overcoming the restriction in [Kon09], this $\mathrm{SL}_2(\mathbb{R})$ result is of independent interest, and improves on [KO12, Theorem 1.8] which used methods from homogeneous dynamics to count Pythagorean triples.

In the Kleinian Setting: There are several further modifications and innovations needed to extend the above-described $\mathrm{SL}_2(\mathbb{R})$ approach to the setting of orbits in circle/sphere packings. In the previous setting, the stabilizer of χ_T in (8) was left-N and right-K invariant, and so it was natural to use Iwasawa coordinates $\mathrm{SL}_2(\mathbb{R}) = NAK$. In the setting of sphere packings, one counts spheres in the orbit $S_0\Gamma$ with bend less than T; here S_0 is a fixed $(n-1)$-sphere in $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\} = \partial\mathbb{H}^{n+1}$ and $\Gamma < G = \mathrm{SO}(n+1,1)$ is a symmetry group of the packing, acting on the right by Möbius transformations. The analogous function χ_T is given by:

$$\chi_T : G \to \mathbb{R} : g \mapsto \mathbb{1}_{\{b(S_0 g)<T\}},$$

where again $b(S)$ is the bend of a sphere S. This function is left-H invariant, where $H = \mathrm{Stab}_G(S_0) \cong \mathrm{SO}(n,1)$; it is also right invariant under the group L of rigid affine motions, since neither translating nor rotating a sphere changes its bend. The latter decomposes further as $L = UM$, where U is a one-parameter unipotent group (which controls the co-bend, defined to be the bend of the inversion of a sphere through the unit sphere), and $M \cong \mathrm{SO}(n)$ rotates the sphere about the origin. It turns out that $H \cap M \cong \mathrm{SO}(n-1)$, and thus we set $M_1 := M/(H\cap M) \cong \mathbb{S}^{n-1}$. The subgroup of G which directly controls the bend is also a one-parameter unipotent group we call $\overline{U}$, leading to the map:

$$H \times \overline{U} \times U \times M_1 \to G,$$

which is an isomorphism in a neighborhood of the identity; see §4.2 for details. An important feature of this decomposition is the fact that the Haar measure of G in these coordinates decomposes as a product of H-Haar measure on the H component, times the M_1-Haar measure on the M_1 component, and times something depending only on U and $\overline{U}$; see §4.4. Moreover, in the proof, we only need the Casimir operator restricted to left-H- and right-M_1-invariant functions, for which we derive a nice, concise explicit expression in any dimension; see §4.5.

Let $\Gamma_1 := \Gamma \cap H$ denote the stabilizer of S_0 in Γ. It follows from the Structure Theorem for Kleinian packings [KK23, Theorem 22] that Γ_1 is a *lattice* in H (that is, it acts with finite covolume); this fact will be used crucially in our analysis. The finiteness of the volume of $\Gamma_1 \backslash H$ is analogous in the $\mathrm{SL}_2(\mathbb{R})$ setting to the finiteness of the volume of K, though the latter is trivial since K is compact. Note that we acted on the left in $\mathrm{SL}_2(\mathbb{R})$ and it is more convenient to act on the right for sphere packings.

In the $\mathrm{SL}_2(\mathbb{R})$ setting, the N direction was unbounded and required a cut-off. Analogously here, the U-direction is unbounded; this can be controlled via a similar truncation procedure by invoking the fact that the limit set is bounded in the U direction, see Lemma 64.

Remark 10. *Note that if the stabilizer of $\mathcal{P}$ contains a full rank unipotent subgroup then the methods of [Kon09] may be applied directly. For the existence of such, see [KN19].*

1.2 Organization

In section 2 we collect some preliminaries. In section 3, we warm up to the counting theorem and illustrate the main ideas by proving a result analogous to Theorem 2 in the $\mathrm{SL}_2(\mathbb{R})$ setting. In section 4, we switch to the general $\mathrm{SO}(n+1,1)$ setting, and derive the Haar measure, and Casimir operator in the above-described coordinate system. Finally, in section 5 we prove Theorem 2.

2 Preliminaries

2.1 Lie algebras and the Casimir Operator

We collect here some standard facts about the group $G = SO^{\circ}(n+1,1)$ (where $\circ$ denotes the connected component of the identity), and its Lie algebra $\mathfrak{g} = Lie(G)$ of dimension $d := \dim(\mathfrak{g}) = (n+1)(n+2)/2$.

In general, Casimir operators generate the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. In our rank-one setting, we can compute the Casimir operator $\mathcal{C}$ as follows. Let $X_1, \dots, X_d$ be a basis for the Lie algebra $\mathfrak{g}$, and let $X_1^*, \dots, X_d^*$ be a dual basis with respect to the Killing form:

$$B(X,Y) = \mathrm{Tr}(\mathrm{Ad}(X) \circ \mathrm{Ad}(Y)),$$

that is, $B(X_i, X_j^*) = \mathbb{1}_{\{i=j\}}$. Then the Casimir operator can be expressed as

$$\mathcal{C} = \sum_{i=1}^{d} X_i X_i^*.$$

Since the elements of the Lie algebra act like first order differential operators, the Casimir operator acts as a second order differential operator on smooth functions on G; see §4.5 for a detailed calculation in our setting.

Let $K \cong SO(n+1) < G$ be a maximal compact subgroup. When restricted to K-invariant smooth functions on G, the Casimir operator $\mathcal{C}$ agrees (up to constant) with the hyperbolic Laplacian Δ under an identification $G/K \cong \mathbb{H}^{n+1}$.

2.2 Decomposition of $L^2(\Gamma\backslash G)$ into irreducibles

Let Γ be a discrete, geometrically finite, Zariski dense subgroup of G. Given Iwasawa coordinates $G = NAK$, let $M \cong \mathrm{SO}(n)$ be the centralizer of A in K. The group G acts by the right-regular representation on the Hilbert space $\mathcal{H} := L^2(\Gamma\backslash G)$ of square-integrable Γ-automorphic functions. Recalling the standing assumption that $\mathcal{H}$ does not weakly contain any nonspherical complementary series representations, the space $\mathcal{H}$ decomposes into components as follows:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k \oplus \mathcal{H}^{tempered}. \tag{11}$$

Here each $\mathcal{H}_j$ is the G-span of the eigenfunction corresponding to the eigenvalue $\lambda_j = s_j(n-s_j)$ in (1), each of which is an irreducible spherical complementary series representation with parameter $s_j > n/2$, and $\mathcal{H}^{tempered}$ denotes tempered spectrum. Moreover, the subspace $\mathcal{H}_j^K$ of K-fixed vectors in $\mathcal{H}_j$ is 1-dimensional, and spanned by the corresponding eigenfunction in $L^2(\Gamma\backslash\mathbb{H}^{n+1}) \cong L^2(\Gamma\backslash G)^K$. In general, nonspherical complementary series representations can only occur if $n \geq 3$ and parameter $s \leq n-1$ [Kna01].

2.3 Abstract Spectral Theorem

We recall the abstract spectral theorem (see for example [Rud73, Ch. 13]) for unbounded self-adjoint operators. Let L be a self-adjoint, positive semidefinite operator on the Hilbert space $\mathcal{H}$. In our applications $\mathcal{H}$ will be either $L^2(\Gamma\backslash G)$ or a subspace thereof and L will be the Casimir operator.

Theorem 12 (Abstract Spectral Theorem)**.** *There exists a spectral measure ν on $\mathbb{R}$ and a unitary spectral operator $\widehat{\ }: \mathcal{H} \to L^2([0,\infty), d\nu)$ such that:*

[label =)]Abstract Parseval's Identity: for $\phi_1, \phi_2 \in \mathcal{H}$

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}} = \langle \widehat{\phi_1}, \widehat{\phi_2} \rangle_{L^2([0,\infty),d\nu)}; \tag{13}$$

The spectral operator is diagonal with respect to L: for $\phi \in \mathcal{H}$ and $\lambda \geq 0$,

$$\widehat{L\phi}(\lambda) = \lambda \widehat{\phi}(\lambda); \tag{14}$$

Moreover, if λ is in the point specturm of L with associated eigenspace $\mathcal{H}_\lambda$, then for any $\psi_1, \psi_2 \in \mathcal{H}$ one has

$$\widehat{\psi_1}(\lambda)\overline{\widehat{\psi_2}}(\lambda) = \langle \mathrm{Proj}_{\mathcal{H}_\lambda} \psi_1, \mathrm{Proj}_{\mathcal{H}_\lambda} \psi_2 \rangle, \tag{15}$$

where Proj refers to the projection to the subspace $\mathcal{H}_\lambda$. In the special case that $\mathcal{H}_\lambda$ is one-dimensional and spanned by the normalized eigenfunction ϕ_λ, we have that

$$\widehat{\psi_1}(\lambda)\overline{\widehat{\psi_2}}(\lambda) = \langle \psi_1, \phi_\lambda \rangle \langle \phi_\lambda, \psi_2 \rangle. \tag{16}$$

3 The $\mathrm{SL}_2(\mathbb{R})$ Case

In this section, let $G := \mathrm{SL}_2(\mathbb{R})$ and $\Gamma < G$ be a Zariski dense, finitely generated, discrete subgroup with $\delta_\Gamma > 1/2$. The goal of this section is to prove Theorem 18 below, that is, to improve on [KO12, Theorem 1.11] by extending the proof of [Kon09, Theorem 1.3 (1)] to the setting where Γ_∞ is trivial. This will serve as a model for the method that we will generalize to higher dimensions in the rest of the paper.

Again, from work of Lax-Phillips [LP82] and Patterson [Pat76] we have that the Laplace spectrum below $1/4$ consists of a finite number of discrete eigenvalues

$$\lambda_0 = \delta_\Gamma(1-\delta_\Gamma) < \lambda_1 \leq \cdots \leq \lambda_k < 1/4. \tag{17}$$

Write $\lambda_j = s_j(1-s_j)$ with $s_j \in (\frac{1}{2}, 1)$.

Theorem 18. *Let $G := \mathrm{SL}_2(\mathbb{R})$ and $\Gamma < G$ a Zariski dense, finitely generated, discrete subgroup with $\delta_\Gamma > 1/2$. Assume that ∞ is either a cusp for Γ with stabilizer Γ_∞ or ∞ is not in the limit set of Γ. Then there exist constants $c_0 > 0$, $c_1, \ldots, c_k$, and $\eta > 0$ such that as $T \to \infty$*

$$\begin{aligned} N_\Gamma(T) :&= \#\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma : c^2 + d^2 < T\} \\ &= c_0 T^{\delta_\Gamma} + c_1 T^{s_1} + \cdots + c_k T^{s_k} + O(T^\eta \log^{1/2} T), \end{aligned} \tag{19}$$

with $\eta = \frac{1}{2}(\delta_\Gamma + \frac{1}{2})$. (Of course some of the lower order "main terms", if any, may be dominated by the error term.)

Remark 20. *The case where ∞ is a cusp is the content of [Kon09, Theorem 1.3 (1.5)], so we assume below that ∞ is not in the limit set of Γ.*

Remark 21. *Note that in this case, since we are counting points in $\mathbb{H}^2$ (which is K invariant) we can extract all lower order terms corresponding to eigenvalues other than the base.*

Remark 22. *The corresponding error term in [KO12, §4.1] is significantly worse (and not even explicitly specified) as compared to Theorem 18, due in part to much worse dependence on Sobolev norms of the corresponding test vectors.*

To begin the proof, we proceed as described in the Introduction. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, let

$$\chi_T(g) := \begin{cases} 1 & \text{if } c^2 + d^2 < T, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Under the identification $G/K \sim \mathbb{H}$ (where $K = SO(2)$), $g \mapsto z = gi$, χ_T corresponds to the condition $\Im(z) > 1/T$.

Now average χ_T over the group Γ:

$$F_T(g) := \sum_{\gamma \in \Gamma} \chi_T(\gamma g). \tag{24}$$

such that the count in (19) can be written as $N_\Gamma(T) = F_T(e)$.

To access the value of F_T at the identity, we follow the standard procedure of smoothing the count. To this end, we fix once and for all a smooth, even, nonnegative, compactly supported bump function $\psi_1 \in C_0^\infty(\mathbb{R})$ with unit total mass, $\int_\mathbb{R} \psi_1 dx = 1$. Given $\varepsilon > 0$, we set $\psi_\varepsilon(x) := \frac{1}{\varepsilon}\psi_1(\frac{x}{\varepsilon})$. Write Iwasawa coordinates as $n_x := \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in N$, $a_y := \mathrm{Diag}(\sqrt{y}, 1/\sqrt{y}) \in A$, and $k \in K = \mathrm{SO}(2)$, and by abuse of notation, write $\psi : G \to \mathbb{R}_+$ as follows:

$$\psi(n_x a_y k) := \psi_\varepsilon(x)\psi_\varepsilon(\log y). \tag{25}$$

Clearly ψ is right-K-invariant and depends on ε (although we omit this from the notation). It is easy to compute that $\int_G \psi(n_x a_y k)\frac{dx\,dy\,dk}{y^2} = 1 + O(\varepsilon)$.

We automorphise ψ by setting:

$$\Psi(g) = \Psi_\varepsilon(g) := \sum_{\gamma \in \Gamma} \psi(\gamma g),$$

and consider the smoothed count:

$$\widetilde{N}_\Gamma(T) = \widetilde{N}_\Gamma^\varepsilon(T) := \langle F_T, \Psi \rangle.$$

After unfolding F_T, we see that

$$\widetilde{N}_\Gamma(T) := \sum_{\gamma \in \Gamma} w_T(\gamma),$$

where $w_T = w_{T,\varepsilon} : \mathbb{H} \to [0,1]$ is given by

$$w_T(g) := \int_G \chi_T(gh)\psi(h)dh. \tag{26}$$

The following theorem, from which Theorem 18 follows by optimizing error terms, gives an asymptotic expansion for the smoothed count in the $\mathrm{SL}_2(\mathbb{R})$-setting.

Theorem 27. *Assume that $\varepsilon > 0$ is small enough. Then there exist constants $c_{\Gamma,\varepsilon}^{(i)}$ for $i = 0, 1, \ldots, k$ such that*

$$\widetilde{N}_\Gamma(T) = c_{\Gamma,\varepsilon}^{(0)} T^{\delta_\Gamma} + c_{\Gamma,\varepsilon}^{(1)} T^{s_1} + \cdots + c_{\Gamma,\varepsilon}^{(k)} T^{s_k} + O\left(\frac{1}{\varepsilon} T^{1/2} \log T\right) \tag{28}$$

with $c_{\Gamma,\varepsilon}^{(0)} > 0$, and the implied constant depending only on Γ. Moreover $c_{\Gamma,\varepsilon}^{(i)} = c^{(i)}(1 + O(\varepsilon))$ for all $i = 0, 1, \ldots, k$.

It remains to prove Theorem 27.

3.1 Inserting the Laplacian

The smooth count above is an L^2 inner product of the indicator function F_T with a smooth bump function Ψ. The key idea now is to forget about Ψ and analyze the structure of the inner product of F_T with *any* smooth L^2 function using the hyperbolic Laplacian $\Delta = -y^2(\partial_{xx} + \partial_{yy})$.

Following [Kon09, §3] let

$$K_T(s) := \frac{T^s b^{1-s} - T^{1-s} b^s}{b^{1-s} - b^s}, \qquad L_T(s) := \frac{T^{1-s} - T^s}{b^{1-s} - b^s}, \tag{29}$$

where $1 < b < 3$ is a constant depending on T which ensures that K_T and L_T have no poles (see [Kon09, (3.7)] for an exact computation). With that, if ϕ *were* an eigenfunction of the Laplacian then we would have $\langle F_T, \phi\rangle = A_\phi T^s + B_\phi T^{1-s}$. In this case, we could conclude that

$$\langle F_T, \phi\rangle = K_T(s)\langle F_1, \phi\rangle + L_T(s)\langle F_b, \phi\rangle.$$

Thus we would like to show that

$$F_T = K_T(\Delta)F_1 + L_T(\Delta)F_b, \tag{30}$$

where $K_T(\Delta)$ (and $L_T(\Delta)$) is defined via (29) in the same way that one defines matrix exponentiation (via Taylor expansion). However, there is a problem created by the fact that F_T is not in $L^2(\Gamma\backslash\mathbb{H})$. To get around this, we will first restrict the support of F_T in the x-direction before applying various smoothing arguments to conclude that a version of (30) holds for the modified F_T. Since ∞ is not in the limit set, it is in the free boundary, and hence there exists a fixed an $X = X(\Gamma) > 0$ such that the full region $((-\infty, -X] \cup [X, \infty)) \times [0, \infty) \subset \mathbb{H}$ is contained in a single fundamental domain, see Figure 2.

Restricting the Real Direction: Define the following counting function

$$F_{T,X}(z) := \sum_{\gamma\in\Gamma} \chi_T(\gamma z)\widetilde{\chi}_X(\gamma z)$$

where

$$\widetilde{\chi}_X(x+iy) := \begin{cases} 1 & \text{if } x \in [-X, X], \\ 0 & \text{if } x \notin [-X, X]. \end{cases}$$

Let $\mathfrak{I}_X := [-X, X]$ denote the support of $\widetilde{\chi}_X$. Note that by our choice of X, we still have that $N_\Gamma(T) = F_T(i) = F_{T,X}(i)$.

Now, consider the difference operator

$$G_{T,X} := F_{T,X} - K_T(\Delta)F_{1,X} - L_T(\Delta)F_{b,X}.$$

Our goal is to show, that for any $\Psi \in L^2(\Gamma\backslash\mathbb{H})$ we have $\langle G_{T,X}, \Psi\rangle_\Gamma = 0$. This implies the following identity

Proposition 31. *For any values of T and X large enough depending on the group Γ, we have*

$$F_{T,X} = K_T(\Delta)F_{1,X} + L_T(\Delta)F_{b,X}. \tag{32}$$

where K_T and L_T are the differential operators defined above and b fixed.

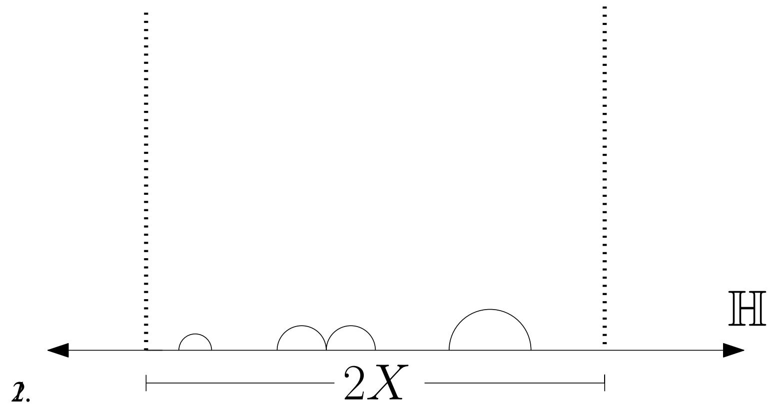

Figure 2: A fundamental domain of Γ which extends to infinity in the real direction, and the cut-off due to $\widetilde{\chi}_X$ which is large enough such that $[-X,X]$ contains the entire limit set.

Proving this proposition will be somewhat involved, so we break it down into several steps. First, we unfold the inner product

$$\langle G_{T,X},\Psi\rangle = \int_{\mathbb{H}} (\chi_T(z)\widetilde{\chi}_X(z) - K_T(\Delta)\chi_1(z)\widetilde{\chi}_X(z) - L_T(\Delta)\chi_b(z)\widetilde{\chi}_X(z))\Psi(z)\frac{dxdy}{y^2}.$$

Smoothing χ_T: Now we smooth the function χ_T so that we can move the Laplacian over to $\widetilde{\chi}_X$ via self-adjointness. Let $\sigma > 0$, and define the following smooth cut-off function

$$\chi_{1,\sigma}(x+iy) := \begin{cases} 1 & \text{if } y > 1, \\ 0 & \text{if } y < (1-\sigma), \end{cases}$$

and smooth, in between the two bounds. Now let $\chi_{T,\sigma}(x+iy) := \chi_{1,\sigma}(x+iTy)$.

Let $G^{\sigma}_{T,X}$ be defined similarly to $G_{T,X}$, with χ_T replaced by $\chi_{T,\sigma}$. That is, let

$$G^{\sigma}_{T,X}(\Psi) := \int_0^{\infty}\int_{x\in\mathfrak{I}_X} \widetilde{\chi}_X(z) \\ (\chi_{T,\sigma}(z)\Psi(z) - \chi_{1,\sigma}(z)(K_T(\Delta)\Psi)(z) - \chi_{b,\sigma}(z)(L_T(\Delta)\Psi)(z))\,dx\frac{dy}{y^2}.$$

Note that by construction, for any fixed t,

$$\int_0^{\infty}\int_{x\in\mathfrak{I}_X} |\chi_{t,\sigma}(z) - \chi_t|^2\,dx\frac{dy}{y^2} \ll_{t,X} \sigma.$$

Thus, using Cauchy-Schwarz, we have that for any $\Psi \in L^2(\Gamma\backslash\mathbb{H})$

$$\lim_{\sigma\to 0} G^{\sigma}_{T,X}(\Psi) = \langle G_{T,X},\Psi\rangle_{\Gamma}.$$

Note also that, by the support of the $\chi_{*,\sigma}$, we may also write, when convenient,

$$G^{\sigma}_{T,X}(\Psi) := \int_{1/2T}^{\infty}\int_{x\in\mathcal{I}_X}\widetilde{\chi}_X(z)$$
$$(\chi_{T,\sigma}(z)\Psi(z)-\chi_{1,\sigma}(z)(K_T(\Delta)\Psi)(z)-\chi_{b,\sigma}(z)(L_T(\Delta)\Psi)(z))\,dx\frac{dy}{y^2}.$$

Now our goal, is to establish the following lemma, note that we have already fixed T, X large enough, $\sigma>0$ small enough.

Lemma 33. *For any function $\Psi\in L^2(\Gamma\backslash\mathbb{H})$ we have that $G^{\sigma}_{T,X}(\Psi)=0$.*

To prove this lemma, fix σ, then we will show that for any $\varepsilon>0$ small enough $\left|G^{\sigma}_{T,X}(\Psi)\right|<\varepsilon$.

Periodizing and Smoothing Ψ: Now we periodize and smooth Ψ in order to do spectral theory on $\mathcal{I}_X\times(0,\infty)$ which we call the space $\mathcal{F}_X$. Note that $\mathcal{F}_X\cong\Xi_\infty\backslash\mathbb{H}$ for the elementary group $\Xi_\infty:=\left(\begin{smallmatrix}1 & 2X\mathbb{Z}\\ 0 & 1\end{smallmatrix}\right)$, but this coordinate description is more useful when we generalize to higher dimensions. Let $L^2(\mathcal{F}_X)$ denote the L^2 space with measure $\frac{dxdy}{y^2}$ and periodic with respect to Ξ_∞. Let $\widetilde{\Psi}(z)$ denote a function which agrees with Ψ on

$$[-X,X-\eta)\times(1/2T,\infty)\subset\mathbb{H},$$

where $\eta>0$ is to be chosen later. From $x=X-\eta$ to $x=X-\eta/2$, $\widetilde{\Psi}(z)$ smoothly interpolates between the values of $\Psi(z)$ and $\Psi(z-2X)$, and from $x=X-\eta/2$ up to $x=X$, $\widetilde{\Psi}$ is exactly equal to $\Psi(z-2X)$, ensuring that all derivatives of $\widetilde{\Psi}$ at the boundary values $x=X$ and $x=-X$ agree. We also impose that $\widetilde{\Psi}$ decays rapidly to zero below $y=1/4T$, so that $\widetilde{\Psi}\in L^2(\mathcal{F}_X)$.

Note that the cost of moving from Ψ to $\widetilde{\Psi}$ is small. Indeed,

$$\begin{aligned}\operatorname{diff}(\Psi) &:= \int_{1/2T}^{\infty}\int_{X-\eta}^{X}\left|\widetilde{\Psi}(z)-\Psi(z)\right|^2dx\frac{dy}{y^2}\\ &\ll \int_{1/2T}^{\infty}\int_{X-\eta}^{X}|\Psi(z-2X)|^2+|\Psi(z)|^2dx\frac{dy}{y^2}.\end{aligned}$$

Since $\Psi\in L^2(\Gamma\backslash\mathbb{H})$, as $\eta\to0$, this integral goes to zero, and hence by choosing η small enough, we can make the difference less than ε^2.

Now note that

$$\begin{aligned}&\int_{1/2T}^{\infty}\int_{\mathcal{I}_X}(\chi_{T,\sigma}(z)\widetilde{\chi}_X(z)-K_T(\Delta)\widetilde{\chi}_X(z)\chi_{1,\sigma}(z)-L_T(\Delta)\widetilde{\chi}_X(z)\chi_{b,\sigma}(z))\Psi(z)dx\frac{dy}{y^2}\\ &=\int_{1/2T}^{\infty}\int_{\mathcal{I}_X}(\chi_{T,\sigma}(z)\widetilde{\chi}_X(z)-K_T(\Delta)\chi_{1,\sigma}(z)\widetilde{\chi}_X(z)-L_T(\Delta)\chi_{b,\sigma}(z)\widetilde{\chi}_X(z))(\Psi(z)-\widetilde{\Psi}(z))dx\frac{dy}{y^2}\\ &+\int_{1/2T}^{\infty}\int_{\mathcal{I}_X}(\chi_{T,\sigma}(z)\widetilde{\chi}_X(z)-K_T(\Delta)\chi_{1,\sigma}(z)\widetilde{\chi}_X(z)-L_T(\Delta)\chi_{b,\sigma}(z)\widetilde{\chi}_X(z))\widetilde{\Psi}(z)dx\frac{dy}{y^2},\end{aligned}$$

and by Cauchy Schwarz and the above argument

$$\int_{1/2T}^{\infty}\int_{\mathcal{I}_X}(\chi_{T,\sigma}(z)\widetilde{\chi}_X(z)-K_T(\Delta)\chi_{1,\sigma}(z)\widetilde{\chi}_X(z)-L_T(\Delta)\chi_{b,\sigma}(z)\widetilde{\chi}_X(z))(\Psi(z)-\widetilde{\Psi}(z))dx\frac{dy}{y^2}$$
$$\ll (\mathrm{diff}(\Psi))^{1/2}\left[\int_{1/2T}^{\infty}\int_{\mathcal{I}_X}\Big(|(\chi_{T,\sigma}(z)\widetilde{\chi}_X(z)|^2+|K_T(\Delta)\chi_{1,\sigma}(z)\widetilde{\chi}_X(z)|^2\right.$$
$$\left.+|L_T(\Delta)\chi_{b,\sigma}(z)\widetilde{\chi}_X(z))|^2\Big)dx\frac{dy}{y^2}\right]^{1/2}$$

Since $\mathrm{diff}(\Psi)$ is bounded by ε^2, it remains to show that the integral in the brackets is finite. Taking, say, the middle term (the others being similar), we note that, since the region $[1/2T,\infty)\times\mathcal{I}_X$ is contained in a finite number (depending on T) of translates of a fixed fundamental domain, $\mathcal{F}$, for Γ,

$$\int_{1/2T}^{\infty}\int_{\mathcal{I}_X}|K_T(\Delta)\chi_{1,\sigma}(z)\widetilde{\chi}_X(z)|^2dx\frac{dy}{y^2}\leq\sum_{\gamma\in\Gamma}\int_{\mathcal{F}}|K_T(\Delta)\chi_{1,\sigma}(\gamma z)\widetilde{\chi}_X(\gamma z)|^2\frac{dxdy}{y^2},$$

where only finitely many values of γ give nonzero contributions to the sum. For each contributing γ, we can extend the function $h_\gamma(z)=h_{\gamma,T,X,\sigma}(z):=\chi_{1,\sigma}(\gamma z)\widetilde{\chi}_X(\gamma z)$ supported on $\mathcal{F}$ to a Γ-automorphic function $H_\gamma(z)$ on $\mathbb{H}$, which agrees with $h_\gamma(z)$ on $\mathcal{F}$. Applying the Abstract Spectral Theorem (13), (14) in $L^2(\Gamma\backslash\mathbb{H})$, we get

$$\int_{\Gamma\backslash\mathbb{H}}|K_T(\Delta)H_\gamma(z)|^2\frac{dxdy}{y^2}=\int_{\lambda\in[0,\infty)}\left|K_T(\lambda)\widehat{H_\gamma}(\lambda)\right|^2d\nu(\lambda)$$
$$\ll\ T^2\,\|H_\gamma\|^2_{L^2(\Gamma\backslash\mathbb{H})}\ \ll_{T,X,\sigma}\ 1,$$

where ν is the abstract spectral measure, and we used the bounds $K_T(\lambda)\ll T^s\leq T^1$.

With that, the difference

$$\left|G^{\sigma}_{T,X}(\Psi)-G^{\sigma}_{T,X}(\widetilde{\Psi})\right|\ll_{T,X,\sigma}\varepsilon \tag{34}$$

for any value of $\varepsilon>0$. Thus it remains to show that $G^{\sigma}_{T,X}(\widetilde{\Psi})=0$. Since $\widetilde{\Psi}(x+iy)$ is periodic on $x\in\mathcal{I}_X$, we can use self-adjointness to write

$$G^{\sigma}_{T,X}(\widetilde{\Psi})=\int_0^{\infty}\int_{\mathcal{I}_X}(\chi_{T,\sigma}(z)-K_T(\Delta)\chi_{1,\sigma}(z)-L_T(\Delta)\chi_{b,\sigma}(z))\widetilde{\Psi}(z)dx\frac{dy}{y^2}. \tag{35}$$

Working on $\mathcal{F}_X$: To prove (35) let Now, to prove that $G^{\sigma}_{T,X}(\widetilde{\Psi})=0$, let

$$g^{\sigma}_{T,X}(z):=\chi_{T,\sigma}(z)-K_T(\Delta)\chi_{1,\sigma}(z)-L_T(\Delta)\chi_{b,\sigma}(z).$$

Thus, we have

$$G^{\sigma}_{T,X}(\widetilde{\Psi})=\langle g^{\sigma}_{T,X},\widetilde{\Psi}\rangle_{\mathcal{F}_X}.$$

Lemma 33 will then follow from the following lemma, which shows that, for an arbitrary $\psi\in L^2(\mathcal{F}_X)$ the inner product $\langle g^{\sigma}_{T,X},\psi\rangle$ is not correlated with any almost eigenfunction:

Lemma 36. *Fix T, X, and σ as above, then for any $\psi \in L^2(\mathcal{F}_X)$ and any $\lambda \geq 0$, we have*

$$\langle g^{\sigma}_{T,X}, \psi\rangle_{\mathcal{F}_X} \ll_{\lambda,T,\sigma,X} \|(\Delta-\lambda)\psi\|_{\mathcal{F}_X}. \tag{37}$$

Proof of Lemma 36. Fix $\psi \in L^2(\mathcal{F}_X)$ and consider

$$\langle \chi_{T,\sigma}, \psi\rangle_{\mathcal{F}_X} = \int_0^\infty \chi_{T,\sigma}(y) \int_{\mathcal{I}_X} \psi(z) dx \frac{dy}{y^2},$$

and define $f(y) := \int_{\mathcal{I}_X} \psi(z)dx$ and $h(y) := \int_{\mathcal{I}_X} (\Delta-\lambda)\psi(z)dx$. Then we note that, by periodicity in the x-direction,

$$\begin{aligned} h(y) &= \int_{\mathcal{I}_X} (\Delta-\lambda)\psi(z)dx \\ &= -y^2\partial_{yy} f(y) - \lambda f(y). \end{aligned}$$

Thus, [Kon09, Lemma B.1] (which is a simple application of the method of variation of parameters) implies

$$f(y) = \alpha y^s + \beta y^{1-s} + y^s u(y) + y^{1-s} v(y), \tag{38}$$

where

$$u(y) := (1-2s)^{-1} \int_{(1-\sigma)/T}^{y} w^{-1-s} h(w) dw,$$

$$\text{and} \;\; v(y) := (1-2s)^{-1} \int_{(1-\sigma)/T}^{y} w^{s-2} h(w) dw,$$

if $\lambda = s(1-s) \neq 1/4$. Note that the choice of integration boundary for u and v corresponds to the bottom of the support of $\chi_{T,\sigma}$; this will be convenient later on. Moreover when $\lambda = 1/4$

$$f(y) = \alpha y^{1/2} + \beta y^{1/2} \log y + y^{1/2} u(y) + v(y) y^{1/2} \log y, \tag{39}$$

where

$$u(y) := \int_{(1-\sigma)/T}^{y} w^{-3/2} \log(w) h(w) dw,$$

$$\text{and} \qquad v(y) := -\int_{(1-\sigma)/T}^{y} w^{-3/2} h(w) dw.$$

Therefore (assuming $\lambda \neq 1/4$ for simplicity) integrating the y variable gives

$$\langle \chi_{T,\sigma}, \psi\rangle_{\Xi_\infty} = \int_0^\infty \chi_{T,\sigma}(y)(\alpha y^s + \beta y^{1-s}) \frac{dy}{y^2} + I + II$$

where $I := \int_0^\infty \chi_{T,\sigma}(y)y^{s-2}u(y)dy$ and $II := \int_0^\infty \chi_{T,\sigma}(y)y^{-1-s}v(y)dy$. Now we can use Cauchy-Schwarz (as in [Kon09, (B.5)]) to establish that $I, II \ll \|(\Delta-\lambda)\psi\|_{\mathcal{F}_X}$. In fact, this is the crucial reason why we needed to work on $\mathcal{F}_X$ which is compact in the x direction, and thus $\chi_{T,\sigma}$ is (square) integrable. Thus, we may conclude

$$\langle \chi_{T,\sigma}, \psi\rangle_{\mathcal{F}_X} = A_\sigma T^s + B_\sigma T^{1-s} + O(\|(\Delta-\lambda)\psi\|_{\mathcal{F}_X})$$

where $A_\sigma = \beta \int_0^\infty \chi_{1,\sigma}(y)y^{-1-s}dy$ and $B_\sigma := \alpha \int_0^\infty \chi_{1,\sigma}(y)y^{s-2}dy$.

Given our choice of K_T and L_T from (29) we have that

$$K_T(s), L_T(s) \ll \begin{cases} T^s & \text{if } s \in (1/2, 1], \\ T^{1/2}\log T & \text{if } s = 1/2+it. \end{cases} \tag{40}$$

Then, by the analysis in [Kon09, Proposition 3.5] we have

$$\langle g^\sigma_{T,X}, \psi\rangle_{\mathcal{F}_X} \ll_{T,\lambda,X,\sigma} \|(\Delta-\lambda)\psi\|_{\mathcal{F}_X} \tag{41}$$

for any choice of $\psi \in L^2(\mathcal{F}_X)$. □

From there we can choose ψ to be as in [Kon09, Proof of Theorem 3.2] to establish that $g^\sigma_{T,X}$ is almost everywhere 0. Note that for this argument to work one only needs the bound (41), a Hilbert space (here $L^2(\mathcal{F}_X)$), an unbounded self-adjoint operator (i.e Δ), and the abstract spectral theorem.

From there we conclude that

$$\langle g^\sigma_{T,X}, \widetilde{\Psi}\rangle_{\mathcal{F}_X} = 0$$

From there, we conclude that $G^\sigma_{T,X}(\widetilde{\Psi}) = 0$. Then thanks to (34) we conclude

$$\begin{aligned} G^\sigma_{T,X}(\Psi) &= \int_0^\infty \int_{\mathcal{I}_X} \widetilde{\chi}_X(z)(\chi_{T,\sigma}(z) - K_T(\Delta)\chi_{1,\sigma}(z) - L_{T,\sigma}(\Delta)\chi_{b,\sigma}(z))\Psi(z)dx\frac{dy}{y^2} \\ &\ll_{\sigma,T,X} \varepsilon \end{aligned}$$

for any value of $\varepsilon > 0$. Taking ε to 0 establishes Lemma 33. Since we have that for any $\Psi \in L^2(\Gamma\backslash\mathbb{H})$

$$\lim_{\sigma\to 0} G^\sigma_{T,X}(\Psi) = \langle G_{T,X}, \Psi\rangle_\Gamma$$

we conclude that $\langle G_{T,X}, \Psi\rangle_\Gamma = 0$. Which is exactly Proposition 31. □

3.2 Proof of Theorem 27

For the proof of Theorem 27, we return to the smoothed count

$$\widetilde{N}_\Gamma(T) = \langle F_{T,X}, \Psi\rangle_\Gamma.$$

Now apply the abstract Parseval's identity (13)

$$\begin{aligned}\langle F_{T,X}, \Psi\rangle_\Gamma &= \langle \widehat{F_{T,X}}, \widehat{\Psi}\rangle_{\mathrm{Spec}(\Gamma)} \\ &= \widehat{F_{T,X}}(\lambda_0)\widehat{\Psi}(\lambda_0) + \int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \widehat{F_{T,X}}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\nu(\lambda),\end{aligned} \tag{42}$$

where for ease of exposition, we assume that λ_0 is the only eigenvalue below $1/4$; in general, the other eigenvalues are dealt with similarly.

Addressing the first term in (42), we can use the abstract spectral theorem, and the fact that $F_{T,X}$ and Ψ are K-fixed, to say

$$\widehat{F_{T,X}}(\lambda_0)\widehat{\Psi}(\lambda_0) = \langle F_{T,X}, \phi_0\rangle\langle \Psi, \phi_0\rangle,$$

where ϕ_0 is the base eigenfunction. Note that, for the first factor we can apply the main identity Proposition 31, and the definition to conclude

$$\widehat{F_{T,X}}(\lambda_0)\widehat{\Psi}(\lambda_0) = T^\delta c\langle \phi_0, \Psi\rangle_\Gamma + O(T^{1/2})$$

for some constant c independent of T. As for the second factor, by the mean value theorem (see [Kon09, (4.17)] for details) we have

$$\langle \phi_0, \Psi_\varepsilon\rangle_\Gamma = \phi_0(i) + O(\varepsilon).$$

It remains to bound the contribution to (42) from the remainder of the spectrum (assuming here that there are no isolated eigenvalues apart from the base). Using Proposition 31 we have that

$$\begin{aligned}\mathrm{Err} &:= \int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \widehat{F}_{T,X}(\lambda)\widehat{\Psi}(\lambda)d\nu \\ &= \int_{\mathrm{Spec}(\Gamma\setminus\{\lambda_0\}} \left(K_T\widehat{(\Delta)F}_{1,X}(\lambda) + L_T\widehat{(\Delta)F}_{b,X}(\lambda)\right)\widehat{\Psi}(\lambda)d\nu.\end{aligned}$$

By the abstract spectral theorem and (40) we have $K_T\widehat{(\Delta)F}_{1,X}(\lambda) \ll T^{1/2}\log T\widehat{F_{1,X}}(\lambda)$. Thus, by Cauchy-Schwarz, positivity, and Parseval's identity

$$\begin{aligned}&\int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} K_T(\lambda)\widehat{F_{1,X}}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\nu \ll T^{1/2}\log T\int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \widehat{F_{1,X}}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\nu \\ &\ll T^{1/2}\log T\left(\int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \left|\widehat{F_{1,X}}(\lambda)\right|^2\mathrm{d}\nu\right)^{1/2}\left(\int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \left|\widehat{\Psi}(\lambda)\right|^2\mathrm{d}\nu\right)^{1/2} \\ &\ll T^{1/2}\log T\|F_{1,X}\|_\Gamma\|\Psi\|_\Gamma,\end{aligned}$$

Since Ψ is an ε-approximation to the identity, and since the term involving L_T can be treated similarly, we thus conclude

$$\mathrm{Err} \ll \frac{X^{1/2}}{\varepsilon}T^{1/2}\log T,$$

note that X does not go to ∞, it is kept fixed. This completes the proof of Theorem 27.

4 Kleinian Sphere Packings

Turning now to the sphere packing setting, let $\mathcal{P}$ be a fixed bounded Kleinian sphere packing. Given a sphere $S \in \mathcal{P}$ let $b(S)$ denote the bend of S. Then our aim is to establish the asymptotic, (5) for

$$N_{\mathcal{P}}(T) := \#\{S \in \mathcal{P} : b(S) < T\}.$$

4.1 Preliminaries on Inverse Coordinate Systems and the Symmetry Group

We now give a model of hyperbolic space, and develop an inversive coordinate system for the spheres on the ideal boundary of such; see, e.g., [KK23, §3.1] for background. To begin, we fix a real quadratic form Q of signature $(n+1,1)$. For concreteness, we can change variables over $\mathbb{R}$ to the "standard" example of $Q = -x_0x_{n+1} + x_1^2 + \cdots + x_n^2$ which has half-Hessian

$$Q = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & I_n & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}. \tag{43}$$

Then the quadratic space $(V = \mathbb{R}^{n+1,1}, Q)$ with product $v \star w = v \cdot Q \cdot w^t$ contains the cone $V_0 := \{v : Q(v) = 0\}$, the one-sheeted hyperboloid $V_1 = \{v : Q(v) = 1\}$, and the two-sheeted hyperboloid $V_{-1} := \{v : Q(v) = -1\}$; fix either component of the latter for our model of $\mathbb{H}^{n+1}$. The group $O_Q(\mathbb{R})$ acts on V_{-1}, and its subgroup $G = O_Q^\circ(\mathbb{R})$, that is, the connected component of the identity, fixes the components.

There is a 1-1 correspondence between vectors $v \in V_1$ and (oriented) spheres in $\partial\mathbb{H}^{n+1}$, obtained as follows. Given such a v, the orthogonal space $v^\perp := \{w \in V : v \star w = 0\}$ intersects the fixed component of V_{-1} at a hyperplane $\cong \mathbb{H}^n$, and the ideal boundary of the latter is the desired sphere.

This geometric correspondence is made algebraic after a choice of (inversive) coordinates on V_1 as follows. Let $V^* := \{v^* : V \to \mathbb{R}, \text{ linear}\}$ be the dual space to V, and Q^* the dual form, so that $v^* \star w^* = v \star w$. Fix a non-zero null covector $b^* \in V^*$, that is, $Q^*(b^*) = 0$. Also fix a null covector $\hat{b}^*$, with $b^* \star \hat{b}^* = -2$. For the case of the standard form in (43), one can make the choice $b^* = (0,\ldots,0,-2)$ and $\hat{b}^* = (-2,0,\ldots,0)$. Then one picks an orthonormal system, $bx_1^*,\ldots,bx_n^*$, for the orthogonal complement to the span of b^* and $\hat{b}^*$. Then the sphere corresponding to the vector $v \in V_1$ has bend (that is, inversive radius)

$$b^*(v), \tag{44}$$

and center $\frac{1}{b^*(v)}(bx_1^*(v),\ldots,bx_n^*(v))$. When $b^*(v) = 0$, the sphere is a plane (which has no "center"), so the expression $(bx_1^*(v),\ldots,bx_n^*(v))$ is a unit normal to the plane. The "co-bend" of a sphere is defined as the bend of the image of the sphere on inversion through the unit sphere at the origin. The co-bend of the sphere corresponding to v is given by $\hat{b}^*(v)$. Therefore the tuple $(b^*(v), bx_1^*(v),\ldots,bx_n^*(v),\hat{b}^*(v))$ gives a complete inversive coordinate system on V_1. It is sometimes convenient to isolate the 'bend-center', comprising the coordinates not including the first and last; so we define:

$$bz^*(v) := (bx_1^*(v),\ldots,bx_n^*(v)). \tag{45}$$

A Kleinian sphere packing decomposes into finitely many Γ-orbits (by the Structure Theorem [KK23, Theorem 22]), which in the above coordinate system corresponds simply to orbits $v_0 \cdot \Gamma$, with $v_0 \in V_1$,

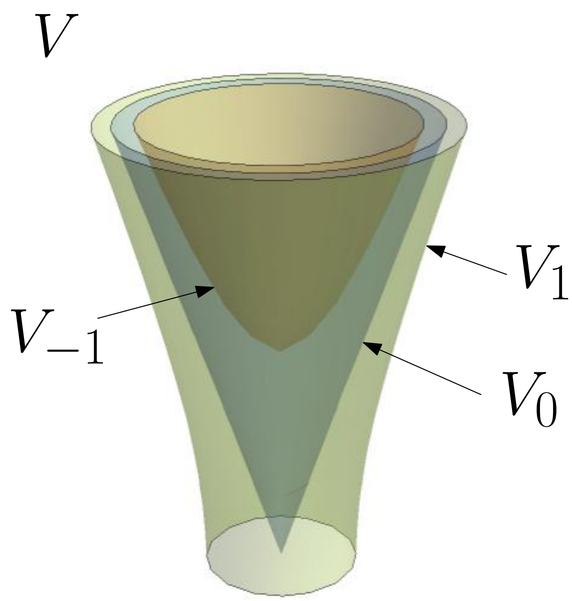

Figure 3: The quadratic space V, and (the upper parts of) its components V_0, V_1 and V_{-1}.

and the bends of such are measured by $b^*(v)$, for $v \in v_0 \cdot \Gamma$. We count the whole packing by counting one orbit at a time.

4.2 Decomposition of G

Fix v_0 as above. Let $\chi_T(g) = \mathbb{1}\{b^*(v_0 g) < T\}$ denote the indicator function of the vector $v_0 g$ having bend at most T, where $g \in G$. This function is left-invariant under $H := \mathrm{Stab}_G(v_0) = \{g \in G : v_0 g = v_0\}$, and also right-invariant under $L := \mathrm{Stab}_{b^*} = \{g \in G : b^*(vg) = b^*(v), \text{ for all } v \in V\}$.

It will be useful to decompose $G = O_Q^\circ(\mathbb{R})$ (for Q given by (43)) and its Lie algebra $\mathfrak{g} := Lie(G)$ as follows. First we decompose $\mathfrak{g}$ as:

$$\mathfrak{g} = \mathfrak{h} \oplus \bar{\mathfrak{u}} \oplus \mathfrak{u} \oplus \mathfrak{m}_1.$$

Here $\mathfrak{h} = Lie(H)$, and with

$$M := \begin{pmatrix} 1 & 0 & 0 \\ 0 & O(n) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have that $M \cap H \cong O(n-1)$. Then set $\mathfrak{m} := Lie(M)$ and $\mathfrak{m}_1 := \mathfrak{m} \ominus Lie(M \cap H)$. (Note that $\mathfrak{h} \cap \mathfrak{m}$) is

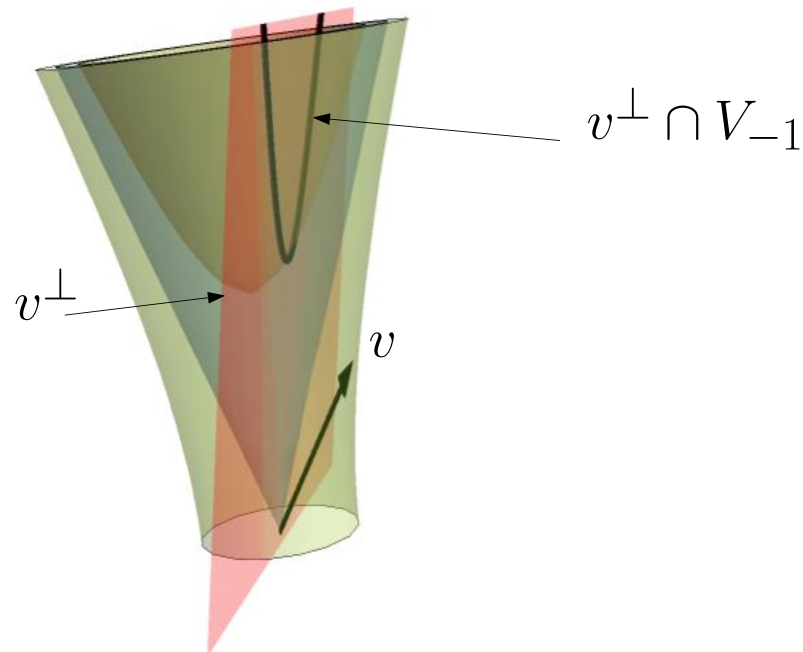

Figure 4: A vector $v \in V_1$, its orthogonal space $v^\perp$ and the intersection this orthogonal space with V_{-1} which corresponds to an ideal sphere.

trivial when $G \cong O(3,1)$.) Set $M_1 := \exp \mathfrak{m}_1 < G$. The one-parameter Lie algebras $\mathfrak{u}$ and $\bar{\mathfrak{u}}$ are given by:

$$\mathfrak{u} = \left\{ \begin{pmatrix} 0 & 0_{n-1} & w & 0 \\ 0 & 0 & 0 & 0_{n-1} \\ 0 & 0 & 0 & 2w \\ 0 & 0 & 0 & 0 \end{pmatrix} : w \in \mathbb{R} \right\},$$

and

$$\bar{\mathfrak{u}} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0_{n-1} & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0_{n-1} & y/2 & 0 \end{pmatrix} : y \in \mathbb{R} \right\},$$

which exponentiate to the groups

$$U := \left\{ u(w) := \begin{pmatrix} 1 & 0_{n-1} & w & w^2 \\ 0 & I_{n-1} & 0 & 0_{n-1} \\ 0 & 0 & 1 & 2w \\ 0 & 0 & 0 & 1 \end{pmatrix} : w \in \mathbb{R} \right\},$$

and

$$\overline{U} := \left\{ \overline{u}(y) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0_{n-1} & I_{n-1} & 0 & 0 \\ y & 0 & 1 & 0 \\ y^2/4 & 0_{n-1} & y/2 & 1 \end{pmatrix} : y \in \mathbb{R} \right\},$$

respectively.

This gives the corresponding decomposition:

$$H \times \overline{U} \times U \times M_1 \to G : (h, \overline{u}, u, m_1) \mapsto h\overline{u}um_1.$$

Geometrically this decomposition is in fact very natural. H is the stabilizer of v_0, corresponding to the sphere whose orbit we are considering. The action of $\overline{U}$ changes the bend of the sphere, while UM_1 changes the co-bend by moving the center via a polar coordinate description of the plane (U as the radial coordinate and $M_1 \cong O(n)/O(n-1)$ giving a rotation).

Let $d := \dim(\mathfrak{g}) = (n+1)(n+2)/2$. Let $p := \dim(\mathfrak{h}) = n(n+1)/2$, and let $\ell := \dim(\mathfrak{m}_1) = n-1$. Hence $d = p + \ell + 2$.

4.3 An Explicit Basis.

Before proceeding, it will help our calculations to fix a basis for $\mathfrak{h}$ and $\mathfrak{m}_1$. To ease notation, we will describe a basis in a special case when $n = 4$, so $d = 15$, $p = 10$ and $\ell = 3$; the general case will be clear by analogy. Take:

$$\left. X_1 := \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&-1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, X_2 := \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \\ 0&-1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, X_3 := \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&-1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \right\} \in \mathfrak{h} \cap \mathfrak{m}$$

$$\left. X_4 := \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&\frac{1}{2}&0&0 \end{pmatrix}, X_5 := \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&\frac{1}{2}&0&0&0 \end{pmatrix}, X_6 := \begin{pmatrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&\frac{1}{2}&0&0&0&0 \end{pmatrix}, \right\} \text{Lower Triangular in } \mathfrak{h}$$

$$\left. X_7 := \begin{pmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&2 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, X_8 := \begin{pmatrix} 0&0&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&2 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, X_9 := \begin{pmatrix} 0&1&0&0&0&0 \\ 0&0&0&0&0&2 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \right\} \text{Upper Triangular in } \mathfrak{h}$$

$$X_{10} := \begin{pmatrix} 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&-1 \end{pmatrix}, \text{ -- Diagonal in } \mathfrak{h}$$

$$X_{11} := \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&\frac{1}{2}&0 \end{pmatrix} \in \overline{\mathfrak{u}}, \; X_{12} := \begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&2 \\ 0&0&0&0&0&0 \end{pmatrix} \in \mathfrak{u},$$

$$X_{13} := \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&-1&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, X_{14} := \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&-1&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, X_{15} := \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&-1&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix} \Bigg\} \qquad \in \mathfrak{m}_1.$$

To move back to group elements we denote

$$\begin{aligned} h_j(x_j) &:= \exp(x_j X_j) \in H \qquad && \text{for } j \le p, \\ \overline{u}(y) &:= \exp(y X_{p+1}) \in \overline{U}, && \\ u(w) &:= \exp(w X_{p+2}) \in U, && \\ m_j(\varphi_j) &:= \exp(\varphi_j X_{j+p+2}) \in M_1 && \text{for } 1 \le j \le \ell. \end{aligned}$$

It will also be convenient to record the following components of H:

$$H_M := \prod_{X_j \in \mathfrak{h} \cap \mathfrak{m}} \exp(x_j X_j), \; H_+ := \prod_{X_j \text{ “Upper Triangular”}} \exp(x_j X_j), \tag{46}$$

$$H_- := \prod_{X_j \text{ “Lower Triangular”}} \exp(x_j X_j), \text{ and } H_A := \prod_{X_j \text{ “Diagonal”}} \exp(x_j X_j).$$

4.4 Calculating the Haar Measure

Given that decomposition of the Lie algebra, we now derive an explicit form of the Haar measure in our chosen coordinate system. Let

$$\mathbf{z} := (x_1, \ldots, x_p, y, w, \varphi_1, \ldots, \varphi_\ell)$$

denote a set of coordinates and let

$$\mathcal{J} : \mathbb{R}^d \to G : \mathbf{z} \mapsto h_1(x_1) h_2(x_2) \ldots h_p(x_p) \overline{u}(y) u(w) m_1(\varphi_1) \ldots m_\ell(\varphi_\ell)$$

map our coordinate space to G. Then, in these new coordinates we have the following decomposition of the Haar measure

Theorem 47. *[Haar Measure Structure Theorem] Let $\rho(\mathbf{z})$ denote the density of the Haar measure in $\mathbf{z}$ coordinates. Then*

$$\rho(\mathbf{z}) = \rho_H(x_1, \ldots x_p) \rho_{\overline{U}U}(y, w) \rho_{M_1}(\varphi_1, \ldots, \varphi_\ell), \tag{48}$$

where ρ_H and ρ_{M_1} are (respectively) the densities for the Haar measure on H and M_1. Moreover we have that $\rho_{\overline{U}U}(y, w) = |1 + wy|^{n-1}$.

Proof. Given a linear differential operator $T_{\mathbf{z}}$ acting on functions of G, we can represent it as

$$T_{\mathbf{z}} = \eta_1 \partial_{x_1} + \cdots + \eta_p \partial_{x_p} + \eta_{p+1} \partial_y + \eta_{p+2} \partial_w + \eta_{p+3} \partial_{\varphi_1} + \cdots + \eta_d \partial_{\varphi_\ell}.$$

Consider an element in the Lie algebra $\mathfrak{g}$

$$Z = b_1 X_1 + b_2 X_2 + \cdots + b_d X_d.$$

Then Z corresponds to a linear differential operator via the exponential map:

$$T_{\mathbf{z}}(f(g)) = \frac{d}{dt} f(ge^{Zt})\big|_{t=0}.$$

Hence we have two representations of the linear differential operator corresponding to Z (one in terms of the b variables and one in terms of the η variables), our goal is to recover the η variables from the b variables.

To first order, we can write the differential $T_{\mathbf{z}}$ acting on a function depending on $g = h_1 \cdots h_p \overline{n} n m_1 \cdots m_\ell$ as

$$\begin{aligned} &h_1(I+\eta_1 X_1)\cdots h_p(I+\eta_p X_p)\overline{u}(I+\eta_{p+1}X_{p+1}) \\ &\quad \cdot u(I+\eta_{p+2}X_{p+2}) m_1(I+\eta_{p+3})\cdots m_\ell(I+\eta_d) \\ &= h_1\cdots h_p \overline{n} n m_1 \cdots m_\ell \left\{ I + \eta_1 \operatorname{Ad}((h_2\cdots h_p \overline{u} u m_1 \cdots m_\ell)^{-1}) X_1 \right. \\ &\quad \left. + \eta_2 \operatorname{Ad}((h_3\cdots h_p \overline{u} u m_1\cdots m_\ell)^{-1})X_2 + \cdots + \eta_{d-1}\operatorname{Ad}(m_\ell^{-1})X_{d-1} + \eta_d X_d \right\}. \end{aligned}$$

Giving us the differential operator

$$\begin{aligned} D(\mathfrak{J}) &: \eta_1 \partial_{x_1} + \cdots + \eta_p \partial_{x_p} + \eta_{p+1}\partial_y + \eta_{p+2}\partial_w + \eta_{p+3}\partial_{\varphi_1} + \cdots + \eta_d \partial_{\varphi_\ell} \\ &\mapsto \eta_1 \operatorname{Ad}((h_2\cdots h_p \overline{u} u m_1 \cdots m_\ell)^{-1})X_1 + \\ &\eta_2 \operatorname{Ad}((h_3 \cdots h_p \overline{u} u m_1 \cdots m_\ell)^{-1})X_2 + \cdots + \eta_{p+\ell+1}\operatorname{Ad}(m_\ell^{-1})X_{d-1} + \eta_d X_d. \end{aligned}$$

Thus, if we want to apply the differential operator X_j to a function $f(g)$, then we simply solve for η_i on the right hand side of this map. Then the left hand side tells us the action on the coordinates. Let us denote the vector of such η_i's by $(\eta_{j1}, \eta_{j2}, \ldots, \eta_{jd})$. Thus

$$\begin{aligned} X_j &= \eta_{j1}\operatorname{Ad}((h_2\cdots h_p \overline{n} n m_1 \cdots m_\ell)^{-1})X_1 + \eta_{j2}\operatorname{Ad}((h_3 \cdots h_p \overline{u} u m_1 \cdots m_\ell)^{-1})X_2 \\ &\quad + \cdots + \eta_{j(d-1)}\operatorname{Ad}(m_\ell^{-1})X_{d-1} + \eta_{jd}X_d, \end{aligned}$$

for every $j = 1, \ldots, d$.

Now to calculate the Haar measure, we proceed by the following methodology: define the right multiplication operator $R_A(\mathbf{z}) := \mathfrak{J}^{-1}(\mathfrak{J}(\mathbf{z}) \cdot A)$ and its Jacobian:

$$\left[R_A'(\mathbf{z})\right]_{jp} := \frac{\partial^j R_A(\mathbf{z})}{d\mathbf{z}_p}.$$

Then our goal is to find $\rho(\mathbf{z})$ such that:

$$\int f(\mathfrak{J}(\mathbf{z}))\rho(\mathbf{z})d\mathbf{z} = \int f(\mathfrak{J}(R_A(\mathbf{z})))\rho(\mathbf{z})d\mathbf{z}. \tag{49}$$

Changing variables $\mathbf{y} = R_{A^{-1}}(\mathbf{z})$, on the left hand side gives:

$$\int f(\mathfrak{J}(\mathbf{z}))\rho(\mathbf{z})d\mathbf{z} = \int f(\mathfrak{J}(R_A(\mathbf{y})))\rho(R_A(\mathbf{y}))\left|\det R_A'(\mathbf{y})\right| d\mathbf{y}$$

which is equal to the right hand side of (49). Hence we want to find ρ such that:

$$\rho(R_A(\mathbf{y})) = \frac{\rho(\mathbf{y})}{\left|\det R'_A(\mathbf{y})\right|}$$

for all choices of $\mathbf{y}$ and $A \in G$. In particular, we may choose $\mathbf{y} = 0$, and $A = \mathcal{J}(\mathbf{z})$, which gives

$$\rho(\mathbf{z}) = \frac{1}{\left|\det R'_{\mathcal{J}(\mathbf{z})}(0)\right|}. \tag{50}$$

Note that since the Haar measure is only unique up to a constant, we set $\rho(0) = 1$ without loss of generality. Using our decomposition of the Lie algebra we can write:

$$[R'_{\mathcal{J}(\mathbf{z})}(0)]_{ij} = \left[\frac{\partial}{\partial_{t_i}}\mathcal{J}(\mathbf{z})e^{\sum_{i=1}^{d} t_i X_i}\Big|_{\mathbf{t}=0}\right]_j$$

Now we can linearize the exponential and then find the corresponding coordinate description of the differential operator as we did above. The above argument implies

$$\rho(\mathbf{z})^{-1} = \det\left|[R'_{\mathcal{J}(\mathbf{z})}(0)]_{ij}\right| = \det|\eta_{ij}|.$$

Now to simplify matters, we express each adjoint as a linear combination of elements in the basis:

$$\begin{aligned}
\mathrm{Ad}((h_2\cdots h_p\bar{u}um_1\cdots m_\ell)^{-1})X_1 &= \sum_{i=1}^{d}\mu_{1i}X_i\\
\mathrm{Ad}((h_3\cdots h_p\bar{u}um_1\cdots m_\ell)^{-1})X_2 &= \sum_{i=1}^{d}\mu_{2i}X_i\\
&\cdots\\
X_d &= \sum_{i=1}^{d}\mu_{di}X_i.
\end{aligned}$$

Thus, to calculate the Haar measure, by linearity of the adjoint operator, we need to find a $d \times d$ matrix η such that

$$\eta\mu\mathbf{X} = \mathbf{X}$$

where $\mathbf{X} := (X_1, X_2, \ldots X_d)^T$. Hence, since the determinant is multiplicative, we have that $\rho(\mathbf{z}) = \det[\mu]$.

Now suppose we wanted to calculate the Haar measure of H, then we can write

$$\begin{aligned}
\mathrm{Ad}((h_2\cdots h_p)^{-1})X_1 &= \sum_{i=1}^{d} \nu_{1i}X_i\\
\mathrm{Ad}((h_3\cdots h_p)^{-1})X_2 &= \sum_{i=1}^{d} \nu_{2i}X_i\\
&\cdots\\
\mathrm{Ad}(h_p^{-1})X_{p-1} &= \sum_{i=1}^{d} \nu_{(p-1)i}X_i\\
X_p &= \sum_{i=1}^{d} \nu_{pi}X_i.
\end{aligned}$$

For completeness write $\nu_{ji} = \delta_{i,j}$ for $j > p$. Then the above argument implies that one can write the Haar measure on H as $\rho_H(\mathbf{z}) = \det[\nu]$.

Now the key observation is to write

$$\begin{aligned}
\mathrm{Ad}((\bar{u}um_1\cdots m_\ell)^{-1})X_1 &= \sum_{i=1}^{d} c_{1i}X_i\\
&\cdots\\
\mathrm{Ad}((\bar{u}um_1\cdots m_\ell)^{-1})X_p &= \sum_{i=1}^{d} c_{pi}X_i\\
\mathrm{Ad}((um_1\cdots m_\ell)^{-1})X_{p+1} &= \sum_{i=1}^{d} c_{(p+1)i}X_i\\
&\cdots\\
X_d &= \sum_{i=1}^{d} c_{di}X_i.
\end{aligned}$$

By linearity of the adjoint operator we have that $\mu = \nu c$, and hence $\det(\mu) = \det(\nu)\det(c)$. Or rather, $\rho(\mathbf{z}) = \rho_H(x_1,\ldots,x_p)f(w,y,\varphi_1,\ldots,\varphi_\ell)$ for some function f.

Moreover, since the Haar measure is unimodular, and since M_1 is a group, we can apply the same argument to M_1 on the right, and show that ρ satisfies the product structure from (48).

Finally, thanks to this product structure, we can take $m_1,\ldots,m_l$ equal to the identity when calculating

$\rho_{\overline{U}U}$. Thus, let

$$\begin{aligned}
\mathrm{Ad}((\bar{u}u)^{-1})X_j &= \sum_{i=1}^{d} d_{ji}X_i, \qquad \text{for } j=1,\dots,p \\
\mathrm{Ad}((u)^{-1})X_{p+1} &= \sum_{i=1}^{d} d_{(p+1)i}X_i \\
X_j &= \sum_{i=1}^{d} d_{ji}X_i, \qquad \text{for } j=p+2,\dots,d.
\end{aligned} \tag{51}$$

Hence it remains to find $\det(d)$.

No matter the dimension, our basis elements are each one of seven types (in $\mathfrak{m}\cap\mathfrak{h}$, lower triangular in $\mathfrak{h}$, upper triangular in $\mathfrak{h}$, diagonal, in $\bar{\mathfrak{u}}$, in $\mathfrak{u}$, or in $\mathfrak{m}_1$). Depending on the type of X_i, we can calculate d_{ij} explicitly independent of dimension via an inductive argument. From which it follows that the d matrix has the following form:

$$d = \begin{pmatrix} I_{(n-1)(n-2)/2} & 0 & 0 & 0 & 0 \\ 0 & I_{n-1} & \frac{w^2}{2}I_{n-1} & 0 & wI_{n-1} \\ 0 & \frac{y^2}{2}I_{n-1} & \frac{1}{4}(2+wy)^2 I_{n-1} & 0 & \frac{1}{2}w(2+wy)I_{n-1} \\ 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & I_{n-1} \end{pmatrix} \tag{52}$$

where B is the 3×3 matrix given by

$$B := \begin{pmatrix} 1+wy & -w & \frac{1}{2}(2+wy) \\ -y & 1 & -\frac{y^2}{2} \\ 0 & 0 & 1. \end{pmatrix}$$

Hence, one can determine explicitly thanks to a block matrix decomposition that $\det(d) = |1+wy|^{n-1}$. This completes the proof. $\square$

4.5 Calculating the Casimir Operator

As in §2, given our basis, $X_1, X_2 \dots, X_d$, for the Lie algebra in §4.3, there is a corresponding dual basis $X_1^*, X_2^*, \dots X_d^*$ with respect to the Killing form. Then the Casimir operator can be written as the following second order differential operator:

$$\mathcal{C} := \sum_{i=1}^{p+2+\ell} X_i X_i^*. \tag{53}$$

Using the above argument, in $\mathbf{z}$ coordinates, we can express $X_i = \sum_{j=1}^{d} \eta_{ij}\partial_j$. Likewise, we can express $X_i^* = \sum_{j=1}^{d} \eta_{ij}^*\partial_j$.

Now define μ^* analogously to how we defined μ, that is

$$\begin{aligned}
\mathrm{Ad}((h_2\cdots h_p\overline{u}um_1\cdots m_\ell)^{-1})X_1 &= \sum_{i=1}^{d}\mu^*_{1i}X^*_i\\
\mathrm{Ad}((h_3\cdots h_p\overline{u}um_1\cdots m_\ell)^{-1})X_2 &= \sum_{i=1}^{d}\mu^*_{2i}X^*_i\\
&\cdots\\
X_d &= \sum_{i=1}^{d}\mu^*_{di}X^*_i.
\end{aligned}$$

We then have that $\eta^*\cdot\mu^*\mathbf{X}^* = \mathbf{X}^*$, and hence $\eta^* = (\mu^*)^{-1}$. Thus, if we define $\underline{\partial} := (\partial_{x_1},\dots,\partial_{n_d})^T$, then as a differential operator

$$\mathcal{C} = \left(\mu^{-1}\underline{\partial}\right)\cdot\left((\mu^*)^{-1}\underline{\partial}\right).$$

The Casimir operator in all d variables is, of course, rather unwieldy. However, fortunately we shall only need the Casimir operator to act on left H-invariant, and right M_1-invariant functions. Hence, if we write $\mu^* = \nu^*c^*$, with ν^* and c^* defined analogously to as above, then when acting on such functions, we have

$$\mathcal{C} = \left(c^{-1}\underline{\partial}\right)\cdot\left((c^*)^{-1}\underline{\partial}\right).$$

Using this decomposition, we can explicitly calculate the Casimir operator in $\mathbf{z}$ coordinates, when acting on left H-invariant and right M_1-invariant function.

Theorem 54. *[Structure Theorem for the Casimir Operator] Let $f : H\backslash G/M_1 \to \mathbb{C}$. Then in the $\mathbf{z}$-coordinate system, f is only a function of (y,w), that is, the $\overline{U}$ and U variables, and the Casimir operator acting on f has the following form:*

$$\mathcal{C}f(y,w) = -\frac{1}{2}\left(y^2\partial_y^2 + (n+1)y\partial_y + 2\partial_{yw} + \frac{(n-1)|w|\partial_w}{|1+yw|}\right)f(y,w). \tag{55}$$

Proof. By definition of c^* we have

$$Y_j := \mathrm{Ad}((\overline{u}um_1\cdots m_\ell)^{-1})X_j = \sum_{i=1}^{d}c^*_{pi}X^*_i \text{ for } i = 1,\dots,p$$

and

$$\begin{aligned}
Y_{p+1} := \mathrm{Ad}((um_1\cdots m_\ell)^{-1})X_{p+1} &= \sum_{i=1}^{d}c^*_{(p+1)i}X^*_i\\
&\cdots\\
Y_d := X_d &= \sum_{i=1}^{d}c_{di}X^*_i.
\end{aligned}$$

Thus, $(c^*)^{-1}\mathbf{Y} = \mathbf{X}^*$. Since our function depends only on w and y, we are only interested in the $(p+1)^{th}$ and $(p+2)^{th}$ columns of $(c^*)^{-1}$. We can explicitly calculate Y_i depending on which type of basis vector is X_i, then we can explicitly solve for $(c^*)^{-1}_{ij}$. Using an inductive argument we then have that $(c^*)^{-1}\underline{\partial}$ has the following explicit form: Let

$$v_M := (\sin(\varphi_1), \cos(\varphi_1)\sin(\varphi_2), \cos(\varphi_1)\cos(\varphi_2)\sin(\varphi_3), \\ \ldots, \cos(\varphi_1)\cdots\cos(\varphi_{\ell-1})\sin(\varphi_\ell))^T \tag{56}$$

then we have (in fact, we only need the fourth and fifth row of the below)

$$(c^*)^{-1}\underline{\partial} = \frac{1}{2}\begin{pmatrix} 0_{(n-1)n/2} \\ -v_M\partial_w \\ \frac{1}{2}v_M(w^2\partial_w - 2(1+wy)\partial_y) \\ (y\partial_y - w\partial_w) \\ \cos(\varphi_1)\ldots\cos(\varphi_\ell)\partial_w \\ \frac{1}{2}\cos(\varphi_1)\ldots\cos(\varphi_\ell)(-w^2\partial_w + 2(1+wy)\partial_y) \\ 0_{n-2} \end{pmatrix}. \tag{57}$$

Let $(A_1, \ldots, A_d) := (c^*)^{-1} \cdot \underline{\partial}$, then our aim is to evaluate

$$\mathcal{C} = \sum_{j=1}^{d}\sum_{i=1}^{d}(c^{-1})_{ji}\partial_i A_j.$$

However, since we are considering the Casimir acting on right M_1 invariant functions, we may set the last ℓ coordinates equal to 0, that is

$$\mathcal{C}f(y,w) = \left(\sum_{j=1}^{d}\sum_{i=1}^{d}\left[(c^{-1})_{ji}\right]_{\varphi=0}\left[\partial_i A_j\right]_{\varphi=0}\right)f(y,w),$$

however $\left[(c^{-1})_{ji}\right]_{\varphi=0} = (d^{-1})_{ji}$ where d is the matrix of coefficients defined in (51).

Since the d matrix is explicitly described in (52), and the vector $(A_1, \ldots, A_d)$ is given explicitly in (57), then (55) can be found through direct computation. □

5 Counting

We turn now to the proof of Theorem 2. For simplicity, we assume the packing is bounded; similar methods apply for counting in regions. We also assume that the packing is the orbit of a single sphere S_1 under the action of a symmetry group Γ; in general, the counting problem reduces to a finite sum of such [KK23, Theorem 22]. Let the sphere S_1 be represented in the inversive coordinate system described in §4 by the vector $v_1 \in V$. We begin in the same way as we did in the $\mathrm{SL}_2(\mathbb{R})$ case, writing the count as

$$\begin{aligned} N_{\mathcal{P}}(T) &:= \#\{S \in \mathcal{P} \mid b(S) < T\} \\ &= \#\{v \in v_1 \cdot \Gamma \mid b^*(v) < T\} \\ &= \sum_{\gamma \in \Gamma_1 \backslash \Gamma} \chi_T(\gamma), \end{aligned}$$

where b^* is the "bend" covector (as in (44)), $\Gamma_1 := \mathrm{Stab}_{v_1}(\Gamma)$, and for $g \in G$

$$\chi_T(g) := \begin{cases} 1 & \text{if } b^*(v_1 \cdot g) < T, \\ 0 & \text{else.} \end{cases} \tag{58}$$

We observe that χ_T is a function on $H\backslash G/UM_1$, where H is the stabalizer of v_1 (so that $\Gamma_1 = \Gamma \cap H$).

Now we automorphize χ_T, that is define

$$F_T(g) := \sum_{\gamma \in \Gamma_1 \backslash \Gamma} \chi_T(\gamma g).$$

Hence F_T is left Γ-invariant and $N_{\mathcal{P}}(T) = F_T(e)$. Again, rather than trying to evaluate the discontinuous function F_T at the origin, consider its inner product with a smooth approximation of the identity. However, we want to restrict as few directions of our smooth approximation as possible, to optimize the resulting error terms. To this end, fix an $\varepsilon > 0$, and let

$$\psi = \psi_\varepsilon \in L^2(\Gamma_1 \backslash G/M_1) \tag{59}$$

be given as follows. Let $\mathcal{F}$ be a fundamental domain described in $\mathbf{z}$-coordinates, of $\Gamma_1 \backslash G/M_1$. Then on $\mathcal{F}$ we let ψ be of the form

$$\psi(\mathbf{z}) = \psi_1(x_1) \cdots \psi_p(x_p) \psi_U(y) \psi_{\overline{U}}(w)$$

, with all components nonnegative and unit total mass,

$$\int_{\Gamma_1 \backslash G/M_1} \psi dg = 1,$$

and satisfying the following conditions.

For the $n+1$ variables in the components H_+, H_A (in the notation of (46)) and $\overline{U}$, we restrict the coefficients to ε-balls around 0. The other variables are restricted only to compact regions of bounded size around 0. We can choose such a ψ have L^2 mass bounded by:

$$\|\psi\|_{L^2(\Gamma_1 \backslash G/M_1)} \ll \varepsilon^{-(n+1)/2}.$$

Now, let $\mathcal{F}_0$ denote the fundamental domain for Γ_1 containing the origin, and define the function $w_T = w_{T,\varepsilon} : \Gamma_1 \backslash G \to [0,1]$ by

$$w_T(g) := \int_{\mathcal{F}_0} \chi_T(gh) \psi_\varepsilon(h) dh. \tag{60}$$

Our main counting theorem (Theorem 2) will follow from the following smooth effective count.

Theorem 61. *Let the Laplace eigenvalues of $\Gamma\backslash\mathbb{H}^{n+1}$ be as in (1). Then there exist constants $c^i_{\Gamma,\varepsilon}$ for $i = 0, 1, \ldots, k$, with $c^0_{\Gamma,\varepsilon} > 0$, such that*

$$\widetilde{N}_{\Gamma,\varepsilon}(T) := \sum_{\gamma \in \Gamma_1 \backslash \Gamma} w_T(\gamma) = c^0_{\Gamma,\varepsilon} T^\delta + c^1_{\Gamma,\varepsilon} T^{s_1} + \cdots + c^k_{\Gamma,\varepsilon} T^{s_k} + O\left(\frac{1}{\varepsilon^{(n+1)/2}} T^{n/2} \log(T)\right) \tag{62}$$

where the implied constant depends only on Γ. Moreover $c^0_{\Gamma,\varepsilon} = c^0_\Gamma(1 + O(\varepsilon))$, and for $i \geq 1$ (if any exist), we have that $c^i_{\Gamma,\varepsilon} \ll \varepsilon^{-(n+1)/2}$.

Remark 63. *Note that this smoothed counting theorem is optimal in that the error goes all the way to the tempered spectrum.*

5.1 Proof of Theorem 2

An explicit calculation shows that the inversive coordinates of a sphere which has been transformed by g are

$$(y, -(1+wy)v_M^T, (1+wy)\cos(\varphi_1)\cdots\cos(\varphi_\ell), w(2+wy))$$

where v_M is the vector defined in (56). Thus the 'bend radius' (i.e the distance of the center of the sphere to the origin multiplied by the bend) is $(1+wy)$. Since this distance is necessarily bounded from above (since the packing is bounded) and since y is bounded from below (because it controls the bend, and the packing has a largest sphere), we know that w is bounded from above. In particular we have

Lemma 64. *Let $\gamma \in \Gamma_1\backslash\Gamma$ then the w coordinate of γ, $\left|w_\gamma\right|$ is bounded.*

Now, assuming Theorem 61 we present the proof of Theorem 2.

Proof of Theorem 2. Consider

$$w_T(\gamma) = \int_{\mathcal{F}_0} \chi_T(\gamma g)\psi_\varepsilon(g)dg.$$

We write $\gamma = \mathcal{J}(x_{1,\gamma},\ldots,x_{k,\gamma},y_\gamma,w_\gamma,\varphi_{1,\gamma},\ldots,\varphi_{\ell,\gamma})$ and note that χ_T is left H-invariant and right UM_1 invariant. Hence

$$\chi_T(\gamma g) = \chi_T(\overline{u}(y_\gamma)u(w_\gamma)m_1(\varphi_\gamma)h(\mathbf{x})\overline{n}(y)).$$

now we can write h as a product of one dimensional components of H from (46) (here, it is convenient to change the order of the decomposition). That is, write

$$h(\mathbf{x}) = h_+(\mathbf{x}_+)h_A(\mathbf{x}_A)h_-(\mathbf{x}_-)h_M(\mathbf{x}_M).$$

Since ψ restricts the $\mathbf{x}_-$ and $\mathbf{x}_A$ coordinates to balls of radius ε, we can apply adjoints to move those factors to the left of $\overline{u}(y_\gamma)$ and the other factors will be perturbed by an ε error. Then we can use left H-invariance of χ_T. We can do the same for the $\overline{u}(y)$ factor which gets absorbed in the $\overline{u}(y)$ factor. Thus, for g in the support of ψ_ε

$$\begin{aligned}\chi_T(\gamma g) &= \chi_T(\overline{u}(y_\gamma + O(\varepsilon))u(w_\gamma + O(\varepsilon))m_1(\varphi_\gamma + O(\varepsilon))h_+(\mathbf{x}_+ + O(\varepsilon))h_M(\mathbf{x}_M + O(\varepsilon)) \\ &= \chi_T(\overline{u}(y_\gamma + O(\varepsilon))),\end{aligned}$$

where the second line follows from right-M invariance, and right-H_+ invariance.

Thus,

$$\chi_T(\gamma g) = \begin{cases} 1 & \text{if } y_\gamma < \frac{T}{1+c\varepsilon} \\ 0 & \text{if } y_\gamma > \frac{T}{1-c\varepsilon} \end{cases}$$

for some absolute constant c.

Hence, since $\widetilde{N}_{\Gamma,\varepsilon}(T) = \sum_{\Gamma_1\backslash\Gamma} w_T(\gamma)$, then our count satisfies:

$$\widetilde{N}_{\Gamma,\varepsilon}(T(1-c\varepsilon)) \leq N_{\mathcal{P}}(T) \leq \widetilde{N}_{\Gamma,\varepsilon}(T(1+c\varepsilon)) \tag{65}$$

Now, assuming there are no other eigenvalues, we apply Theorem 61 to find:

$$\widetilde{N}_{\Gamma,\varepsilon}(T(1\pm\varepsilon)) = c^0(1+O(\varepsilon))T^{\delta} + O(\frac{1}{\varepsilon^{(n+1)/2}}T^{n/2}\log(T)).$$

Then choosing $\varepsilon = T^{\frac{2}{n+3}(n/2-\delta)}\log(T)^{2/(n+3)}$ optimizes this inequality. Thus

$$\widetilde{N}_{\Gamma,\varepsilon}(T(1\pm\varepsilon)) = c^0T^{\delta} + O(T^{\frac{2}{n+3}(n/2-\delta)+\delta}\log(T)^{\frac{2}{n+3}}).$$

The general case follows similarly. □

5.2 Inserting the Casimir Operator

Once again the smooth count is the inner product of F_T with an L^2 function. Consider the inner product of F_T with a general $\Psi \in L^2(\Gamma\backslash G/M_1)$. That is, given a function ψ on $\Gamma_1\backslash G/M_1$ we automorphize it

$$\Psi(z) := \sum_{\gamma\in\Gamma_1\backslash\Gamma} \psi(\gamma g).$$

Let

$$K_T(s) := \frac{T^s b^{n-s} - T^{n-s} b^s}{b^{n-s} - b^s}, \qquad L_T(s) := \frac{T^{n-s} - T^s}{b^{n-s} - b^s},$$

while for $s = n/2 + it$ we have

$$K_T(s) := T^{n/2}\frac{\sin(t\log T/\log b)}{\sin(t\log b)}, \qquad L_T(s) := \left(\frac{T}{b}\right)^{n/2}\frac{\sin(t\log T)}{\sin(t\log b)}. \tag{66}$$

Again, by choosing an appropriate choice of b one can ensure that

$$K_T(s), L_T(s) \ll \begin{cases} T^s & \text{if } s \in (n/2, n], \\ T^{n/2}\log T & \text{if } s = n/2 + it. \end{cases} \tag{67}$$

In the $\mathrm{SL}_2(\mathbb{R})$ case, we decomposed the real direction, since $\Gamma\backslash\mathbb{H}$ was not compact in the real direction. Analogously in the current setting, we have the group decomposition $(\Gamma_1\backslash H)\overline{U}UM_1$. Since $(\Gamma_1\backslash H)$ has finite H-Haar measure, and M_1 is compact, and we have imposed a cut-off in the $\overline{U}$-direction, we are again faced with a one dimensional non-compact direction, the U-direction. To that end, since ∞ lies outside the limit set, from Lemma 64 it follows that there exists an X large enough, such that

$$N_{\mathcal{P}} = F_{T,X}(e)$$

where

$$F_{T,X}(g) := \sum_{\Gamma_1\backslash\Gamma} \chi_T(\gamma g)\widetilde{\chi}_X(\gamma g), \qquad \text{and}$$

$$\widetilde{\chi}_X(g) := \begin{cases} 1 & \text{if } |bz^*(v_1 \cdot g)|/b(c_1 \cdot g) < X, \\ 0 & \text{otherwise.} \end{cases}$$

where bz^* denotes the bend center, as in (45) and b denotes the bend. Thus, note that $\widetilde{\chi}_X$ is also left Γ_1 invariant, and right M_1 invariant (since it depends only on the distance to the origin of the center, not the polar coordinate angles).

Moreover, a calculation shows that, in the $\mathbf{z}$-coordinates, $\widetilde{\chi}_X$ can be written as:

$$\widetilde{\chi}_X(g) := \begin{cases} 1 & \text{if } -X - 1/y < w < X - 1/y, \\ 0 & \text{otherwise.} \end{cases} \tag{68}$$

The Difference Operator: Again, we will prove an identity in terms of K_T and L_T for $F_{T,X}$. To that end, consider the difference operator

$$G_{T,X} := F_{T,X} - K_T(\mathcal{C})F_{1,X} - L_T(\mathcal{C})F_{b,X}.$$

By self-adjointness of $\mathcal{C}$, for any $\Psi \in L^2(\Gamma\backslash G/M_1)$ we have

$$\begin{aligned} G_{T,X}(\Psi) &:= \langle G_{T,X}, \Psi\rangle_{\Gamma\backslash G} \\ &= \int_{\Gamma\backslash G} [F_{T,X}(g)\Psi(g) - F_{1,X}(g)(K_T(\mathcal{C})\Psi)(g) - F_{b,X}(g)(L_T(\mathcal{C})\Psi)(g)]\, dg, \end{aligned}$$

which we can unfold to

$$\begin{aligned} &G_{T,X}(\Psi) = \\ &\int_{\Gamma_1\backslash G} \widetilde{\chi}_X(g)\,(\chi_T(g)\Psi(g) - \chi_1(g)(K_T(\mathcal{C})\Psi)(g) - \chi_b(g)(L_T(\mathcal{C})\Psi)(g))\, dg. \end{aligned} \tag{69}$$

It is more convenient to work using $\mathbf{z}$-coordinates describing the group G. Now fix a fundamental domain for $\Gamma_1\backslash G/M_1$, this fundamental domain can be written in coordinates as $\mathcal{F} := P_{\Gamma_1} \times [0,\infty) \times \mathbb{R} \times [-\pi,\pi]^{n-1}$, where P_{Γ_1} is a (finite H-volume) fundamental domain for the action of Γ_1 on H. Given a function $f : G \to \mathbb{C}$ we abuse notation and write $f(\mathbf{z}) = f(g_{\mathbf{z}})$, thus if we let ρ denote density of the Haar measure, we can write (69) as:

$$\begin{aligned} &G_{T,X}(\Psi) = \\ &\int_{\mathcal{F}} \widetilde{\chi}_X(y,w)\,(\chi_T(y)\Psi(\mathbf{z}) - \chi_1(y)(K_T(\mathcal{C})\Psi)(\mathbf{z}) - \chi_b(y)(L_T(\mathcal{C})\Psi)(\mathbf{z}))\,\rho(\mathbf{z})d\mathbf{z}. \end{aligned}$$

Note that by definition $\chi_T(\mathbf{z})$ is a function of y and $\widetilde{\chi}_X(\mathbf{z})$ is a function of w (and y), for clarity we ignore the dependence on the other variables.

Let $\mathcal{F}_X := \{z \in \mathcal{F} \,:\, w \in \mathcal{I}_X\}$ where $\mathcal{I}_X := [w_{-X}(y), w_X(y)]$; here we have introduced the notation

$$w_r(y) := r - 1/y,$$

motivated by (68). Now write

$$G_{T,X}(\Psi) = \int_{\mathcal{F}_X} (\chi_T(y)\Psi(\mathbf{z}) - \chi_1(y)(K_T(\mathcal{C})\Psi)(\mathbf{z}) - \chi_b(y)(L_T(\mathcal{C})\Psi)(\mathbf{z}))\,\rho(\mathbf{z})d\mathbf{z}.$$

By showing that $G_{T,X}(\Psi) = 0$ for any choice of Ψ we will prove the following.

Proposition 70. *For Γ and $F_{T,X}$ as above we have that*

$$F_{T,X} = K_T(\mathcal{C})F_{1,X} + L_T(\mathcal{C})F_{b,X} \tag{71}$$

where K_T and L_T are the differential operators defined in (66).

Once again our goal is to work on the fundamental domain of a group (rather than working with discontinuous cut-offs). Thus we will perform the same smoothing as we did in the $\mathrm{SL}_2(\mathbb{R})$ case.

Smoothing χ_T: Let $\sigma > 0$ and let

$$\chi_{1,\sigma}(\mathbf{z}) := \begin{cases} 1 & \text{if } y < 1, \\ 0 & \text{if } y > (1+\sigma), \end{cases}$$

and let $\chi_{1,\sigma}$ interpolate smoothly for all values in between. Now let $\chi_{T,\sigma}(\mathbf{z}) := \chi_{1,\sigma}(Ty)$.

Let

$$G^{\sigma}_{T,X}(\Psi) := \int_{\mathcal{F}_X} (\chi_{T,\sigma}(y)\Psi(\mathbf{z}) - \chi_{1,\sigma}(y)(K_T(\mathcal{C})\Psi)(\mathbf{z}) - \chi_{b,\sigma}(y)(L_T(\mathcal{C})\Psi)(\mathbf{z}))\,\rho(\mathbf{z})d\mathbf{z}.$$

Now by construction

$$\int_{\mathcal{F}_X} |\chi_{t,\sigma}(y) - \chi_t(y)|^2 \rho(\mathbf{z})d\mathbf{z} \ll_{t,X} \sigma$$

Thus by Cauchy-Schwarz we have that $\lim_{\sigma\to 0} G^{\sigma}_{T,X}(\Psi) = G_{T,X}(\Psi)$. Now our goal is to show that for any fixed $\varepsilon > 0$ we have $G^{\sigma}_{T,X}(\Psi) < \varepsilon$.

Periodizing and Smoothing Ψ: Let

$$\mathcal{I}_{X,\eta} := [w_{-X}(y) + \eta, w_X(y) - \eta]\,.$$

Let $\widetilde{\Psi} : G/M_1 \to \mathbb{R}$ denote a function which agrees with Ψ on

$$\{z \in \mathcal{F} \,:\, w \in \mathcal{I}_{X,\eta}\}$$

for some $\eta > 0$ to be chosen later. When $w \notin \mathcal{I}_{X,\eta/2}$ we impose the condition $\widetilde{\Psi}(w_{-X}(y)) = \widetilde{\Psi}(w_X(y))$ for any value of the other variables, and interpolate smoothly in between.

Using the same Cauchy-Schwarz argument as we employed in the $\mathrm{SL}_2(\mathbb{R})$ case, we can choose η such that the $L^2(\mathcal{F}_X)$ cost of moving from Ψ to $\widetilde{\Psi}$ is less than ε. Thus, Proposition 70 follows if we can prove that

$$G_{T,X,\sigma}(\widetilde{\Psi}) = 0. \tag{72}$$

Working on $\mathcal{F}_X$: Let

$$g^{\sigma}_{T,X}(\mathbf{z}) := \widetilde{\chi}_X(\mathbf{z})(\chi_{T,\sigma}(\mathbf{z}) - K_T(\mathcal{C})\chi_{1,\sigma}(\mathbf{z}) - L_T(\mathcal{C})\chi_{b,\sigma}(\mathbf{z})).$$

Furthermore, let $L^2(\mathcal{F}_X)$ denote the space of square integrable functions on $\mathcal{F}_X$, which are Γ_1-invariant in the H variables, and invariant under translation by $2X$ in the w variable. Note that $\widetilde{\Psi} \in L^2(\mathcal{F}_X)$. Then (72) follows from the following lemma.

Lemma 73. *For any $\psi \in L^2(\mathcal{F}_X)$ which is independent of the $\varphi \in [-\pi,\pi)^{n-1}$ variables, and any $\lambda \geq 0$, we have*

$$\langle g^{\sigma}_{T,X}, \psi \rangle_{\mathcal{F}_X} \ll_{\lambda,T,\sigma,X} \|(\mathcal{C}-\lambda)\psi\|_{\mathcal{F}_X} \tag{74}$$

where $\mathcal{C}$ denotes the Casimir operator on $L^2(\mathcal{F}_X)$ (i.e the Casimir operator in $\mathbf{z}$-coordinates).

Proof. To prove (74), let ψ be an arbitrary function in $L^2(\mathcal{F}_X)$, which is φ invariant. Now consider

$$\int_{\mathcal{F}_X} \chi_{T,\sigma}(y)\psi(\mathbf{z})\rho(\mathbf{z})d\mathbf{z} = \int_0^{2T} \chi_{T,\sigma}(y) \int_{w_{-X}(y)}^{w_X(y)} \int_{[-\pi,\pi)^{n-1}} \int_{P_{\Gamma_1}} \psi(\mathbf{z})\rho(\mathbf{z})d\mathbf{x}d\varphi dw dy.$$

Since, as shown in Theorem 47, the density of the Haar measure decomposes into a product of densities depending on $\mathbf{x}$, one depending on φ and one depending on w and z, we can use this fact to show

$$\int_{P_{\Gamma_1}} \mathcal{C}\psi(\mathbf{x},y,w)\rho_H(\mathbf{x})d\mathbf{x} = \mathcal{C}\int_{P_{\Gamma_1}} \psi(\mathbf{x},y,w)\rho_H(\mathbf{x})d\mathbf{x}.$$

This follows from the fact that ψ is Γ_1-invariant in the $\mathbf{x}$ variable, and P_{Γ_1} is a fundamental domain for Γ_1 acting on H.

Recall that the Haar measure, in $\mathbf{z}$ coordinates is given by $|1+wy|^{n-1}\rho_H\rho_{M_1}$. Let $\widetilde{\psi}(y,w) := \int_{P_{\Gamma_1}} \psi(\mathbf{x},y,w)\rho_H(\mathbf{x})d\mathbf{x}$ and let

$$h(y,w) := 2\int_{P_{\Gamma_1}} \int_{[-\pi,\pi)^{n-1}} \widetilde{\rho}(\mathbf{z})(\mathcal{C}-\lambda)\psi(\mathbf{z})d\mathbf{x}d\varphi.$$

Then, using the Structure Theorem for the Casimir operator (Theorem 54), we have that

$$\begin{aligned} I(y) &:= \int_{w_{-X}(y)}^{w_X(y)} h(w,y)dw \\ &= -\int_{w_{-X}(y)}^{w_X(y)} |1+wy|^{n-1} \left(y^2\partial_y^2 + (n+1)y\partial_y + 2\partial_{yw} + \frac{(n-1)w\partial_w}{1+yw} + \lambda \right) \widetilde{\psi}(y,w)dw \end{aligned} \tag{75}$$

Now make the change of variables $w \mapsto w' = w + 1/y$ and $y' = y$. This leads to the relations

$$\partial_w = \partial_{w'}, \qquad \text{and} \qquad \partial_y = \partial_{y'} - \frac{1}{y^2}\partial_{w'},$$

and results in

$$I(y) = -\int_{-X}^{X} y^{n-1}\,|w|^{n-1}\left(y^2\partial_{yy} + (n+1)y\partial_y - \frac{1}{y^2}\partial_{ww} - \frac{(n-1)}{wy^2}\partial_w + \lambda\right)\widetilde{\psi}(y,w)dw.$$

Now, we can use integration by parts to eliminate two of the terms

$$I(y) = -y^{n-1}\int_{-X}^{X}|w|^{n-1}\left(y^2\partial_{yy} + (n+1)y\partial_y + \lambda\right)\widetilde{\psi}(y,w)dw.$$

We now let $f(y) := \int_{-X}^{X}|w|^{n-1}\,\widetilde{\psi}(y,w)dw$; then $f(y)$ satisfies the differential equation

$$\left(y^2\partial_{yy} + (n+1)y\partial_y + \lambda\right)f(y) = h(y) \tag{76}$$

with $h(y) = -\int_{w_{-X}(y)}^{w_X(y)} h(w,y)dw$.

The homogeneous solution to this differential equation is then

$$C_1 y^{s-n} + C_2 y^{-s},$$

where we write $\lambda = s(n-s)$. From here we can use the usual variation of parameters argument to conclude

$$\int_0^T |y|^{n-1}\int_{-X}^{X}\widetilde{\psi}dwdy = C_1T^s + C_2T^{n-s} + O_{T,\lambda,\Gamma}(\|(\mathcal{C}-\lambda)\widetilde{\psi}\|).$$

□

Now, with Lemma 73 we use the same argument as in [Kon09, Proof of Theorem 3.2] to conclude that $g^{\sigma}_{T,X}$ is almost everywhere 0. Thus, working our way back up, with the same argument as in the $\mathrm{SL}_2(\mathbb{R})$ setting, we conclude the proof of Proposition 70.

5.3 Proof of Theorem 61

As for the $SL_2(\mathbb{R})$ case, we now return to the smooth count with U-cutoff:

$$\widetilde{N}_\Gamma(T) = \langle F_{T,X}, \Psi\rangle_\Gamma.$$

Now apply the abstract Parseval's identity (13)

$$\begin{aligned}\langle F_{T,X}, \Psi\rangle_\Gamma &= \langle \widehat{F_{T,X}}, \widehat{\Psi}\rangle_{\mathrm{Spec}(\Gamma)} \\ &= \widehat{F_{T,X}}(\lambda_0)\widehat{\Psi}(\lambda_0) + \int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}}\widehat{F_{T,X}}(\lambda)\widehat{\Psi}(\lambda)\mathrm{d}\nu(\lambda).\end{aligned} \tag{77}$$

As with the $\mathrm{SL}_2(\mathbb{R})$ case, we can apply spectral methods to extract the T and ε-dependence. Applying the abstract spectral theorem gives

$$\widehat{F_{T,X}}(\lambda_0)\widehat{\Psi}(\lambda_0) = \langle \mathrm{Proj}_{\mathcal{H}_0}(F_{T,X}), \mathrm{Proj}_{\mathcal{H}_0}(\Psi)\rangle.$$

Then by linearity and our main identity Proposition 70 we conclude

$$\begin{aligned}\widehat{F_{T,X}}(\lambda_0)\widehat{\Psi_\varepsilon}(\lambda_0) &= K_T(\lambda_0)\langle \mathrm{Proj}_{\mathcal{H}_0}(F_{1,X}), \mathrm{Proj}_{\mathcal{H}_0}(\Psi_\varepsilon)\rangle \\ &\quad + L_T(\lambda_0)\langle \mathrm{Proj}_{\mathcal{H}_0}(F_{b,X}), \mathrm{Proj}_{\mathcal{H}_0}(\Psi_\varepsilon)\rangle \\ &= T^\delta \langle H, \mathrm{Proj}_{\mathcal{H}_0}(\Psi_\varepsilon)\rangle + O(T^{n/2}),\end{aligned}$$

where $H := c_1 \mathrm{Proj}_{\mathcal{H}_0}(F_{1,X}) + c_b \mathrm{Proj}_{\mathcal{H}_0}(F_{b,X})$ for some constants c_1, c_b.

The problem remains to determine the ε dependence of the projection operator $\mathrm{Proj}_{\mathcal{H}_0}(\Psi_\varepsilon)$. In general, this projection can be realized in a number of ways, either as a Burger-Roblin-type measure of Ψ_ε (see [MO15, p. 861]), or using representation-theoretic decompositions as in [BKS10, Vin12]. We will give a soft argument that avoids either.

We know from (65) that

$$N_{\mathcal{P}}\left(\frac{T}{1+c\varepsilon}\right) \le \widetilde{N}_{\Gamma,\varepsilon}(T) \le N_{\mathcal{P}}\left(\frac{T}{1-c\varepsilon}\right),$$

for any value of ε and any value of T. However we also know *a priori* (e.g., using [Kim15]) that

$$N_{\mathcal{P}}\left(\frac{T}{1\pm c\varepsilon}\right) = c_{\mathcal{P}}\left(\frac{T}{1\pm c\varepsilon}\right)^\delta (1+o(1)),$$

as $T\to\infty$. Dividing by T^δ then gives

$$c_{\mathcal{P}}\left(\frac{1}{1+c\varepsilon}\right)^\delta - o(1) \le \langle H, \mathrm{Proj}_{\mathcal{H}_0}(\Psi_\varepsilon)\rangle \le c_{\mathcal{P}}\left(\frac{1}{1-c\varepsilon}\right)^\delta + o(1).$$

Now send $T\to\infty$ and Taylor expand $\frac{1}{(1\pm c\varepsilon)^\delta}$ in ε, giving:

$$\langle H, \mathrm{Proj}_{\mathcal{H}_0}(\Psi_\varepsilon)\rangle = C + O(\varepsilon).$$

Hence

$$\widehat{F_{T,X}}(\lambda_0)\widehat{\Psi_\varepsilon}(\lambda_0) = T^\delta c(1+O(\varepsilon)) + O(T^{n/2}) \tag{78}$$

for some constant c independent of ε. (Note that this positivity argument does not apply to the other eigenvalues. Hence with sharp cutoffs, as in Theorem 2, we do not extract lower order terms.)

Turning now to the remainder, after extracting the main term corresponding to λ_0 we are left with

$$\begin{aligned}\mathrm{Err} &:= \int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \widehat{F_{T,X}}(\lambda)\widehat{\Psi}(\lambda)d\nu \\ &= \int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \Big(K_T(\lambda)\widehat{F_{1,X}}(\lambda) + L_T(\lambda)\widehat{F_{b,X}}(\lambda)\Big)\widehat{\Psi}(\lambda)d\nu.\end{aligned}$$

Assume for simplicity that there are no other discrete eigenvalues above the base. Now apply the abstract spectral theorem and the bounds from (67) to conclude that

$$\int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} K_T(\lambda)\widehat{F_{1,X}}(\lambda)\widehat{\Psi}(\lambda)d\nu \ll T^{n/2}\log T \int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \widehat{F_{1,X}}(\lambda)\widehat{\Psi}(\lambda)d\nu.$$

Now apply Cauchy-Schwarz and Parseval to get

$$\begin{aligned}
&\ll\ T^{n/2}\log T\left(\int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \widehat{F_{1,X}}(\lambda)^2 d\nu\right)^{1/2}\left(\int_{\mathrm{Spec}(\Gamma)\setminus\{\lambda_0\}} \widehat{\Psi}(\lambda)^2 d\nu\right)^{1/2}\\
&\leq\ T^{n/2}\log T\left(\int_{\mathrm{Spec}(\Gamma)} \widehat{F_{1,X}}(\lambda)^2 d\nu\right)^{1/2}\left(\int_{\mathrm{Spec}(\Gamma)} \widehat{\Psi}(\lambda)^2 d\nu\right)^{1/2}\\
&=\ T^{n/2}\log T\ \|F_{1,X}\|_\Gamma\ \|\Psi\|_\Gamma.
\end{aligned}$$

Finally, note that since ψ_ε is normalized to have unit L^1-mass, we have that $\|\Psi\|_\Gamma \ll \varepsilon^{-(n+1)/2}$. In the case of other eigenvalues, we replace the bound $T^{n/2}\log T$ above with T^{s_1}. This completes the proof of Theorem 61. □

Remark 79. *If we remove our assumption that $L^2(\Gamma\backslash G)$ does not weakly contain any nonspherical complementary series representations, then, after removing contributions from Laplace eigenvalues in* (77)*, the remainder would not necessarily be tempered. So instead of getting an error of order $T^{n/2}\log T$, we would only be able to bound what remains by $O(T^{n-1})$, corresponding to the spectral parameter of any potential nonspherical complimentary series. It is likely possible to improve on this bound, see Remark 7.*

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Log-concave poset inequalities

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Abstract: We study combinatorial inequalities for various classes of set systems: matroids, polymatroids, poset antimatroids, and interval greedoids. We prove log-concave inequalities for counting certain weighted feasible words, which generalize and extend several previous results establishing *Mason conjectures* for the numbers of independent sets of matroids. Notably, we prove matching equality conditions for both earlier inequalities and our extensions.

In contrast with much of the previous work, our proofs are combinatorial and employ nothing but linear algebra. We use the language formulation of greedoids which allows a linear algebraic setup, which in turn can be analyzed recursively. The underlying noncommutative nature of matrices associated with greedoids allows us to proceed beyond polymatroids and prove the equality conditions. As further application of our tools, we rederive both *Stanley's inequality* on the number of certain linear extensions, and its equality conditions, which we then also extend to the weighted case.

Key words and phrases: Combinatorial inequalities, Algebraic aspects of posets

1 Introduction

1.1 Foreword

It is always remarkable and even a little suspicious, when a nontrivial property can be proved for a large class of objects. Indeed, this says that the result is "global", i.e. the property is a consequence of the underlying structure rather than individual objects. Such results are even more remarkable in combinatorics, where the structures are weak and the objects are plentiful. In fact, many reasonable

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conjectures in the area fail under experiments, while some are ruled out by theoretical considerations (cf. §16.1 and §17.1).

This paper is concerned with log-concavity results for counting problems in the general context of posets, and is motivated by a large body of amazing recent work in area, see a survey by Huh [Huh18]. Surprisingly, these results involve deep algebraic tools which go much beyond previous work on the subject, see earlier surveys [Brä15, Bre89, Bre94, Sta89]. This leads to several difficult questions, such as:

- How far do these inequalities generalize?
- How do we extend/develop new algebraic tools to prove these generalizations?

We aim to answer the first question in as many cases as we can, both generalizing the inequalities to larger classes of posets and strengthening these inequalities to match equality conditions which we also prove. We do this by sidestepping the second question, or avoiding it completely.

There is a very long and only partially justified tradition in combinatorics of looking for purely combinatorial proofs of combinatorial results. Although the very idea of using advanced algebraic tools to prove combinatorial inequalities is rather mesmerizing, one wonders if these tools are really necessary. Are they giving us a *true insight* into the nature of these inequalities that we were missing for so long? Or, perhaps, the absence of purely combinatorial proofs is a reflection of our continuing *lack of understanding*?

We posit that, in fact, all poset inequalities can be obtained by elementary means (cf. §1.21). We show how this can be done for a several large families of inequalities, and intend to continue this work in the future (see §17.17). There are certain tradeoffs, of course, as we need to introduce a technical linear algebraic setup (see §1.20), which allows us to quickly reprove both classical and recently established poset inequalities. The advantage of our approach is its flexibility and noncommutative nature, making it amenable to extend and generalize these inequalities in several directions.

Of course, none of what we did takes anything away from the algebraic proofs of poset inequalities which remained open for decades — the victors keep all the spoils (see Section 16). We do, however, hope the reader will appreciate that our combinatorial tools are indeed *more powerful* than the algebraic tools, at least in the cases we consider (cf. §§17.8–17.11).

1.2 What to expect now

A long technical paper deserves a long technical introduction. Similarly, a friendly and accessible paper deserves a friendly and accessible introduction. Naturally, we aim to achieve both somewhat contradictory goals.

Below we present our main results and applications, all of which require definitions which are standard in the area, but not a common knowledge in the rest of mathematics. We make an effort to have the introduction thorough yet easily accessible, at the expense of brevity.[1]

In addition, rather than jump to the most general and thus most involved results, we begin slowly, and take time to introduce the reader to the world of poset inequalities. Essentially, the rest of the introduction

[1] In an effort to streamline the presentation, some basic notation is collected in a short Section 2, which we encourage the reader to consult whenever there is an apparent misunderstanding or ambiguity.

can be viewed as an extensive survey of our own results interspersed with a few examples and some earlier results directly related to our work. The reader well versed in the greedoid literature can speed read a few early subsections.

We say very little about our tools at this stage, even though we consider them to be our main contribution (see §1.20 and §1.21). These are fully presented in the following sections, which in turn are followed by the proofs of all the results. As we mentioned above, our tools are elementary but technical, and are best enjoyed when the reader is convinced they are worth delving into.

Similarly, in the introduction, we say the bare minimum about the rich history of the subject and the previous work on poset inequalities. This is rather unfair to the many experts in the area whose names and contributions are mentioned only at the end of the paper. Our choice was governed by the effort to keep the introduction from exploding in size. We beg forgiveness on this point, and try to mitigate it by a lengthy historical discussion in Section 16, with quick pointer links sprinkled throughout the introduction.

1.3 Matroids

A (finite) *matroid* $\mathcal{M}$ is a pair $(X, \mathcal{I})$ of a *ground set* X, $|X| = n$, and a nonempty collection of *independent sets* $\mathcal{I} \subseteq 2^X$ that satisfies the following:

- (*hereditary property*) $S \subset T,\ T \in \mathcal{I} \ \Rightarrow\ S \in \mathcal{I}$, and
- (*exchange property*) $S, T \in \mathcal{I},\ |S| < |T| \ \Rightarrow\ \exists x \in T \setminus S$ s.t. $S + x \in \mathcal{I}$.

Rank of a matroid is the maximal size of the independent set: $\mathrm{rk}(\mathcal{M}) := \max_{S \in \mathcal{I}} |S|$. A *basis* of a matroid is an independent set of size $\mathrm{rk}(\mathcal{M})$. Finally, let $\mathcal{I}_k := \{S \in \mathcal{I},\ |S| = k\}$, and let $\mathrm{I}(k) = |\mathcal{I}_k|$ be the *number of independent sets* in $\mathcal{M}$ of size k, $0 \le k \le \mathrm{rk}(\mathcal{M})$.

Theorem 1.1 (*Log-concavity for matroids*, [AHK18, Thm 9.9 (3)], formerly *Welsh–Mason conjecture*)**.** *For a matroid $\mathcal{M} = (X, \mathcal{I})$ and integer $1 \le k < \mathrm{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \ \ge\ \mathrm{I}(k-1) \cdot \mathrm{I}(k+1)\,. \tag{1.1}$$

See §16.5 for the historical background. The *log-concavity* in (1.1) classically implies *unimodality* of the sequence $\{\mathrm{I}(k)\}$:

$$\mathrm{I}(0) \ \le\ \mathrm{I}(1) \ \le\ \ldots\ \le\ \mathrm{I}(k) \ \ge\ \mathrm{I}(k+1) \ \ge\ \ldots\ \ge\ \mathrm{I}(m)\,, \quad \text{where} \ \ m \ =\ \mathrm{rk}(\mathcal{M})\,.$$

It was noted in [Lenz11, Lem. 4.2] that other results in [AHK18] imply that the inequalities (1.1) are *always strict* (see §16.6). Further improvements to (1.1) have been long conjectured by Mason [Mas72] and were recently established in quick succession.

Theorem 1.2 (*One-sided ultra-log-concavity for matroids*, [HSW22, Cor. 9], formerly *weak Mason conjecture*)**.** *For a matroid $\mathcal{M} = (X, \mathcal{I})$ and integer $1 \le k < \mathrm{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \ \ge\ \left(1 + \frac{1}{k}\right) \mathrm{I}(k-1)\ \mathrm{I}(k+1)\,. \tag{1.2}$$

Theorem 1.3 (*Ultra-log-concavity for matroids*, [ALOV18, Thm 1.2] and [BH20, Thm 4.14], formerly *strong Mason conjecture*). *For a matroid $\mathcal{M} = (X, \mathcal{I})$, $|X| = n$, and integer $1 \le k < \mathrm{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \ge \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) \mathrm{I}(k-1)\, \mathrm{I}(k+1). \tag{1.3}$$

Equation (1.3) is a reformulation of *ultra-log-concavity* of the sequence $\{\mathrm{I}(k)\}$:

$$\mathrm{i}(k)^2 \ge \mathrm{i}(k-1) \cdot \mathrm{i}(k+1), \quad \text{where} \quad \mathrm{i}(m) := \frac{\mathrm{I}(m)}{\binom{n}{m}}$$

can be viewed as the probability that random m-subset of X is independent in $\mathcal{M}$.

1.4 More matroids

For an independent set $S \in \mathcal{I}$ of a matroid $\mathcal{M} = (X, \mathcal{I})$, denote by

$$\mathrm{Cont}(S) := \big\{x \in X \setminus S \,:\, S + x \in \mathcal{I}\big\} \tag{1.4}$$

the set of *continuations* of S. For all $x, y \in \mathrm{Cont}(S)$, we write $x \sim_S y$ when $S + x + y \notin \mathcal{I}$ or when $x = y$. Note that "$\sim_S$" is an equivalence relations, see Proposition 4.1. We call an equivalence class of the relation $\sim_S$ a *parallel class* of S, and we denote by $\mathrm{Par}(S)$ the set of parallel classes of S.

For every $0 \le k < \mathrm{rk}(\mathcal{M})$, define the *k-continuation number* of a matroid $\mathcal{M}$ as the maximal number of parallel classes of independent sets of size k:

$$\mathrm{p}(k) := \max\big\{ \left|\mathrm{Par}(S)\right| \,:\, S \in \mathcal{I}_k \big\}. \tag{1.5}$$

Clearly, $\mathrm{p}(k) \le n - k$.

Theorem 1.4 (*Refined log-concavity for matroids*). *For a matroid $\mathcal{M} = (X, \mathcal{I})$ and integer $1 \le k < \mathrm{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \ge \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}(k-1) - 1}\right) \mathrm{I}(k-1)\, \mathrm{I}(k+1). \tag{1.6}$$

Clearly, Theorem 1.4 implies Theorem 1.3. This is our first result of the long series of generalizations that follow. Before we proceed, let us illustrate the power of this refinement in a special case.

Example 1.5 (*Graphical matroids*). Let $G = (V, E)$ be a connected graph with $|V| = \mathrm{N}$ edges. The corresponding *graphical matroid* $\mathcal{M}_G = (E, \mathcal{I})$ is defined to have independent sets to be all *spanning forests* in G, i.e. spanning subgraphs without cycles. Then $\mathrm{I}(k)$ is the number of spanning forests with k edges, bases are *spanning trees* in G, and $\mathrm{rk}\big(\mathcal{M}_G\big) = \mathrm{N} - 1$.

Let $k = \mathrm{N} - 2$ in Theorem 1.4. Observe that $\mathrm{p}(\mathrm{N}-3) \le 3$ since $T - e - e'$ can have at most three connected components, for every spanning tree T in G and edges $e, e' \in E$. Then (1.6) gives:

$$\frac{\mathrm{I}(\mathrm{N}-2)^2}{\mathrm{I}(\mathrm{N}-3) \cdot \mathrm{I}(\mathrm{N}-1)} \ge \frac{3}{2}\left(1 + \frac{1}{\mathrm{N}-2}\right) \to \frac{3}{2} \quad \text{as} \quad \mathrm{N} \to \infty. \tag{1.7}$$

This is both numerically and asymptotically better than (1.3), cf. §17.12. For example, when $|E| - \mathrm{N} \to \infty$, we have:

$$\frac{\mathrm{I}(\mathrm{N}-2)^2}{\mathrm{I}(\mathrm{N}-3) \cdot \mathrm{I}(\mathrm{N}-1)} \geq_{(1.3)} \left(1 + \frac{1}{|E| - \mathrm{N} + 2}\right) \left(1 + \frac{1}{\mathrm{N}-2}\right) \to 1 \quad \text{as} \quad \mathrm{N} \to \infty.$$

1.5 Weighted matroid inequalities

Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid, and let $\omega : X \to \mathbb{R}_{>0}$ be a positive *weight function* on the ground set X. We extend the weight function to every independent set $S \in \mathcal{I}$ as follows:

$$\omega(S) := \prod_{x \in S} \omega(x).$$

For all $1 \leq k < \mathrm{rk}(\mathcal{M})$, define

$$\mathrm{I}_\omega(k) := \sum_{S \in \mathcal{I}_k} \omega(S).$$

Theorem 1.6 (*Refined weighted log-concavity for matroids*)**.** *Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid on $|X| = n$ elements, let $\omega : X \to \mathbb{R}_{>0}$ be a weight function, and let $1 \leq k < \mathrm{rk}(\mathcal{M})$. Then:*

$$\mathrm{I}_\omega(k)^2 \geq \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\mathrm{p}(k-1) - 1}\right) \mathrm{I}_\omega(k-1)\, \mathrm{I}_\omega(k+1). \tag{1.8}$$

Remark 1.7. In this theorem, the setup is more important than the result as it can be easily reduced to Theorem 1.4. Indeed, note that one can take multiple copies of elements in a matroid $\mathcal{M}$. This implies the result for integer valued ω. The full version follows by homogeneity and continuity. This natural approach fails for the equality conditions as strict inequalities are not necessarily preserved in the limit, and for many generalizations below where we have constraints on the weight function. See §16.11 for some background.

1.6 Equality conditions for matroids

For a matroid $\mathcal{M} = (X, \mathcal{I})$ on $|X| = n$ elements, define $\mathrm{girth}(\mathcal{M}) := \min\left\{k : \mathrm{I}(k) < \binom{n}{k}\right\}$. By analogy with graph theory, *girth of a matroid* is the size of the smallest circuit in $\mathcal{M}$.

Theorem 1.8 (*Equality for matroids*, [MNY21, Cor. 1.2])**.** *Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid on $|X| = n$ elements, and let $1 \leq k < \mathrm{rk}(\mathcal{M})$. Then:*

$$\mathrm{I}(k)^2 = \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{n-k}\right) \mathrm{I}(k-1)\, \mathrm{I}(k+1) \tag{1.9}$$

if and only if $\mathrm{girth}(\mathcal{M}) > (k+1)$.

See §16.12 for some background on equality conditions. The theorem says that in order to have equality (1.9), we must have probabilities $\mathrm{i}(k-1) = \mathrm{i}(k) = \mathrm{i}(k+1) = 1$. Now we present a weighted version of Theorem 1.8. We say that weight function $\omega : X \to \mathbb{R}_{>0}$ is *uniform* if $\omega(x) = \omega(y)$ for all $x, y \in X$.

Theorem 1.9 (*Weighted equality for matroids*). *Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid on $|X| = n$ elements, let $1 \le k < \mathrm{rk}(\mathcal{M})$, and let $\omega : X \to \mathbb{R}_{>0}$ be a weight function. Then:*

$$\mathrm{I}_\omega(k)^2 = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) \mathrm{I}_\omega(k-1)\, \mathrm{I}_\omega(k+1) \tag{1.10}$$

if and only if $\mathrm{girth}(\mathcal{M}) > (k+1)$, *and the weight function ω is uniform.*

The uniform condition in the theorem is quite natural for integer weight functions, as it basically says that in order to have (1.10) all elements have to be repeated the same number of times. In other words, weighted inequalities do not have a substantially larger set of equality cases.

Theorem 1.10 (*Refined equality for matroids*). *Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid, $1 \le k < \mathrm{rk}(\mathcal{M})$, and let $\omega : X \to \mathbb{R}_{>0}$ be a weight function. Then:*

$$\mathrm{I}_\omega(k)^2 = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}(k-1)-1}\right) \mathrm{I}_\omega(k-1)\, \mathrm{I}_\omega(k+1) \tag{1.11}$$

if and only if there exists $\mathrm{s}(k-1) > 0$, such that for every $S \in \mathcal{I}_{k-1}$ we have:

$$\big|\mathrm{Par}(S)\big| = \mathrm{p}(k-1), \quad \textit{and} \tag{ME1}$$
$$\sum_{x \in \mathcal{C}} \omega(x) = \mathrm{s}(k-1) \quad \textit{for every } \mathcal{C} \in \mathrm{Par}(S). \tag{ME2}$$

Condition (ME1) says that the $(k-1)$-continuation number is achieved on all independent sets $S \in \mathcal{I}_{k-1}$. When the weight function is uniform, condition (ME2) is saying that all parallel classes $\mathcal{C} \in \mathrm{Par}(S)$ have the same size.

1.7 Examples of matroids

First, we prove that the equality conditions are rarely satisfied for graphical matroids, see Example 1.5. More precisely, we prove that the refined log-concavity inequality (1.7) is an equality only for cycles:

Proposition 1.11 (*Equality for graphical matroids*). *Let $G = (V, E)$ be a simple connected graph on $|V| = \mathrm{N}$ vertices, and let $\mathrm{I}(k)$ be the number of spanning forests with k edges. Then*

$$\frac{\mathrm{I}(\mathrm{N}-2)^2}{\mathrm{I}(\mathrm{N}-3) \cdot \mathrm{I}(\mathrm{N}-1)} \ge \frac{3}{2}\left(1 + \frac{1}{\mathrm{N}-2}\right), \tag{1.12}$$

and the equality holds if and only if G is an N-cycle.

We now show that the equality conditions in Theorem 1.10 have a rich family of examples (see §16.7 for more on these examples). The weight function is uniform in all these cases: $\omega(x) = 1$ for every $x \in X$.

Example 1.12 (*Finite field matroids*). Let $\mathbb{F}_q$ be a finite field with q elements, let $m \geq 1$, and let $X = \mathbb{F}_q^m$. Let $\mathcal{I}$ be a set of subsets $S \subset \mathbb{F}_q^m$ which are linearly independent as vectors. Finally, let $\mathcal{M}(m,q) = (X, \mathcal{I})$ be a matroid of vectors in $\mathbb{F}_q^m$ of rank m.

Let $1 \leq k < m$ and let $S \in \mathcal{I}_{k-1}$, so we have $\dim_{\mathbb{F}_q} \langle S \rangle = k-1$. For all parallel classes $\mathcal{C} \in \mathrm{Par}(S)$ we then have $|\,\mathcal{C}\,| = q^{k-1}$. Therefore,

$$\left|\mathrm{Par}(S)\right| \, = \, \frac{q^m \, - \, q^{k-1}}{q^{k-1}} \, = \, q^{m-k+1} \, - \, 1 \, . \tag{1.13}$$

The conditions (ME1) and (ME2) are then satisfied with $\mathrm{p}(k-1) = q^{m-k+1} - 1$ and $\mathrm{s}(k-1) = q^{k-1}$. We conclude that (1.6) is an equality for $\mathcal{M}(m,q)$, for all $1 \leq k < m$. Curiously, the equality (1.13) is optimal for matroids over $\mathbb{F}_q$, and we have the following result (see §10.6 for the proof).

Corollary 1.13. *Let $X \subseteq \mathbb{F}_q^m$ be a set of n vectors which span $\mathbb{F}_q^m$, and let $\mathcal{M} = (X, \mathcal{I})$ be the corresponding matroid of rank $m = \mathrm{rk}(\mathcal{M})$. Then, for all $1 \leq k < m$, we have:*

$$\mathrm{I}(k)^2 \, \geq \, \left(1 \, + \, \frac{1}{k}\right) \left(1 \, + \, \frac{1}{q^{m-k+1} \, - \, 2}\right) \, \mathrm{I}(k-1) \, \mathrm{I}(k+1) \, .$$

Example 1.14 (*Steiner system matroids*). Fix integers $t < m < n$ and a ground set X, with $|X| = n$. A *Steiner system* $\mathrm{Stn}(t,m,n)$ is a collection $\mathcal{B}$ of m-subsets $B \subset X$ called *blocks*, such that each t-subset of X is contained in exactly one block $B \in \mathcal{B}$.

Let $\mathcal{M}(\mathcal{B}) = (X, \mathcal{I})$ be a matroid with $\mathrm{rk}(\mathcal{M}) = \mathrm{girth}(\mathcal{M}) = (t+1)$, where the bases are $(t+1)$-subsets of X that are not contained in any block of the Steiner system. It is easy to see that this indeed defines a matroid, cf. §16.7. Note that (1.8) is trivially an equality for all $1 \leq k < t$.

Let $S \in \mathcal{I}_{t-1}$ be an independent set of size $(t-1)$. The parallel classes of S are given by $B_1 \setminus S, \ldots, B_\ell \setminus S$, where $B_1, \ldots, B_\ell \in \mathcal{B}$ are blocks of the Steiner system that contain S, and $\ell = \frac{n-t+1}{m-t+1}$. Then we have:

$$|\,\mathrm{Par}(S)| \, = \, \ell \, , \qquad \text{and} \qquad |\,\mathcal{C}\,| \, = \, m-t+1 \quad \text{for every } \ \mathcal{C} \in \mathrm{Par}(S).$$

Since the choice of S is arbitrary, the conditions (ME1) and (ME2) are satisfied with $\mathrm{p}(t-1) = \ell$ and $\mathrm{s}(t-1) = m-t+1$. We conclude that (1.6) is also an equality for $k = t$.

1.8 Morphism of matroids

For a matroid $\mathcal{M} = (X, \mathcal{I})$, the *rank function* $f : 2^X \to \mathbb{R}_{>0}$ is defined by

$$f(S) \, := \, \max \left\{ |A| \, : \, A \subseteq S, A \in \mathcal{I} \right\}.$$

Note that $\mathrm{rk}(\mathcal{M}) = f(X)$. There is an equivalent definition of a matroid in terms of monotonic submodular rank functions, see e.g. [Wel76].

Let $\mathcal{M} = (X,\mathcal{I})$ and $\mathcal{N} = (Y,\mathcal{J})$ be two matroids with rank functions f and g, respectively. Let $\Phi : X \to Y$ be a function that satisfies

$$g\big(\Phi(T)\big) - g\big(\Phi(S)\big) \ \leq \ f(T) - f(S) \quad \text{for every } \ S \subseteq T \subseteq X\,. \tag{1.14}$$

In this case we say that Φ is a *morphism* of matroids, write $\Phi : \mathcal{M} \to \mathcal{N}$. A subset $S \in \mathcal{I}$ is said to be a *basis* of Φ if $g(\Phi(S)) = \mathrm{rk}(\mathcal{N})$. In other words, S is contained in a basis of $\mathcal{M}$, and $\Phi(S)$ contains a basis of $\mathcal{N}$. Denote by $\mathcal{B}$ the set of bases of $\Phi : \mathcal{M} \to \mathcal{N}$, and let $\mathcal{B}_k := \mathcal{B} \cap \mathcal{I}_k$.

Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function on the ground set X. As before, for every $0 \leq k \leq \mathrm{rk}(\mathcal{M})$, let

$$\mathrm{B}_\omega(k) \ := \sum_{S \in \mathcal{B}_k} \omega(S)\,, \qquad \text{where} \qquad \omega(S) \ := \prod_{x \in S} \omega(x).$$

Theorem 1.15 (*Log-concavity for morphisms*, [EH20, Thm 1.3])**.** *Let $\mathcal{M} = (X,\mathcal{I})$ and $\mathcal{N} = (Y,\mathcal{J})$ be matroids, let $n := |X|$, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids. In addition, let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function, and let $1 \leq k < \mathrm{rk}(\mathcal{M})$. Then:*

$$\mathrm{B}_\omega(k)^2 \ \geq \ \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) \mathrm{B}_\omega(k-1)\,\mathrm{B}_\omega(k+1)\,. \tag{1.15}$$

Note that when $Y = \{y\}$ and $\mathcal{N} = (Y, \varnothing)$ is defined by $g(y) = 0$, we have condition (1.14) holds trivially and $\mathcal{B} = \mathcal{I}$. Thus, the theorem generalizes Theorem 1.3 to the morphism of matroids setting. We now give the corresponding generalization of Theorem 1.6.

Recall the equivalence relation "$\sim_S$" on the set $\mathrm{Cont}(S) \subseteq X \setminus S$ of continuations of $S \in \mathcal{I}$, see (1.4). Similarly, recall the set $\mathrm{Par}(S)$ of parallel classes of S, see (1.5). For every $1 \leq k \leq \mathrm{rk}(\mathcal{M})$, let

$$\mathrm{p}(k) \ := \ \max\left\{ \big|\mathrm{Par}(S)\big| \ : \ S \in \mathcal{B}_k \right\},$$

the maximum of the number of parallel classes of bases of morphism Φ of size k.

Theorem 1.16 (*Refined log-concavity for morphisms*)**.** *Let $\mathcal{M} = (X,\mathcal{I})$ and $\mathcal{N} = (Y,\mathcal{J})$ be matroids, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids. In addition, let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function, and let $1 \leq k < \mathrm{rk}(\mathcal{M})$. Then:*

$$\mathrm{B}_\omega(k)^2 \ \geq \ \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}(k-1)\, -\, 1}\right) \mathrm{B}_\omega(k-1)\,\mathrm{B}_\omega(k+1)\,. \tag{1.16}$$

As before, since $\mathrm{p}(k-1) \leq n-k+1$, the theorem is an extension of Theorem 1.15.

Remark 1.17. The notion of morphism of matroids generalizes many classical notions in combinatorics such as *graph coloring*, *graph embeddings*, *graph homomorphism*, *matroid quotients*, and are a special case of the *induced matroids*. We refer to [EH20] for a detailed overview and further references (see also §16.8).

1.9 Equality conditions for morphisms of matroids

We start with the following characterization of equality in Theorem 1.15, which resolves an open problem in [MNY21, Question 5.7].

Theorem 1.18 (*Equality for morphisms*). *Let $\mathcal{M} = (X,\mathcal{I})$ and $\mathcal{N} = (Y,\mathcal{J})$ be matroids, let $n := |X|$, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids. In addition, let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function, and let $1 \le k < \mathrm{rk}(\mathcal{M})$. Suppose $\mathrm{B}_\omega(k) > 0$. Then:*

$$\mathrm{B}_\omega(k)^2 \ = \ \left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right)\mathrm{B}_\omega(k-1)\,\mathrm{B}_\omega(k+1)\,. \tag{1.17}$$

if and only if $\mathrm{girth}(\mathcal{M}) > k+1$, *weight function ω is uniform, and* $g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N})$ *for all* $S \in \mathcal{I}_{k-1}$.

Our next result is the following characterization of equality in Theorem 1.16.

Theorem 1.19 (*Refined equality for morphisms*). *Let $\mathcal{M} = (X,\mathcal{I})$ and $\mathcal{N} = (Y,\mathcal{J})$ be matroids, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids. In addition, let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function, and let $1 \le k < \mathrm{rk}(\mathcal{M})$. Suppose $\mathrm{B}_\omega(k) > 0$. Then:*

$$\mathrm{B}_\omega(k)^2 \ \geq \ \left(1+\frac{1}{k}\right)\left(1+\frac{1}{\mathrm{p}(k-1)\,-\,1}\right)\mathrm{B}_\omega(k-1)\,\mathrm{B}_\omega(k+1)\,. \tag{1.18}$$

if and only if there exists $\mathrm{s}(k-1) > 0$, *such that for every* $S \in \mathcal{I}_{k-1}$ *we have:*

$$\big|\mathrm{Par}_S\big| \ = \ \mathrm{p}(k-1), \tag{MME1}$$

$$\sum_{x\in\mathcal{C}} \omega(x) \ = \ \mathrm{s}(k-1) \qquad \textit{for every } \ \mathcal{C} \in \mathrm{Par}(S), \textit{ and} \tag{MME2}$$

$$g\big(\Phi(S)\big) \ = \mathrm{rk}(\mathcal{N}). \tag{MME3}$$

1.10 Discrete polymatroids

A *discrete polymatroid*[2] $\mathcal{D}$ is a pair $([n],\mathcal{J})$ of a ground set $[n] := \{1,\ldots,n\}$ and a nonempty finite collection $\mathcal{J}$ of integer points $\boldsymbol{a} = (a_1,\ldots,a_n) \in \mathbb{N}^n$ that satisfy the following:

- (*hereditary property*) $\boldsymbol{a} \in \mathcal{I}$, $\boldsymbol{b} \in \mathbb{N}^n$ s.t. $\boldsymbol{b} \leqslant \boldsymbol{a} \ \Rightarrow \ \boldsymbol{b} \in \mathcal{I}$, and
- (*exchange property*) $\boldsymbol{a},\boldsymbol{b} \in \mathcal{I}$, $|\boldsymbol{a}| < |\boldsymbol{b}| \ \Rightarrow \ \exists i \in [n]$ s.t. $a_i < b_i$ and $\boldsymbol{a}+\boldsymbol{e}_i \in \mathcal{J}$.

Here $\boldsymbol{b} \leqslant \boldsymbol{a}$ is a componentwise inequality, $|\boldsymbol{a}| := a_1+\ldots+a_n$, and $\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n\}$ is a standard linear basis in $\mathbb{R}^n$. When $\mathcal{J} \subseteq \{0,1\}^n$, discrete polymatroid $\mathcal{D}$ is a matroid. One can think of a discrete polymatroid as a set system where multisets are allowed, so we refer to $\mathcal{J}$ as *independent multisets* and to $|\boldsymbol{a}|$ as *size* of the multiset $\boldsymbol{a}$.

[2] *Discrete polymatroids* are related but should not to be confused with *polymatroids*, which is a family of convex polytopes, see e.g. [Sch03, §44] and §16.9.

The role of bases in discrete polymatroids is played by *maximal elements* with respect to the order "$\leqslant$"; they are called *M-convex sets* in [BH20, §2]. Define $\operatorname{rk}(\mathcal{D}) := \max\{\,|\boldsymbol{a}| \,:\, \boldsymbol{a} \in \mathcal{J}\,\}$. For $0 \leq k \leq \operatorname{rk}(\mathcal{D})$, denote by $\mathcal{J}_k := \{\boldsymbol{a} \in \mathcal{J} \,:\, |\boldsymbol{a}| = k\}$ the subcollection of independent multisets of size k, and let $\mathrm{J}(k) := |\mathcal{J}_k|$.

Let $\omega : [n] \to \mathbb{R}_{>0}$ be a positive *weight function* on $[n]$. We extend weight function ω to all $\boldsymbol{a} \in \mathcal{J}$ as follows:

$$\omega(\boldsymbol{a}) \;:=\; \omega(1)^{a_1} \cdots \omega(n)^{a_n}.$$

For every $0 \leq k \leq \operatorname{rk}(\mathcal{D})$, define

$$\mathrm{J}_\omega(k) \;:=\; \sum_{\boldsymbol{a} \in \mathcal{J}_k} \frac{\omega(\boldsymbol{a})}{\boldsymbol{a}!}\,, \quad \text{where} \quad \boldsymbol{a}! := a_1! \cdots a_n!$$

Theorem 1.20 (*Log-concavity for polymatroids*, [BH20, Thm 3.10 (4) ⇔ (7)])**.** *Let $\mathcal{D} = ([n], \mathcal{J})$ be a discrete polymatroid, and let $\omega : [n] \to \mathbb{R}_{>0}$ be a positive weight function. For every $1 \leq k < \operatorname{rk}(\mathcal{M})$, we have:*

$$\mathrm{J}_\omega(k)^2 \;\geq\; \left(1 + \frac{1}{k}\right) \mathrm{J}_\omega(k-1)\, \mathrm{J}_\omega(k+1)\,. \tag{1.19}$$

We now give a common generalization of Theorem 1.6 and Theorem 1.20. Fix $t \in [0,1]$, and let

$$\pi(\boldsymbol{a}) \;:=\; \sum_{i=1}^{n} \binom{a_i}{2}.$$

For every $0 \leq k \leq \operatorname{rk}(\mathcal{D})$, define

$$\mathrm{J}_{\omega,t}(k) \;:=\; \sum_{\boldsymbol{a} \in \mathcal{J}_k} t^{\pi(\boldsymbol{a})}\, \frac{\omega(\boldsymbol{a})}{\boldsymbol{a}!}\,.$$

Note that $\binom{a}{2} = 0$ for $a \in \{0,1\}$, so $\pi(\boldsymbol{a}) = 0$ for all independent sets $\boldsymbol{a} \in \mathcal{I}$ in a matroid.

For an independent multiset $\boldsymbol{a} \in \mathcal{J}$ of a discrete polymatroid $\mathcal{D} = ([n], \mathcal{J})$, denote by

$$\operatorname{Cont}(\boldsymbol{a}) := \bigl\{ i \in [n] \,:\, \boldsymbol{a} + \boldsymbol{e}_i \in \mathcal{J} \bigr\}. \tag{1.20}$$

the set of *continuations* of $\boldsymbol{a}$. For all $i, j \in \operatorname{Cont}(\boldsymbol{a})$, we write $i \sim_{\boldsymbol{a}} j$ when $\boldsymbol{a} + \boldsymbol{e}_i + \boldsymbol{e}_j \notin \mathcal{J}$ or $i = j$. This is an equivalence relation again, see Proposition 4.2. We call an equivalence class of the relation $\sim_{\boldsymbol{a}}$ a *parallel class* of $\boldsymbol{a}$, and we denote by $\operatorname{Par}(\boldsymbol{a})$ the set of parallel classes of $\boldsymbol{a}$.

For every $0 \leq k < \operatorname{rk}(\mathcal{D})$, define the *k-continuation number* of a discrete polymatroid $\mathcal{D}$ as the maximal number of parallel classes of independent multisets of size k:

$$\mathrm{p}(k) \;:=\; \max\bigl\{\, |\operatorname{Par}(\boldsymbol{a})| \,:\, \boldsymbol{a} \in \mathcal{J}_k \,\bigr\}. \tag{1.21}$$

For matroids, this is the same notion as defined above in §1.4.

Theorem 1.21 (*Refined log-concavity for polymatroids*)**.** *Let $\mathcal{D} = ([n], \mathcal{J})$ be a discrete polymatroid, and let $\omega : [n] \to \mathbb{R}_{>0}$ be a positive weight function. For every $t \in [0,1]$ and $1 \leq k < \operatorname{rk}(\mathcal{M})$, we have:*

$$\mathrm{J}_{\omega,t}(k)^2 \;\geq\; \left(1 + \frac{1}{k}\right) \left(1 + \frac{1-t}{\mathrm{p}(k-1) - 1 + t}\right) \mathrm{J}_{\omega,t}(k-1)\, \mathrm{J}_{\omega,t}(k+1)\,. \tag{1.22}$$

When $t = 1$, this gives Theorem 1.20. When $\mathcal{D}$ is a matroid and $t = 0$, this gives Theorem 1.6. For general discrete polymatroids $\mathcal{D}$ and $0 < t < 1$, this is a stronger result.

Example 1.22 (*Hypergraphical polymatroids*)**.** Let $\mathcal{H} = (V,E)$ be a *hypergraph* on the finite set of vertices V, with *hyperedges* $E = \{e_1,\ldots,e_n\}$, where $e_i \subseteq V$, $e_i \neq \varnothing$. Let $W = \{w_1,\ldots,w_n\}$ be a collection of subsets of V, such that $w_i \subseteq e_i$, $w_i \neq \varnothing$, and every vertex $v \in V$ belongs to some w_i. A *hyperpath* is an alternating sequence $v \to w_i \to v' \to w_j \to v'' \to \ldots \to u$, where $v, v' \in w_i$, $v', v'' \in w_j$, etc., and the vertices $v, v', v'', \ldots, u \in V$ are not repeated.

A *spanning hypertree* in $\mathcal{H}$ is a collection W as above, such that every two vertices $v, u \in V$ are connected by exactly one such hyperpath. Similarly, a *spanning hyperforest* in $\mathcal{H}$ is a collection W as above, such that every two vertices are connected by at most one hyperpath. In the case all $|e_i| = 2$, we get the usual notions of (undirected) graphs, paths, spanning trees and spanning forests. We say that $\boldsymbol{d} = (d_1,\ldots,d_n)$, where $d_i = |w_i| - 1 \geq 0$, is a *degree sequence* of W. Note that in the graphical case, we have $d_i \in \{0,1\}$, so a forest is determined by its degree sequence. In general hypergraphs this is no longer true.

Finally, a *hypergraphical polymatroid* corresponding to $\mathcal{H}$ is a discrete polymatroid $\mathcal{D}_{\mathcal{H}} = ([n],\mathcal{J})$, where $\mathcal{J}$ is a set of degree sequences of spanning hyperforests in $\mathcal{H}$. Similarly to graphical matroids (Example 1.29), the maximal elements are degree sequences of spanning hypertrees in $\mathcal{H}$. Therefore, Theorems 1.20 and 1.21 give log-concavity for the weighted sum $\mathrm{J}_{\omega,t}(k)$ over degree sequences with total degree $d_1 + \ldots + d_n = k$. See §16.10 for the background of this example.

1.11 Equality conditions for polymatroids

A discrete polymatroid $\mathcal{D} = ([n],\mathcal{J})$ is called *nondegenerate* if $\mathbf{e}_i \in \mathcal{J}$ for every $i \in [n]$. Define $\operatorname{polygirth}(\mathcal{D}) := \min\bigl\{k : \mathrm{J}(k) < \binom{n+k-1}{k-1}\bigr\}$. Observe that $\boldsymbol{a} \in \mathcal{J}$ for all $\boldsymbol{a} \in \mathbb{N}^k$, $|\boldsymbol{a}| < \operatorname{polygirth}(\mathcal{D})$. Note that the *polygirth* of a discrete polymatroid does not coincide with the girth of a matroid. In fact, $\operatorname{polygirth}(\mathcal{D}) = 2$ when $\mathcal{D}$ is a matroid with more than one element.

To get the equality conditions for (1.22), we separate the cases $t = 0$, $0 < t < 1$, and $t = 1$. The case $t = 0$ coincides with equality conditions for matroids given in Theorem 1.10. Examples in §1.7 show that this is a difficult condition with many nontrivial examples. The other two cases are in fact much less rich.

Theorem 1.23 (*Refined equality for polymatroids*, $t = 1$ case)**.** *Let $\mathcal{D} = ([n],\mathcal{J})$ be a nondegenerate discrete polymatroid, let $\omega : [n] \to \mathbb{R}_{>0}$ be a positive weight function, and let $1 \leq k < \operatorname{rk}(\mathcal{M})$. Then:*

$$\mathrm{J}_\omega(k)^2 = \left(1 + \frac{1}{k}\right) \mathrm{J}_\omega(k-1)\, \mathrm{J}_\omega(k+1). \tag{1.23}$$

if and only if $\operatorname{polygirth}(\mathcal{D}) > (k+1)$.

We are giving the equality condition for (1.19) in place of (1.22), since $\mathrm{J}_{\omega,1}(k) = \mathrm{J}_\omega(k)$ for all k.

Theorem 1.24 (*Refined equality for polymatroids*, $0 < t < 1$ case)**.** *Let $\mathcal{D} = ([n], \mathcal{J})$ be a nondegenerate discrete polymatroid, and let $\omega : [n] \to \mathbb{R}_{>0}$ be a positive weight function. Fix $1 \leq k < \mathrm{rk}(\mathcal{M})$ and $0 < t < 1$. Then:*

$$\mathrm{J}_{\omega,t}(k)^2 \;=\; \left(1 + \frac{1}{k}\right) \left(1 + \frac{1-t}{\mathrm{p}(k-1) - 1 + t}\right) \mathrm{J}_{\omega,t}(k-1)\, \mathrm{J}_{\omega,t}(k+1)\,. \tag{1.24}$$

if and only if $k = 1$, $\mathrm{polygirth}(\mathcal{D}) > 2$*, and ω is uniform.*

Remark 1.25. The reason the case $t = 0$ is substantially different, is because the combined weight function $t^{N(\boldsymbol{a})}\,\omega(\boldsymbol{a})$ is no longer strictly positive. Alternatively, one can view the dearth of nontrivial examples in these theorems as suggesting that the bound in Theorem 1.21 can be further improved for $t > 0$. This is based on the reasoning that Theorem 1.4 sharply improves over Theorem 1.3 because there are only trivial equality conditions for the latter (see Theorem 1.8), when compared with rich equality conditions for the former (see Theorem 1.9).

1.12 Poset antimatroids

Let X be finite set we call *letters*, let $n = |X|$, and let X^* be a set of *finite words* in the alphabet X. A *language over X* is a nonempty finite subset $\mathcal{L} \subset X^*$. A word is called *simple* if it contains each letter at most once; we consider only simple words from this point on. We write $x \in \alpha$ if word $\alpha \in \mathcal{L}$ contains letter x. Finally, let $|\alpha|$ be the length of the word, and denote $\mathcal{L}_k := \big\{\alpha \in \mathcal{L} \,:\, |\alpha| = k\big\}$.

A pair $\mathcal{A} = (X, \mathcal{L})$ is an *antimatroid*, if the language $\mathcal{L} \subset X^*$ satisfies:

- (*nondegenerate property*) every $x \in X$ is contained in at least one $\alpha \in \mathcal{L}$,
- (*normal property*) every $\alpha \in \mathcal{L}$ is simple,
- (*hereditary property*) $\alpha\beta \in \mathcal{L} \;\Rightarrow\; \alpha \in \mathcal{L}$, and
- (*exchange property*) $x \in \alpha$, $x \notin \beta$, and $\alpha, \beta \in \mathcal{L} \;\Rightarrow\; \exists y \in \alpha$ s.t. $\beta y \in \mathcal{L}$.

Note that for every antimatroid $\mathcal{A} = (X, \mathcal{L})$, it follows from the exchange property that

$$\mathrm{rk}(\mathcal{A}) \,:=\, \max\{\,|\alpha| \,:\, \alpha \in \mathcal{L}\} \,=\, n\,.$$

Throughout the paper we use only one class of antimatroids which we now define (cf. §16.14).

Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements. A simple word $\alpha \in X^*$ is called *feasible* if α satisfies:

- (*poset property*) if α contains $x \in X$ and $y \prec x$, then letter y occurs before letter x in α.

A *poset antimatroid* $\mathcal{A}_{\mathcal{P}} = (X, \mathcal{L})$ is defined by the language $\mathcal{L}$ of all feasible words in X. The exchange property is satisfied because one can always take y to be the minimal letter (w.r.t. order $\prec$) that is not in β.

Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function on X. Denote by $\mathrm{Cov}(x) := \{y \in X : x \leftarrow\!\!\prec y\}$ the set of elements which cover x. We assume the weight function ω satisfies the following (*cover monotonicity property*):

$$\omega(x) \geq \sum_{y \in \mathrm{Cov}(x)} \omega(y), \quad \text{for all } x \in X. \tag{CM}$$

Note that when (CM) is equality for all $x \in X$, we have:

$$\omega(x) = \text{number of maximal chains in } \mathcal{P} \text{ starting at } x. \tag{1.25}$$

For all $\alpha \in \mathcal{L}$ and $0 \leq k \leq n$, let

$$\mathrm{L}_\omega(k) := \sum_{\alpha \in \mathcal{L}_k} \omega(\alpha), \quad \text{where} \quad \omega(\alpha) := \prod_{x \in \alpha} \omega(x).$$

Theorem 1.26 (*Log-concavity for poset antimatroids*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements, and let $\mathcal{A}_{\mathcal{P}} = (X, \mathcal{L})$ be the corresponding poset antimatroid. Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function which satisfies* (CM)*. Then, for every integer $1 \leq k < n$, we have:*

$$\mathrm{L}_\omega(k)^2 \geq \mathrm{L}_\omega(k-1) \cdot \mathrm{L}_\omega(k+1). \tag{1.26}$$

Example 1.27 (*Standard Young tableaux of skew shape*)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, be a Young diagram, and let $\mathcal{P}_\lambda = (\lambda, \prec)$ be a poset on squares $\big\{(i-1, j-1) : 1 \leq i \leq \lambda_j, 1 \leq j \leq \ell\big\} \subset \mathbb{N}^2$, with $(i, j) \preccurlyeq (i', j')$ if $i \geq i'$ and $j \geq j'$. Following (1.25), let $\omega(i, j) = \binom{i+j}{i}$. Denote $a_\lambda(k) := \mathrm{L}_\omega(k)$, $0 \leq k \leq |\lambda|$, and we have:

$$a_\lambda(k) = \sum_{\mu \subset \lambda,\, |\lambda/\mu| = k} f^{\lambda/\mu} \prod_{(i,j) \in \lambda/\mu} \binom{i+j}{i},$$

where $f^{\lambda/\mu} = \big|\mathrm{SYT}(\lambda/\mu)\big|$ is the number of *standard Young tableaux* of shape λ/μ (see §16.15). Now Theorem 1.26 proves that the sequence $\big\{a_\lambda(k)\big\}$ is log-concave, for every λ.

This example also shows that the weight function condition (CM) is necessary. Indeed, let λ be a $m \times m$ square, $n = m^2$, and let $\omega(i, j) = 1$. Then, for all $k \leq m$, we have:

$$b(k) := \mathrm{L}_\omega(k) = \big|\mathcal{L}_k\big| = \sum_{\mu \vdash k} f^\mu .$$

The sequence $\{b_k\}$ is the *number of involutions in S_k*, see e.g. [OEIS, A000085], which satisfies $\log b_k = \frac{1}{2} n \log n + O(n)$, and is actually *log-convex*, see e.g. [Mező20, §4.5.2].

1.13 Equality conditions for poset antimatroids

Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements, and let $\mathcal{A}_{\mathcal{P}} = (X, \mathcal{L})$ be the corresponding poset antimatroid.

For a word $\alpha \in \mathcal{L}$, denote by

$$\mathrm{Cont}(\alpha) := \{x \in X \,:\, \alpha x \in \mathcal{L}\}$$

the set of *continuations* of the word α. Define an equivalence relation "$\sim_\alpha$" on $\mathrm{Cont}(\alpha)$ by setting $x \sim_\alpha y$ if $\alpha x y \notin \mathcal{L}$, see Proposition 4.3. We call the equivalence classes of "$\sim_\alpha$" the *parallel classes* of α, and denote by $\mathrm{Par}(\alpha)$ the set of these parallel classes.

Let $\alpha \in \mathcal{L}$ and $x \in \mathrm{Cont}(\alpha)$. We say that $y \in X$ is a *descendent* of x with respect to α if $\alpha x y \in \mathcal{L}$ and $\alpha y \notin \mathcal{L}$. Denote by $\mathrm{Des}_\alpha(x)$ the set of descendants of x with respect to α. We omit α when the word is clear from the context.

Theorem 1.28 (*Equality for poset antimatroids*). *Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements, and let $\mathcal{A}_{\mathcal{P}} = (X, \mathcal{L})$ be the corresponding poset antimatroid. Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function which satisfies* (CM)*, and fix an integer $1 \le k < n$. Then:*

$$\mathrm{L}_\omega(k)^2 \;=\; \mathrm{L}_\omega(k-1) \cdot \mathrm{L}_\omega(k+1) \tag{1.27}$$

if and only if there exists $\mathrm{s}(k-1) > 0$, such that for every $\alpha \in \mathcal{L}_{k-1}$ and $x \in \mathrm{Cont}(\alpha)$, we have:

$$\sum_{x \in \mathrm{Cont}(\alpha)} \omega(x) \;=\; \mathrm{s}(k-1), \tag{AE1}$$

$$\mathrm{Des}_\alpha(x) \;=\; \mathrm{Cov}(x), \quad \textit{and} \tag{AE2}$$

$$\sum_{y \in \mathrm{Cov}(x)} \omega(y) \;=\; \omega(x). \tag{AE3}$$

The following is an example of a poset that satisfies conditions of Theorem 1.26.

Example 1.29 (*Tree posets*). Let $\mathrm{T} = (V, E)$ be a finite *rooted tree* with root at $R \in V$, and the set of leaves $S \subset V$. Suppose further, that all leaves $v \in S$ are at distance h from R. Consider a poset $\mathcal{P}_{\mathrm{T}} = (V, \prec)$ with $v \prec v'$ if the shortest path $v' \to R$ goes through v, for all $v, v' \in V$. We call $\mathcal{P}_{\mathrm{T}}$ the *tree poset* corresponding to T. Denote by $S(v) := S \cap \{v' \in V \,:\, v' \succcurlyeq v\}$ the subset of leaves in the order ideal of v.

Let $\omega : X \to \mathbb{R}_{>0}$ be defined by (1.25). Observe that $\omega(v) = \big|S(v)\big|$, since maximal chains in $\mathcal{P}_{\mathrm{T}}$ are exactly the shortest paths in T towards one of the leaves, i.e. of the form $v \to w$ for some $w \in S$. Note that $S(v) \supseteq S(v')$ for all $v \prec v'$, $S(v) \cap S(v') = \varnothing$ for all v and v' that are incomparable, and $\sum_{x \in \mathrm{Cov}(v)} |S(x)| = |S(v)|$ for all $v \notin S$. These imply (AE1)–(AE3) for all $k \le h$, with $\mathrm{s}(k-1) = |S|$. By Theorem 1.26, we get an equality (1.27) in this case.

The following result shows the importance of tree posets for the equality conditions.

Theorem 1.30 (*Total equality for poset antimatroids*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements, and let $\mathcal{A}_{\mathcal{P}} = (X, \mathcal{L})$ be the corresponding poset antimatroid. Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function which satisfies* (CM)*. Then:*

$$\mathrm{L}_\omega(k)^2 \; = \; \mathrm{L}_\omega(k-1) \cdot \mathrm{L}_\omega(k+1) \quad \textit{for all} \;\; 1 \le k < \mathrm{height}(\mathcal{P}) \tag{1.28}$$

if and only if $\mathcal{P} \cup \widehat{0}$ *is a tree poset* $\mathcal{P}_{\mathrm{T}}$ *with a root at* $\widehat{0}$*, with all leaves at the same distance to the root, and such that* $c\,\omega$ *is defined by* (1.25)*, for some constant multiple* $c > 0$.

1.14 Interval greedoids

Let X be finite set of letters, and let $\mathcal{L} \subset X^*$ be a language over X. A pair $\mathcal{G} = (X, \mathcal{L})$ is a *greedoid*, if the language $\mathcal{L}$ satisfies:

- (*nondegenerate property*) empty word $\varnothing$ is in $\mathcal{L}$,
- (*normal property*) every $\alpha \in \mathcal{L}$ is simple,
- (*hereditary property*) $\alpha\beta \in \mathcal{L} \;\Rightarrow\; \alpha \in \mathcal{L}$, and
- (*exchange property*) $\alpha, \beta \in \mathcal{L}$ s.t. $|\alpha| > |\beta| \;\Rightarrow\; \exists x \in \alpha$ s.t. $\beta x \in \mathcal{L}$.

Let $\mathrm{rk}(\mathcal{G}) := \max\{|\alpha| : \alpha \in \mathcal{L}\}$ be the *rank* of greedoid $\mathcal{G}$. Note that every maximal word in $\mathcal{L}$ has the same length by the exchange property. In the literature, greedoids are also defined via *feasible sets* of letters in $\alpha \in \mathcal{L}$, but we restrict ourselves to the language notation. We use [BZ92, §8.2.B] and [KLS91, §V.5] as our main references on interval greedoids; see also §16.13 for some background.

Greedoid $\mathcal{G} = (X, \mathcal{L})$ is called *interval* if the language $\mathcal{L}$ also satisfies:

- (*interval property*) $\alpha, \beta, \gamma \in X^*$, $x \in X$ s.t. $\alpha x, \alpha\beta\gamma x \in \mathcal{L} \;\;\Rightarrow\;\; \alpha\beta x \in \mathcal{L}$.

It is well known and easy to see that antimatroids are interval greedoids.

Let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be a positive weight function. Let

$$\mathrm{L}_{\mathrm{q}}(k) \, = \sum_{\alpha \in \mathcal{L}_k} \mathrm{q}(\alpha) \, .$$

In the next section, we define the notion of *k-admissible* weight function q, see Definition 3.2. This notion is much too technical to state here. We use it to formulate our first main result:

Theorem 1.31 (*Log-concavity for interval greedoids*, first main theorem)**.** *Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid, let $1 \le k < \mathrm{rk}(\mathcal{G})$, and let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be a k-admissible weight function. Then:*

$$\mathrm{L}_{\mathrm{q}}(k)^2 \;\ge\; \mathrm{L}_{\mathrm{q}}(k-1) \cdot \mathrm{L}_{\mathrm{q}}(k+1) \, . \tag{1.29}$$

This is the first main result of the paper, as it implies all previous inequalities for matroids, polymatroids and antimatroids.

Example 1.32 (*Directed branching greedoids*). Let $G=(V,E)$ be a directed graph on $|V|=n$ vertices strongly connected towards the root $R\in V$. An *arborescence* is a tree in G strongly connected towards the root R. A word $\alpha=e_1\cdots e_\ell\in E^*$ is called *pointed* if every prefix of α consists of edges which form an arborescence. One can think of pointed words as *increasing arborescences* in G (cf. §16.16).

The *directed branching greedoid* $\mathcal{G}_G=(E,\mathcal{L})$ is defined on the ground sets E by the language $\mathcal{L}\subset E^*$ of pointed words. It is well known and easy to see that $\mathcal{G}_G$ is an interval greedoid. When $G=T$ is a rooted tree, greedoid $\mathcal{G}_T$ is the poset antimatroid corresponding to the tree poset $\mathcal{P}_P$ (see Example 1.29). For general graphs, greedoid $\mathcal{G}_G$ is not necessarily a poset antimatroid. Theorem 1.31 in this case proves log-concavity for the numbers $\mathrm{L}_{\mathrm{q}}(k)$ of weighted increasing arborescences, cf. §16.16.

1.15 Equality conditions for interval greedoids

A word $\beta\in X^*$ is called a *continuation* of the word $\alpha\in\mathcal{L}$, if $\alpha\beta\in\mathcal{L}$. Denote by $\mathrm{Cont}_k(\alpha)\subset X^*$ the set of continuations of the word α with $\beta\in X^*$ of length $|\beta|=k$. Note that $\mathrm{Cont}(\alpha)=\mathrm{Cont}_1(\alpha)$. For notational convenience, we define $\mathrm{Cont}(\alpha)=\varnothing$ if $\alpha\notin\mathcal{L}$.

For every $\alpha\in\mathcal{L}$, let

$$\mathrm{L}_{\mathrm{q},\alpha}(k) \ := \sum_{\beta\in\mathrm{Cont}_k(\alpha)} \mathrm{q}(\alpha\beta)\,.$$

Note that $\mathrm{L}_{\mathrm{q}}(k)=\mathrm{L}_{\mathrm{q},\varnothing}(k)$ and $\mathrm{L}_{\mathrm{q},\alpha}(0)=\mathrm{q}(\alpha)$.

Theorem 1.33 (*Equality for interval greedoids*, cf. Theorem 3.3). *Let $\mathcal{G}=(X,\mathcal{L})$ be an interval greedoid, let $1\le k<\mathrm{rk}(\mathcal{G})$, and let $\mathrm{q}:\mathcal{L}\to\mathbb{R}_{>0}$ be a k-admissible weight function. Then:*

$$\mathrm{L}_{\mathrm{q}}(k)^2 \ = \ \mathrm{L}_{\mathrm{q}}(k-1)\cdot\mathrm{L}_{\mathrm{q}}(k+1)$$

if and only if there is $\mathrm{s}(k-1)>0$, such that for every $\alpha\in\mathcal{L}_{k-1}$ we have:

$$\mathrm{L}_{\mathrm{q},\alpha}(2) \ = \ \mathrm{s}(k-1)\,\mathrm{L}_{\mathrm{q},\alpha}(1) \ = \ \mathrm{s}(k-1)^2\,\mathrm{L}_{\mathrm{q},\alpha}(0)\,.$$

This is the second main result of the paper, giving an easy way to check the equality conditions. A more detailed and technical condition is given in Theorem 3.3, which we use to obtain the equality conditions for matroids, polymatroids and antimatroids.

1.16 Linear extensions

Let $\mathcal{P}:=(X,\prec)$ be a poset on $n:=|X|$ elements. A *linear extension* of $\mathcal{P}$ is a bijection $L:X\to\{1,\ldots,n\}$, such that $L(x)<L(y)$ for all $x\prec y$. Fix an element $z\in X$. Denote by $\mathcal{E}:=\mathcal{E}(P)$ the set of linear extensions of $\mathcal{P}$, let $\mathcal{E}_k:=\{L\in\mathcal{E}\,:\,L(z)=k\}$, and let $e(\mathcal{P}):=|\mathcal{E}|$. See §16.17 and §16.18 for some background.

Theorem 1.34 (*Stanley inequality* [Sta81, Thm 3.1])**.** *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements, and let $z \in X$. Denote by $\mathrm{N}(k) := |\mathcal{E}_k|$ the number of linear extensions $L \in \mathcal{E}(P)$, such that $L(z) = k$. Then, for every $1 < k < n$, we have:*

$$\mathrm{N}(k)^2 \ \geq \ \mathrm{N}(k-1) \cdot \mathrm{N}(k+1). \tag{1.30}$$

We now give a weighted generalization of this result. Let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function on X. We say that ω is *order-reversing* if it satisfies

$$x \preccurlyeq y \quad \Rightarrow \quad \omega(x) \geq \omega(y). \tag{Rev}$$

Fix $z \in X$, as above. Define $\omega : \mathcal{E} \to \mathbb{R}_{>0}$ by

$$\omega(L) \ := \prod_{x\,:\,L(x)<L(z)} \omega(x), \tag{1.31}$$

and let

$$\mathrm{N}_\omega(k) \ := \sum_{L \in \mathcal{E}_k} \omega(L), \quad \text{for all } 1 \leq k \leq n. \tag{1.32}$$

Theorem 1.35 (*Weighted Stanley inequality*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements, and let $\omega : X \to \mathbb{R}_{>0}$ be a positive order-reversing weight function. Fix an element $z \in X$. Then, for every $1 < k < n$, we have:*

$$\mathrm{N}_\omega(k)^2 \ \geq \ \mathrm{N}_\omega(k-1) \cdot \mathrm{N}_\omega(k+1), \tag{1.33}$$

where $\mathrm{N}_\omega(k)$ is defined by (1.32).

Remark 1.36. In §14.8, we give further applications of our approach by extending the set of possible weights in Theorem 1.35 to a smaller class of *posets with belts*. We postpone this discussion to avoid cluttering, but the interested reader is encouraged to skip to that subsection which can be read separately from the rest of the paper.[3]

1.17 Two permutation posets examples

It is not immediately apparent that the numbers of linear extensions appear widely across mathematics. Below we present two notable examples from algebraic and enumerative combinatorics, see §16.19 for some background.

[3] In a followup investigation, we use the combinatorial atlas technology in [CP22b] to prove correlation inequalities for the numbers of linear extensions of posets.

Example 1.37 (*Bruhat orders*)**.** Let $\sigma \in S_n$ and define the *permutation poset* $\mathcal{P}_\sigma = ([n], \prec)$ by letting

$$i \preccurlyeq j \quad \Leftrightarrow \quad i \le j \text{ and } \sigma(i) \le \sigma(j).$$

Fix $z \in [n]$. Viewing $\mathcal{E} = \mathcal{E}(\mathcal{P}_\sigma)$ as a subset of S_n, it is easy to see that $\mathcal{E}$ is the lower ideal of σ in the (weak) *Bruhat order* $\mathcal{B}_n = (S_n, \lhd)$. Thus, $\mathcal{E}_k = \{\nu \in S_n : \nu(z) = k, \nu \unlhd \sigma\}$.

Let $\omega(i) = q^i$, where $0 < q < 1$. Then ω is order-reversing. Now (1.31) gives $\omega(\nu) = q^{\beta(\nu)}$, where

$$\beta(\nu) := \sum_{i=1}^{z-1} i \cdot \chi\big(k - \nu(i)\big) \quad \text{and} \quad \chi(t) := \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \le 0 \end{cases}$$

Now Theorem 1.35 gives log-concavity $a_q(k)^2 \ge a_q(k-1) \cdot a_q(k+1)$, where $a_q(k) := \mathrm{N}_\omega(k) \ge 0$ is given by

$$a_q(k) = \sum_{\nu \in S_n : \nu \unlhd \sigma, \nu(z) = k} q^{\beta(\nu)}.$$

Example 1.38 (*Euler–Bernoulli* and *Entringer numbers*)**.** Let $\mathcal{Q}_m = ([2m-1], \prec)$ be a height two poset corresponding to the skew Young diagram δ_m/δ_{m-2}, where $\delta_m := (m, \ldots, 2, 1)$. The linear extensions of $\mathcal{Q}_m$ are in natural bijection with *alternating permutations* $\sigma \in S_{2m-1}$ s.t.

$$\sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) < \ldots$$

Then the numbers $e(\mathcal{Q}_m)$ are the *Euler numbers*, which are closely related to the *Bernoulli numbers*, and have EGF

$$\sum_{m=1}^{\infty} (-1)^{m-1} e(\mathcal{Q}_m) \frac{t^{2m-1}}{(2m-1)!} = \tan(t),$$

see e.g. [OEIS, A000111]. Fix $z = 1$. It is easy to see that triangle of numbers $a(m,k) = \big|\mathcal{E}_k(\mathcal{Q}_m)\big|$ are *Entringer numbers* [OEIS, A008282], and Stanley's Theorem 1.34 proves their log-concavity:

$$a(m,k)^2 \ge a(m,k-1)\, a(m,k+1) \quad \text{for } 1 \le k \le 2m-2.$$

Now, let $\omega(2) = \omega(4) = \ldots = 1$, $\omega(1) = \omega(3) = \ldots = q$, where $0 < q < 1$. Similarly to the previous example, we have $\omega(\sigma) = q^{\gamma(\sigma)}$, where $\gamma(\sigma)$ is the number of permutation entries in the odd positions which are $< k$. Theorem 1.35 then proves log-concavity for the corresponding q-deformation of the Entringer numbers.

1.18 Equality conditions for linear extensions

Let $\mathcal{P} := (X, \prec)$ be a poset on $|X| = n$ elements. Denote by $f(x) := \big|\{y \in X : y \prec x\}\big|$ and $g(x) := \big|\{y \in X : y \succ x\}\big|$ the sizes of lower and upper ideals of $x \in X$, respectively, excluding the element x.

Theorem 1.39 (*Equality condition for Stanley inequality* [SvH20, Thm 15.3])**.** *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements. Let $z \in X$ and let $\mathrm{N}(k)$ be the number of linear extensions $L \in \mathcal{E}(P)$, such that $L(z) = k$. Suppose that $\mathrm{N}(k) > 0$. Then the following are equivalent:*

(a) $\mathrm{N}(k)^2 \,=\, \mathrm{N}(k-1) \cdot \mathrm{N}(k+1)$,

(b) $\mathrm{N}(k+1) \,=\, \mathrm{N}(k) \,=\, \mathrm{N}(k-1)$,

(c) *we have* $f(x) > k$ *for all* $x \succ z$, *and* $g(x) > n-k+1$ *for all* $x \prec z$.

See §16.22 for some background. The weighted version of this theorem is a little more subtle and needs the following *(s,k)-cohesiveness property*:

$$\omega\big(L^{-1}(k-1)\big) \,=\, \omega\big(L^{-1}(k+1)\big) \,=\, \mathrm{s}, \quad \text{for all } \ L \in \mathcal{E}_k\,. \tag{Coh}$$

Note that (Coh) can hold for non-uniform weight functions ω, for example for $\mathcal{P} = A_{k+1} \oplus C_{n-k-1}$, i.e. the *linear sum* of an antichain on which ω is uniform and a chain on which ω can be non-uniform. In fact, if z is an element in A_{k+1}, we can have $\omega(z)$ different from the rest of the antichain.

Theorem 1.40 (*Equality condition for weighted Stanley inequality*). *Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements, and let $\omega : X \to \mathbb{R}_{>0}$ be a positive order-reversing weight function. Fix element $z \in X$ and let $\mathrm{N}_\omega(k)$ be defined as in (1.32). Suppose that $\mathrm{N}_\omega(k) > 0$. Then the following are equivalent:*

(a) $\mathrm{N}_\omega(k)^2 \,=\, \mathrm{N}_\omega(k-1) \cdot \mathrm{N}_\omega(k+1)$,

(b) *there exists* $\mathrm{s} = \mathrm{s}(k,z) > 0$, *s.t.*

$$\mathrm{N}_\omega(k+1) \,=\, \mathrm{s}\,\mathrm{N}_\omega(k) \,=\, \mathrm{s}^2\,\mathrm{N}_\omega(k-1),$$

(c) *there exists* $\mathrm{s} = \mathrm{s}(k,z) > 0$, *s.t.* $f(x) > k$ *for all* $x \succ z$, $g(x) > n-k+1$ *for all* $x \prec z$, *and* (Coh).

1.19 Summary of results and implications

Here is a chain of matroid results from new to known:

Thm 1.6 ⇒ Thm 1.4 ⇒ Thm 1.3 ⇒ Thm 1.2 ⇒ Thm 1.1.

The first two of these introduce the *refined log-concave inequalities*, both weighted and unweighted, and they imply the last three known theorems. For morphisms of matroids and for polymatroids, we have two new results which extend two earlier results:

Thm 1.16 ⇒ Thm 1.15 and Thm 1.21 ⇒ Thm 1.20.

Here is a family of implications of log-concave inequalities across matroid generalizations, from interval greedoids to polymatroids to matroids, and from interval greedoids to poset antimatroids:

Thm 1.31 $\Rightarrow_{\S 4.4}$ Thm 1.21 $\Rightarrow_{\S 1.10}$ Thm 1.6 and Thm 1.31 $\Rightarrow_{\S 4.2}$ Thm 1.26.

All these results are new. Note that both polymatroids and poset antimatroids are different special cases of interval greedoids, while our results on morphisms of matroids are separate and do not generalize.

For the equality conditions, we have a similar chain of implications across matroid generalizations:

$$\text{Thm 3.3} \;\Rightarrow\; \text{Thm 1.33} \;\Rightarrow\; \text{Thm 1.24} \cup \text{Thm 1.23} \;\Rightarrow\; \text{Thm 1.10} \;\Rightarrow\; \text{Thm 1.9} \;\Rightarrow\; \text{Thm 1.8},$$
$$\text{Thm 1.19} \;\Rightarrow\; \text{Thm 1.18} \qquad \text{and} \qquad \text{Thm 3.3} \;\Rightarrow\; \text{Thm 1.28} \;\Rightarrow\; \text{Thm 1.30}.$$

Of these, only Theorem 1.8 was previously known. The most general of these, Theorem 3.3, is too technical to be stated in the introduction. The same holds for Definition 3.2 needed in Theorem 1.31. We postpone both the definition and the general theorem until Section 3.

Finally, for the Stanley inequality and its equality conditions, we have:

$$\text{Thm 1.35} \;\Rightarrow\; \text{Thm 1.34} \qquad \text{and} \qquad \text{Thm 15.1} \;\Rightarrow\; \text{Thm 1.40} \;\Rightarrow\; \text{Thm 1.39}.$$

In both cases, more general results are new and correspond to the case of weighted linear extensions.

Let us emphasize that while some of these implications are trivial or follow immediately from definitions, others are more involved and require a critical change of notation and some effort to verify certain poset and weight function properties. These implications are discussed in Section 4.

1.20 Proof ideas

Although we prove multiple results, the proof of each log-concavity inequality uses the same approach and technology, so we refer to it as "the proof".

At the first level, the proof is an inductive argument proving a stronger claim about eigenvalues of certain matrices associated with the posets. The induction is not over posets of smaller size, but over other matrices which can in fact be larger, but correspond to certain parameters decreasing as we go along. The claim then reduces to the base of induction, which is the only part of the proof requiring a computation. The latter involves checking eigenvalues of explicitly written small matrices, making the proof fully elementary.

Delving a little deeper, we set up a new type of structure which we call a *combinatorial atlas*. In the special case of greedoids, a combinatorial atlas $\mathbb{A}$ associated with a greedoid $\mathcal{G} = (X, \mathcal{L})$, $|X| = n$, is comprised of:

- acyclic digraph $\Gamma_{\mathcal{G}} = (\mathcal{L}, \Theta)$, with the unique source at the empty word $\varnothing \in \mathcal{L}$, and edges corresponding to multiplications by a letter: $\Theta = \big\{(\alpha, \alpha x) \,:\, \alpha, \alpha x \in \mathcal{L},\, x \in X\big\}$,
- each vertex $\alpha \in \mathcal{L}$ is associated with a pair $(\mathbf{M}_\alpha, \mathbf{h}_\alpha)$, where $\mathbf{M}_\alpha = \big(\mathrm{M}_{ij}\big)$ is a nonnegative symmetric $d \times d$ matrix, $\mathbf{h}_\alpha = (\mathrm{h}_1, \ldots, \mathrm{h}_d)$ is a nonnegative vector, and $d = n+1$,
- each edge $(\alpha, \alpha x) \in \Theta$ is associated with a linear transformation $\mathbf{T}_\alpha^{\langle x \rangle} : \mathbb{R}^d \to \mathbb{R}^d$.

The key technical observation is that under certain conditions on the atlas, we have every matrix $\mathbf{M} := \mathbf{M}_\alpha$, $\alpha \in \mathcal{L}$, is *hyperbolic*:

$$\langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle^2 \;\geq\; \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle \quad \text{for every} \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^d, \;\; \text{such that} \;\; \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle > 0. \tag{Hyp}$$

Log-concavity inequalities now follow from (Hyp) for the matrix $\mathbf{M}_\varnothing$, by interpreting the inner products as numbers $\mathrm{L}_{\mathrm{q}}(k)$, $\mathrm{L}_{\mathrm{q}}(k-1)$ and $\mathrm{L}_{\mathrm{q}}(k+1)$, respectively.

We prove (Hyp) by induction, reducing the claim for $\mathbf{M}_\alpha$ to that of $\mathbf{M}_{\alpha x}$, for all $x \in \mathrm{Cont}(\alpha)$. Proving (Hyp) for the base of induction required the eigenvalue interlacing argument, cf. §17.5. This is where our conditions for the weight function ω appear in the calculation. We also need a few other properties of the atlas. Notably, we require every matrix $\mathbf{M}_\alpha$ to be *irreducible* with respect to its support, but that is proved by a direct combinatorial argument.

For other log-concavity inequalities in the paper, we consider similar atlas constructions and similar claims. For the equalities, we works backwards and observe that we need equations (Hyp) to be equalities. These imply the local properties which must hold for certain edges $(\alpha, \alpha x) \in \Theta$. Analyzing these properties gives the equality conditions we present.

1.21 Discussion

Skipping over the history of the subject (see Section 16), in recent years a great deal of progress on the subject was made by Huh and his coauthors. In fact, until the celebrated Adiprasito–Huh–Katz paper [AHK18], even the log-concavity for the number of k-forests (Welsh–Mason conjecture for graphical matroids), remained open. That paper was partially based on the earlier work [Huh12, Huh15, HK12], and paved a way to a number of further developments, most notably [ADH20, BES19, BST20, B+20a, B+20b, HSW22, HW17].

From the traditional order theory point of view, the level of algebra used in these works overwhelms the senses. The inherent rigidity of the original algebraic approach required either to extend the algebra as in the papers above, or to downshift in the technology. The *Lorentzian polynomials* approach developed by Brändén–Huh [BH18, BH20] and by Anari et. al [ALOV18] allowed stronger results such as Theorem 1.3 and led to further results and applications such as [ALOV19, BLP20, HSW22, MNY21]. This paper represented the first major downshift in the technology.

$(\circ)$ A casual reader can be forgiven in thinking of this paper as a successful deconstruction of the Lorentzian polynomials into the terminology of linear algebra. This is the opposite of what happens both mathematically and philosophically. Our approach does in fact contain much of the Lorentzian polynomials approach as a special case (cf. §17.9). This can be made precise, but we postpone that discussion until [CP22a].

However, viewing greedoids and its special cases as *languages* allows us to reach far beyond what the Lorentzian polynomials possibly can.[4] To put this precisely, our maps $\mathbf{T}_\alpha^{\langle x \rangle}$ have a complete flexibility in their definition. In the world of Lorentzian polynomials, the corresponding maps are trivial. We trade the elegance of that approach to more complexity, flexibility and strength.

$(\circ)$ The true origin of our "combinatorial atlas" technology lies in our deconstruction of the Stanley inequality (1.30). This is both one of oldest and the most mysterious results in the area, and our proof is elementary but highly technical, more so than our proof of greedoid results.

To understand the conundrum Stanley's inequality represents, consider the original proof in [Sta81] which is barely a page long via a simple reduction to the classical *Alexandrov–Fenchel inequality*. The latter is a fundamental result on the subject, with many different proofs across the fields, all of them

[4]Lest one think to use a straightforward generalization to noncommutative polynomials, try imagining the right notion of a *partial derivative* which plays a crucial role in [ALOV18, BH20].

difficult (see §16.20). This difficulty represented the main obstacle in obtaining an elementary proof of Stanley's inequality.

($\circ$) Most recently, the new proof of the Alexandrov–Fenchel inequality by Shenfeld and van Handel [SvH19] using "Bochner formulas", renewed our hopes for the elementary proof of Stanley's inequality. Their proof exploits the finiteness of the set of normals to polytope facets in a very different way from Alexandrov's original approach in [Ale38], see discussion in [SvH19, §6.1]. Our next point of inspiration was a most recent paper [SvH20] by Shenfeld and van Handel, where the authors obtain the equality conditions for Stanley's inequality (see Theorem 1.39) with applications to Stanley's inequality (cf. §17.11).

Deconstruction of [SvH19, SvH20] combined with ideas from [BH20, Sta81] and our earlier work [CPP22a, CPP21], led to our "combinatorial atlas" approach. Both the Stanley inequality and the conditions for equality followed from our linear algebra setting and became amenable to generalizations. Part of the reason for this is the explicit construction of maps $\mathbf{T}_\alpha^{\langle x\rangle}$, which for convex polytopes are shown in [SvH19] to exist only indirectly albeit in greater generality, see also §17.6.

($\circ$) Now, once we climbed the mountain of Stanley's inequality by means of the new technology, going down to poset antimatroids, polymatroids and matroids became easier. Our ultimate extension to interval greedoids required additional effort, as evidenced in the technical definitions in Section 3. Furthermore, our approach retained the flexibility of allowing us to match the results with equality conditions.

($\circ$) In conclusion, let us mention that the ultimate goal we set out in [Pak19], remains unresolved. There, we observed that the Adiprasito–Huh–Katz inequalities for graphs and Stanley inequalities for numbers of linear extensions correspond to nonnegative integer functions in GAPP $=$ #P $-$ #P. We asked whether these functions are themselves in #P. This amounts to finding a *combinatorial interpretation* for the difference of the LHS and the RHS of these inequalities. While we use only elementary tools, the eigenvalue based argument is not direct enough to imply a positive answer. See §17.17 for more on this problem.

1.22 Paper structure

We start with basic definitions and notions in Section 2. In the next Section 3 we present the main results of the paper on log-concave inequalities and the matching equality conditions for interval greedoids. We follow in Section 4 with a chain of combinatorial reductions explaining how our greedoids results imply poset antimatroid, polymatroid and matroid results.

In Section 5 we introduce the notion of *combinatorial atlas*, which is the main technical structure of this paper. We then show how to derive log-concave inequalities in this general setting. The key combinatorial properties of the atlases are given in Section 6. In the next Section 7, we show that under additional conditions on the atlas, we can characterize the equality conditions.

From this point on, much of the paper occupy proofs of the results:

- Thm 1.31 (interval greedoids inequality) is proved in Section 8,
- Thm 3.3 (interval greedoids equality conditions) is proved in Section 9,
- Thm 1.6, Thm 1.9, Thm 1.10 (matroid inequality and equality conditions) are proved in Section 10;

in addition, this section includes proof of Prop. 1.11, further results on log-concavity for graphs (§10.5), and examples of combinatorial atlases (§10.7),

- Thm 1.21, Thm 1.23 and Thm 1.24 (discrete polymatroid inequality and equality conditions) are proved in Section 11,
- Thm 1.26, Thm 1.28 and Thm 1.30 (poset antimatroid inequality and equality conditions) are proved in Section 12,
- Thm 1.16, Thm 1.18 and Thm 1.19 (morphism of matroids inequality and equality conditions) are proved in Section 13,
- Thm 1.35 (weighted Stanley's inequality) is proved in Section 14; in addition, this section includes

§14.8 on posets with belts and an example §14.7 of a combinatorial atlas in this case,

- Thm 1.40 (equality condition for weighted Stanley's inequality) is proved in Section 15.

These last two sections are the most technically involved parts of this paper. Note that although Sections 10–13 are somewhat independent, we do recommend the reader start with the matroid proofs in Section 10 because of the examples and as a starting point of generalizations, and antimatroid proofs in Section 12 because it has the shortest and cleanest reduction to the earlier greedoid results.

We conclude the paper with a lengthy historical Section 16 which cover to some degree various background behind results int he introduction. Since the material is so vast, we are somewhat biased towards most recent and general results. We present final remarks and open problems in Section 17.

2 Definitions and notations

2.1 Basic notation

We use $[n] = \{1,\ldots,n\}$, $\mathbb{N} = \{0,1,2,\ldots\}$, $\mathbb{Z}_+ = \{1,2,\ldots\}$, $\mathbb{R}_{\geq 0} = \{x \geq 0\}$ and $\mathbb{R}_{>0} = \{x > 0\}$. For a subset $S \subseteq X$ and element $x \in X$, we write $S+x := S \cup \{x\}$ and $S-x := S \smallsetminus \{x\}$.

2.2 Matrices and vectors

Throughout the paper we denote matrices with bold capitalized letter and the entries by roman capitalized letters: $\mathbf{M} = (\mathrm{M}_{ij})$. We also keep conventional index notations, so, e.g., $\big(\mathbf{M}^3 + \mathbf{M}^2\big)_{ij}$ is the (i,j)-th matrix entry of $\mathbf{M}^3 + \mathbf{M}^2$. We denote vectors by bold small letters, while vector entries by either unbolded uncapitalized letters or vector components, e.g. $\mathbf{h} = (\mathrm{h}_1, \mathrm{h}_2, \ldots)$ and $\mathrm{h}_i = (\mathbf{h})_i$.

A real matrix (resp., a real vector) is *nonnegative* if all its entries are nonnegative real numbers, and is *strictly positive* if all of its entries are positive real numbers. The *support* of a real $d \times d$ symmetric matrix $\mathbf{M}$ is defined as:

$$\operatorname{supp}(\mathbf{M}) \ := \ \big\{\, i \in [d] \, : \, \mathrm{M}_{ij} \neq 0 \ \text{ for some } j \in [d] \,\big\}.$$

In other words, supp(**M**) is the set of indexes for which the corresponding row and column of **M** are nonzero vectors. Similarly, the *support* of a real d-dimensional vector $\mathbf{h}$ is defined as:

$$\operatorname{supp}(\mathbf{h}) \ := \ \{ \, i \in [d] \, : \, \mathrm{h}_i \neq 0 \, \}.$$

For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$, we write $\mathbf{v} \leqslant \mathbf{w}$ to mean the componentwise inequality, i.e. $\mathrm{v}_i \leq \mathrm{w}_i$ for all $i \in [d]$. We write $|\mathbf{v}| := \mathrm{v}_1 + \ldots + \mathrm{v}_d$. We also use $\mathbf{e}_1, \ldots, \mathbf{e}_d$ to denote the standard basis of $\mathbb{R}^d$.

Finally, for a subset $S \subseteq [d]$, the *characteristic vector* of S is the vector $\mathbf{v} \in \mathbb{R}^d$ such that $\mathrm{v}_i = 1$ if $i \in S$ and $\mathrm{v}_i = 0$ if $i \notin S$. We use $\mathbf{0} \in \mathbb{R}^d$ to denote the zero vector.

2.3 Words

For a finite ground set X, we denote by X^* the set of all sequences $x_1 \cdots x_\ell$ $(\ell \geq 0)$ of elements $x_i \in X$ for $i \in [\ell]$. We call an element of X^* a *word* in the *alphabet* X. By a slight abuse of notation we use x_i to also denote the i-th letter in the word α. The *length* of a word $\alpha = x_1 \cdots x_\ell$ is the number of letters ℓ in the word, and is denoted by $|\alpha|$. The *concatenation* $\alpha\beta$ of two words α and β is the string α followed by the string β. In this case α is called a *prefix* of $\alpha\beta$. For every $\alpha = x_1 \cdots x_\ell \in X^*$, we write $z \in \alpha$ if $x_i = z$ for some $i \in [\ell]$.

2.4 Posets

A *poset* $\mathcal{P} = (X, \prec)$ is a pair of *ground set* X and a *partial order* "$\prec$" on X. For $x, y \in X$, we say that y *covers* x in $\mathcal{P}$, write $x \leftarrow\!\!\!\prec y$, if $x \prec y$, and there exists no $z \in X$ such that $x \prec z \prec y$. For $x, y \in X$, we write $x || y$ if x and y are *incomparable* in $\mathcal{P}$. Denote by $\operatorname{inc}(x) \subset X$ the subset of elements $y \in X$ incomparable with x.

A *lower ideal* of $\mathcal{P}$ is a subset $S \subseteq X$ such that, if $x \in S$ and $y \prec x$, then $y \in S$. Similarly, an *upper ideal* of $\mathcal{P}$ is a subset $S \subseteq X$ such that, if $x \in S$ and $y \succ x$, then $y \in X$. The *Hasse diagram* $\mathcal{H} := \mathcal{H}_{\mathcal{P}}$ of $\mathcal{P}$ is the acyclic digraph with X as the vertex set, and with (x, y) as an edge if $x \leftarrow\!\!\!\prec y$.

A *chain* of $\mathcal{P}$ is a subset of X that is totally ordered: $x_1 \prec x_2 \prec \ldots \prec x_\ell$. An antichain is a subset $S \subset X$, such that every two elements in S are incomparable. *Height* of a poset $\operatorname{height}(\mathcal{P})$ is the length of the maximal chain in $\mathcal{P}$. Similarly, *width* of a poset $\operatorname{width}(\mathcal{P})$ is the size of the maximal antichain in $\mathcal{P}$. Element $x \in X$ is called *minimal* if there is no $y \in X$, s.t. $y \prec x$. Define *maximal* elements similarly.

3 Combinatorics of interval greedoids

3.1 Preliminaries

Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid of rank $m := \operatorname{rk}(\mathcal{G})$. Recall the definitions of $\operatorname{Par}(\alpha)$ and $\operatorname{Des}_\alpha(x)$ given in §1.13 above, and note that "$\sim_\alpha$" remains an equivalence relation, see Proposition 4.3.

For all $\alpha \in \mathcal{L}$ and $x, y \in X$, define *passive* and *active non-continuations* as follows:

$$\operatorname{Pas}_\alpha(x,y) \ := \ \big\{ z \in X \ : \ \alpha z \notin \mathcal{L}, \ \alpha x z, \alpha y z \notin \mathcal{L}, \ \alpha x y z \in \mathcal{L} \big\},$$
$$\operatorname{Act}_\alpha(x,y) \ := \ \big\{ z \in X \ : \ \alpha z \notin \mathcal{L}, \ \alpha x z, \alpha y z \in \mathcal{L}, \ \alpha x y z \in \mathcal{L} \big\}.$$

Let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be a positive *weight function*, which we extend to $\mathrm{q} : X^* \to \mathbb{R}$ by setting $\mathrm{q}(\alpha) = 0$ for all $\alpha \notin \mathcal{L}$. Let $\boldsymbol{c} = (c_0, \ldots, c_m) \in \mathbb{R}^{m+1}_{>0}$, where $m = \mathrm{rk}(\mathcal{G})$, be a fixed positive sequence, which we call the *scale sequence*. Consider another weight function $\omega : X^* \to \mathbb{R}$:

$$\omega(\alpha) := \frac{\mathrm{q}(\alpha)}{c_\ell}, \quad \text{where } \ell = |\alpha| \text{ and } \alpha \in X^*, \tag{3.1}$$

which we call the *scaled weight function*.

3.2 Properties

Fix weight function $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ and scale sequence $\boldsymbol{c} \in \mathbb{R}^{m+1}_{>0}$. For every word $\alpha \in \mathcal{L}$ of length $\ell := |\alpha|$, consider the following properties.

1. ***Continuation invariance property:***

$$\mathrm{q}(\alpha x y \beta) \ = \ \mathrm{q}(\alpha y x \beta) \quad \text{for all } x, y \in \mathrm{Cont}(\alpha) \text{ and } \beta \in X^*. \tag{ContInv}$$

Note that by the exchange property, we have $\alpha x y \beta \in \mathcal{L}$ if and only if $\alpha y x \beta \in \mathcal{L}$.

2. ***Passive-active monotonicity property:***

$$\sum_{z \in \mathrm{Pas}_\alpha(x,y)} \sum_{\beta \in \mathrm{Cont}_k(\alpha x y z)} \mathrm{q}\big(\alpha x y z \beta\big) \ \geq \ \sum_{z \in \mathrm{Act}_\alpha(x,y)} \sum_{\beta \in \mathrm{Cont}_k(\alpha x y z)} \mathrm{q}\big(\alpha x y z \beta\big), \tag{PAMon}$$

for all distinct $x, y \in \mathrm{Cont}(\alpha)$, and $k \geq 0$. We also have a stronger property stated in terms of $\mathcal{L}$.

2′. ***Weak local property:***

$$x, y, z \in X \ \text{ s.t. } \ \alpha x z,\, \alpha y z,\, \alpha x y z \in \mathcal{L} \quad \Rightarrow \quad \alpha z \in \mathcal{L}. \tag{WeakLoc}$$

Observe that (WeakLoc) implies that $\mathrm{Act}_\alpha(x,y) = \varnothing$ for all distinct $x, y \in \mathrm{Cont}(\alpha)$, which in turn trivially implies (PAMon). Note also that (WeakLoc) is a property of a greedoid rather than the weight function. Greedoids that satisfy (WeakLoc) are called *weak local greedoids*.[5]

3. ***Log-modularity property:***

$$\omega(\alpha x)\ \omega(\alpha y) \ = \ \omega(\alpha)\ \omega(\alpha x y) \quad \text{for all } x, y \in \mathrm{Cont}(\alpha) \text{ s.t. } \alpha x y \in \mathcal{L}. \tag{LogMod}$$

4. ***Few descendants property:***

$$|\mathcal{C}| \geq 2 \quad \Rightarrow \quad \mathrm{Des}_\alpha(x) \ = \ \varnothing, \quad \text{for every } x \in \mathcal{C} \text{ and } \mathcal{C} \in \mathrm{Par}(\alpha). \tag{FewDes}$$

[5]This is a new class of greedoids which is similar but more general than the *local poset greedoids*. See Section 4 for the properties of weak local greedoids, relationships to other classes, and §17.14 for further background.

Note that (FewDes) is satisfied if $|\mathcal{C}| \leq 1$, or if $\mathrm{Des}_\alpha(x) = \varnothing$.

5. ***Syntactic monotonicity property:***

$$\omega(\alpha x)^2 \;\geq\; \sum_{y \in \mathrm{Des}_\alpha(x)} \omega(\alpha)\, \omega(\alpha x y)\,, \quad \text{for all } \ x \in \mathrm{Cont}(\alpha). \tag{SynMon}$$

For all $\mathcal{C} \in \mathrm{Par}(\alpha)$, define

$$\mathrm{b}_\alpha(\mathcal{C}) \;:=\; \begin{cases} \displaystyle\sum_{y \in \mathrm{Des}_\alpha(x)} \frac{\omega(\alpha)\, \omega(\alpha x y)}{\omega(\alpha x)^2} & \text{if } \ \mathcal{C} = \{x\}, \\ 0 & \text{if } \ |\mathcal{C}| \geq 2. \end{cases} \tag{3.2}$$

Note that properties (FewDes) and (SynMon) imply that $\mathrm{b}_\alpha(\mathcal{C}) \leq 1$ for all $\mathcal{C} \in \mathrm{Par}(\alpha)$. This sets up our final

6. ***Scale monotonicity property:***

$$\left(1 - \frac{c_{\ell+1}^2}{c_\ell\, c_{\ell+2}}\right) \sum_{\mathcal{C} \in \mathrm{Par}(\alpha)} \frac{1}{1 - \mathrm{b}_\alpha(\mathcal{C})} \;\leq\; 1\,, \quad \text{for all } \ \mathcal{C} \in \mathrm{Par}(\alpha). \tag{ScaleMon}$$

We adopt the convention that (ScaleMon) is always satisfied whenever $c_{\ell+1}^2 \geq c_\ell\, c_{\ell+2}$ (because then the LHS is considered nonpositive), and that $\mathrm{b}_\alpha(\mathcal{C}) < 1$ for all $\mathcal{C} \in \mathrm{Par}(\alpha)$ whenever $c_{\ell+1}^2 < c_\ell c_{\ell+2}$ (as otherwise the LHS is considered to be ∞) . In particular, note that (ScaleMon) is satisfied for the uniform scale sequence $\boldsymbol{c} = (1, \ldots, 1)$.

Remark 3.1. The last four properties (LogMod), (FewDes), (SynMon) and (ScaleMon) have a linear algebraic interpretation as certain matrix being *hyperbolic*. We postpone a discussion of this until the next section.

3.3 Admissible weight functions

We can now give the main definition used in the first main result of the paper (Theorem 1.31).

Definition 3.2 (*k-admissible weight functions*)**.** Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid of rank $m := \mathrm{rk}(\mathcal{G})$, and let $1 \leq k < m$. Weight function $\omega : \mathcal{L} \to \mathbb{R}_{>0}$ is called *k-admissible*, if there is a scale sequence $\boldsymbol{c} = (c_0, \ldots, c_m) \in \mathbb{R}_{>0}^{m+1}$, such that properties (ContInv), (PAMon), (LogMod), (FewDes), (SynMon) and (ScaleMon) are satisfied for all $\alpha \in \mathcal{L}$ of length $|\alpha| < k$.

We can also state our second main result of the paper, which gives the third equivalent condition in Theorem 1.33 that is both more detailed and useful in applications.

Theorem 3.3 (*Equality for interval greedoids*, second main theorem). *Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid of rank $m := \mathrm{rk}(\mathcal{G})$, let $1 \le k < m$, and let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be a k-admissible weight function with a scale sequence $\boldsymbol{c} = (c_0, \ldots, c_m) \in \mathbb{R}_{>0}^{m+1}$. Then, the following are equivalent:*

a. *We have:*

$$\mathrm{L}_{\mathrm{q}}(k)^2 \;=\; \mathrm{L}_{\mathrm{q}}(k-1) \cdot \mathrm{L}_{\mathrm{q}}(k+1). \tag{GE-a}$$

b. *There is $\mathrm{s}(k-1) > 0$, such that for every $\alpha \in \mathcal{L}_{k-1}$ we have:*

$$\mathrm{L}_{\mathrm{q},\alpha}(2) \;=\; \mathrm{s}(k-1)\,\mathrm{L}_{\mathrm{q},\alpha}(1) \;=\; \mathrm{s}(k-1)^2\,\mathrm{L}_{\mathrm{q},\alpha}(0). \tag{GE-b}$$

c. *There is $\mathrm{s}(k-1) > 0$, such that for every $\alpha \in \mathcal{L}_{k-1}$ we have:*

$$\sum_{x \in \mathrm{Cont}(\alpha)} \frac{\mathrm{q}(\alpha x)}{\mathrm{q}(\alpha)} \;=\; \mathrm{s}(k-1), \;\; \textit{and} \tag{GE-$c1$}$$

$$\big(1 - \mathrm{b}_\alpha(\mathcal{C})\big) \sum_{x \in \mathcal{C}} \frac{\mathrm{q}(\alpha x)}{\mathrm{q}(\alpha)} \;=\; \mathrm{s}(k-1)\left(1 - \frac{c_k^2}{c_{k-1}\,c_{k+1}}\right) \qquad \textit{for all} \quad \mathcal{C} \in \mathrm{Par}(\alpha), \tag{GE-$c2$}$$

where $\mathrm{b}_\alpha(\mathcal{C})$ is defined in (3.2).

Note that (GE-$c1$) and (GE-$c2$) imply that (ScaleMon) is always an equality for $\alpha \in \mathcal{L}_{k-1}$.

Remark 3.4. Note that the k-admissible property of weight functions q is quite constraining and there are interval greedoid for which there are no such q. Given the abundance of examples where such weight functions are natural, we do not investigate the structural properties they constrain (cf. §16.11).

4 Combinatorial preliminaries

In this section we present basic properties of matroids, polymatroids, poset antimatroids, local poset greedoids and interval greedoids. We include the relations between these classes which will be important in the proofs. Most of these are relatively straightforward, but stated in a different way and often dispersed across the literature. We include the short proofs for completeness and as a way to help the reader get more familiar with the notions. The reader well versed with greedoids can skip this section and come back whenever proofs call for the specific results.

4.1 Equivalence relations

Here we prove that equivalence relations given in the introduction are well defined. We include short proofs both for completeness.

Proposition 4.1. *Let $\mathcal{M} = (X, \mathcal{I})$ be a matroid, and let $S \in \mathcal{I}$ be an independent set. Then the relation "$\sim_S$" defined in §1.4 is an equivalence relation.*

Proof. Observe that $x \sim_S y$ if and only if x and y are *parallel* in the matroid $\mathcal{M}/S$ obtained from $\mathcal{M}$ by contracting over S. $\square$

Proposition 4.2. *Let $\mathcal{D} = ([n], \mathcal{J})$ be a discrete polymatroid, and let $\boldsymbol{a} \in \mathcal{J}$ be an independent multiset. Then the relation "$\sim_{\boldsymbol{a}}$" defined in §1.10 is an equivalence relation.*

Proof. It suffices to prove transitivity of "$\sim_{\boldsymbol{a}}$", as reflexivity and symmetry follow immediately from the definition. Let $i \sim_{\boldsymbol{a}}$ and $j \sim_{\boldsymbol{a}} k$. Suppose to the contrary that $i \not\sim_{\boldsymbol{a}} k$, so $\boldsymbol{a} + \mathbf{e}_i + \mathbf{e}_k \in \mathcal{J}$. On the other hand, $\boldsymbol{a} + \mathbf{e}_j \in \mathcal{J}$ since $j \in \operatorname{Cont}(\boldsymbol{a})$. It then follows from applying the exchange property to $\boldsymbol{a} + \mathbf{e}_j$ and $\boldsymbol{a} + \mathbf{e}_i + \mathbf{e}_k$, that either $\boldsymbol{a} + \mathbf{e}_j + \mathbf{e}_i \in \mathcal{J}$ or $\boldsymbol{a} + \mathbf{e}_j + \mathbf{e}_k \in \mathcal{J}$, both of which give us a contradiction. $\square$

Proposition 4.3. *Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid, and let $\alpha \in \mathcal{L}$ be a fixed word. Then the relation "$\sim_\alpha$" defined in §1.12 is an equivalence relation.*

Proof. Reflexivity follows immediately from the definition. For the symmetry, let $x \sim_\alpha y$ and suppose to the contrary that $y \not\sim_\alpha x$. This is equivalent to $\alpha y x \in \mathcal{L}$. On the other hand, $\alpha x \in \mathcal{L}$ since $x \in \operatorname{Cont}(\alpha)$. It then follows from applying the exchange property to αx and $\alpha y x$ that $\alpha x y \in \mathcal{L}$, which contradicts $x \sim_\alpha y$.

For transitivity, let $x \sim_\alpha y$ and $y \sim_\alpha z$. Suppose to the contrary, that $x \not\sim_\alpha z$, so $\alpha x z \in \mathcal{L}$. On the other hand, $\alpha y \in \mathcal{L}$ since $y \in \operatorname{Cont}(\alpha)$. It then follows from applying the exchange property to αy and $\alpha x z$, that either $\alpha y x \in \mathcal{L}$ or $\alpha y z \in \mathcal{L}$, both of which gives us a contradiction. $\square$

We conclude with another equivalence relation, which will prove important in §13.2. Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids, let f be the rank function for $\mathcal{M} = (X, \mathcal{I})$, and let g be the rank function for $\mathcal{N} = (Y, \mathcal{J})$. For an independent set $S \in \mathcal{I}$, let $H \subseteq X$ be given by

$$H \ := \ \big\{ x \in X \setminus S \ : \ g\big(\Phi(S+x)\big) = \operatorname{rk}(\mathcal{N}) - 1 \big\}. \tag{4.1}$$

Denote by "$\sim_H$" the equivalence relation on H, defined by

$$x \sim_H y \quad \Longleftrightarrow \quad g\big(\Phi(S+x+y)\big) \ = \ \operatorname{rk}(\mathcal{N}) - 1. \tag{4.2}$$

Proposition 4.4. *The relation "$\sim_H$" defined in (4.2) is an equivalence relation.*

Proof. Reflexivity and symmetry follows directly from definition, so it suffices to prove transitivity. Suppose that $x, y, z \in H$ are distinct elements, such that $x \sim_H y$ and $y \sim_H z$. Assume to the contrary, that $x \not\sim_H z$. This implies that $g\big(\Phi(S+x+z)\big) = \operatorname{rk}(\mathcal{N})$. Applying the exchange property for matroid $\mathcal{N}$ to $\Phi(S+y)$ and $\Phi(S+x+z)$, we have that either $g\big(\Phi(S+y+x)\big) = \operatorname{rk}(\mathcal{N})$ or $g\big(\Phi(S+y+z)\big) = \operatorname{rk}(\mathcal{N})$. This contradicts the assumption, and completes the proof. $\square$

4.2 Antimatroids $\subset$ interval greedoids

Note that (*nondegenerate property*) defining the language of a greedoid is vacuously true for poset antimatroids. Also note that two properties defining the language of a greedoid are identical to those defining antimatroids: (*normal property*) and (*hereditary property*). Similarly, the (*exchange property*) for antimatroids is more restrictive than the (*exchange property*) for greedoids.

It remains to show that the (*interval property*) holds for antimatroids. Let $\mathcal{A} = (X, \mathcal{L})$ be an antimatroid. Suppose $\alpha, \beta, \gamma \in X^*$ and $x \in X$, s.t. αx, $\alpha\beta\gamma x \in \mathcal{L}$. Write $\alpha' := \alpha x$ and $\beta' := \alpha\beta$. Then note that $x \in \alpha'$ and $x \notin \beta'$, as otherwise $w := \alpha\beta\gamma x \notin \mathcal{L}$ since w is not a simple word, and $\alpha', \beta' \in \mathcal{L}$. Also note that x is the only letter in α' that is not contained in β'. It then follows from the (*exchange property*) for $\mathcal{A}$, that $\alpha\beta x = \beta' x \in \mathcal{L}$, as desired. $\square$

Proposition 4.5. *Let $\mathcal{P} = (X, \prec)$ be a poset, and let $\mathcal{A} = (X, \mathcal{L})$ be the corresponding antimatroid. Then $\mathcal{A}$ satisfies the* (interval property)*,* (FewDes) *and* (WeakLoc).[6]

Proof. The (*interval property*) is proved above for all antimatroids. For (WeakLoc), let $x, y, z \in X$, s.t. αxz, αyz, $\alpha xyz \in \mathcal{L}$. Since $\alpha xz \in \mathcal{L}$ and $y \notin \alpha xz$, this implies z is incomparable to y in $\mathcal{P}$. Together with $\alpha yz \in \mathcal{L}$, this implies that $\alpha z \in \mathcal{L}$, as desired.

For (FewDes), note that $\mathcal{A}$ satisfies

$$\alpha x,\, \alpha y \in \mathcal{L}, \quad x, y \in X \quad \Longrightarrow \quad \alpha xy \in \mathcal{L}\,. \tag{4.3}$$

Indeed, this is because $\alpha y \in \mathcal{L}$ implies that every element in $\mathcal{P}$ that is less than y is contained in α, so they are also contained in αx. This in turn implies that $\alpha xy \in \mathcal{L}$. Now note that (4.3) implies that $|\mathcal{C}| = 1$ for every parallel class $\mathcal{C} \in \operatorname{Par}(\alpha)$ of $\alpha \in \mathcal{L}$, and thus (FewDes) is satisfied trivially. $\square$

4.3 Matroids $\subset$ greedoids

Given a matroid $\mathcal{M} = (X, \mathcal{I})$, we construct the corresponding greedoid $\mathcal{G} = (X, \mathcal{L})$, where $\mathcal{L}$ is defined as follows:

$$\alpha \;=\; x_1 \cdots x_\ell \in \mathcal{L} \quad \Longleftrightarrow \quad \alpha \;\text{ is simple and } \{x_1, \ldots, x_\ell\} \in \mathcal{I}\,.$$

Observe that (*nondegenerate property*) for $\mathcal{G}$ follows from matroid $\mathcal{M}$ being nonempty, (*normal property*) follows from definition, (*hereditary property*) for $\mathcal{G}$ follows from the (*hereditary property*) for $\mathcal{M}$, and the (*exchange property*) for $\mathcal{G}$ follows from (*exchange property*) for $\mathcal{M}$.

Proposition 4.6. *Given a matroid $\mathcal{M} = (X, \mathcal{I})$, the greedoid $\mathcal{G} = (X, \mathcal{L})$ constructed above satisfies the* (interval property)*,* (FewDes) *and* (WeakLoc).

Proof. Now note that, the greedoid $\mathcal{G}$ satisfies

$$\alpha xy \in \mathcal{L}, \quad x, y \in X \quad \Longrightarrow \quad \alpha y \in \mathcal{L}\,. \tag{4.4}$$

[6] Weak local property does not hold for all antimatroids, but holds for all poset antimatroids.

This follows from commutativity of $\mathcal{L}$ and the (*hereditary property*) of $\mathcal{M}$. The (*interval property*) for $\mathcal{G}$ follows immediately from (4.4).

Now, it follows from (4.4) that $\mathrm{Des}_\alpha(x) = \varnothing$ for every $\alpha \in \mathcal{L}$ and $x \in X$, and (FewDes) then follows trivially. Finally, let $x,y,z \in X$, s.t. αxz, αyz, $\alpha xyz \in \mathcal{L}$. Applying (4.4) to $\alpha xz \in \mathcal{L}$, it then follows that $\alpha z \in \mathcal{L}$. This proves (WeakLoc), and completes the proof. $\square$

4.4 Discrete polymatroids $\subset$ greedoid

Given a discrete polymatroid $\mathcal{D} = ([n], \mathcal{J})$, we construct the corresponding greedoid $\mathcal{G} = (X, \mathcal{L})$ as follows. Let $X := \{x_{i\,j} : 1 \le i, j \le n\}$ be the alphabet.[7]

For every word $\alpha \in X^*$, denote by $\boldsymbol{a}_\alpha = (\mathrm{a}_1, \ldots, \mathrm{a}_n) \in \mathbb{N}^n$ the vector counting the number of occurrences of $x_{i,*}$'s in α, i.e. $\mathrm{a}_i := \big|\{ j \in [n] : x_{i\,j} \in \alpha\}\big|$. The word $\alpha \in X^*$ is called *well-ordered* if for every letter $x_{i\,j}$ in α, letter $x_{i\,j-1}$ is also in α *before* $x_{i\,j}$.

Define $\mathcal{L}$ to be the set of simple well-ordered words $\alpha \in X^*$, such that $\boldsymbol{a}_\alpha \in \mathcal{J}$. Note that, each vector $\boldsymbol{a} \in \mathcal{J}$ corresponds to $\binom{|\boldsymbol{a}|}{\mathrm{a}_1, \ldots, \mathrm{a}_n}$ many feasible words $\alpha \in \mathcal{L}$ for which $\boldsymbol{a}_\alpha = \boldsymbol{a}$. Namely, these are all permutations of the word $x_{1\,1} \cdots x_{1\,\mathrm{a}_1} \cdots x_{n\,1} \cdots x_{n\,\mathrm{a}_n}$ preserving the relative order of letters $x_{i\,1}, \ldots, x_{i\,\mathrm{a}_i}$.

For the greedoid $\mathcal{G} = (X, \mathcal{L})$, the (*nondegenerate property*) and the (*normal property*) follow from definition. On the other hand, the (*hereditary property*) and the (*exchange property*) for $\mathcal{G}$ follows from the corresponding properties for $\mathcal{D}$. This completes the proof. $\square$

Proposition 4.7. *Given a discrete polymatroid $\mathcal{D} = ([n], \mathcal{J})$, the greedoid $\mathcal{G} = (X, \mathcal{L})$ constructed above satisfies the* (interval property)*,* (FewDes) *and* (WeakLoc).

Proof. First, let us show that (*interval property*) holds for $\mathcal{G}$. Let $\alpha, \beta, \gamma \in X^*$, and let $z = x_{i\,j} \in X$ s.t. $\alpha z, \alpha\beta\gamma z \in \mathcal{L}$. Since $\alpha\beta\gamma z \in \mathcal{L}$, this implies that $x_{i\,j+1}, \ldots, x_{i\,n} \notin \beta$. Since $\alpha z \in \mathcal{L}$, this implies that $\alpha\beta z$ is well-ordered. On the other hand, by applying the (*hereditary property*) of $\mathcal{D}$ to the word $\alpha\beta\gamma z$, it then follows that $\boldsymbol{a}_{\alpha\beta z} \in \mathcal{J}$. Hence, the word $\alpha\beta z \in \mathcal{L}$, which proves the (*interval property*).

Now, note that $\mathcal{G}$ satisfies

$$\mathrm{Des}_\alpha(x_{ij}) \subseteq \{x_{i\,j+1}\} \qquad \text{for every} \quad \alpha \in \mathcal{L} \quad \text{and} \quad x_{ij} \in \mathrm{Cont}(\alpha). \tag{4.5}$$

For (WeakLoc), let $x,y,z \in X$, s.t. αxz, αyz, $\alpha xyz \in \mathcal{L}$. Suppose to the contrary, that $\alpha z \notin \mathcal{L}$. Since $\alpha xz \in \mathcal{L}$ and $\alpha yz \in \mathcal{L}$, this implies that $z \in \mathrm{Des}_\alpha(x)$ and $z \in \mathrm{Des}_\alpha(y)$. On the other hand, this intersection is empty by (4.5). This gives a contradiction, and proves (WeakLoc).

For (FewDes), let $\boldsymbol{a} = \boldsymbol{a}_\alpha$ where $\alpha \in \mathcal{L}$, and let $x, y \in \mathrm{Cont}(\alpha)$ be distinct elements s.t. $x \sim_\alpha y$. Let $i, j \in [n]$ be such that $\boldsymbol{a}_{\alpha x} = \boldsymbol{a} + \mathbf{e}_i$ and $\boldsymbol{a}_{\alpha y} = \boldsymbol{a} + \mathbf{e}_j$. Note that $i \ne j$ and $\boldsymbol{a} + \mathbf{e}_i, \boldsymbol{a} + \mathbf{e}_j \in \mathcal{J}$. Suppose to the contrary, that (FewDes) is not satisfied, so we can assume that $\mathrm{Des}_\alpha(x) \ne \varnothing$. By (4.5), this implies that $\boldsymbol{a} + 2\mathbf{e}_i \in \mathcal{J}$. Now, by applying the polymatroid exchange property to $\boldsymbol{a} + \mathbf{e}_j$ and $\boldsymbol{a} + 2\mathbf{e}_i$, we then have $\boldsymbol{a} + \mathbf{e}_i + \mathbf{e}_j \in \mathcal{J}$. This contradicts the assumption that $x \sim_\alpha y$, and proves (FewDes). $\square$

[7] Unlike the rest of the paper, here $|X| = n^2$.

4.5 Exchange property for morphism of matroids

We will also need the following basic result.

Proposition 4.8. *Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids $\mathcal{M} = (X, \mathcal{I})$ and $\mathcal{N} = (Y, \mathcal{J})$. Let $S, T \subset X$, $|S| = |T|$ be two distinct bases of Φ. Then there exists $z \in S \setminus T$ and $w \in T \setminus S$ such that $S - z + w$ is also a basis of Φ.*

Proof. Fix an arbitrary $z \in S \setminus T$. We split the proof into two cases. First, suppose that $\Phi(S - z)$ contains a basis of $\mathcal{N}$. Applying the exchange property of $\mathcal{M}$ to the independent sets $S - z$ and T, there exists $w \in T \setminus S$ such that $S' := S - z + w$ is an independent set of $\mathcal{M}$. Note that $\Phi(S') \supset \Phi(S - z)$ contains a basis of $\mathcal{N}$ by assumption, so S' is a basis of Φ, as desired.

Second, suppose that $\Phi(S - z)$ does not contain a basis of $\mathcal{N}$. Applying the exchange property of $\mathcal{N}$ to $\Phi(S - z)$ and $\Phi(T)$, there exists $w \in T \setminus S$ such that $\Phi(S - z + w)$ contains a basis of $\mathcal{N}$. Since Φ is a morphism of matroid, we have

$$f(S - z + w) \, - \, f(S - z) \;\geq\; g\big(\Phi(S - z + w)\big) \, - \, g\big(\Phi(S - z)\big) \;=\; 1,$$

where f and g are rank functions in $\mathcal{M}$ and $\mathcal{N}$, respectively. This implies that $S - z + w$ is an independent set of $\mathcal{M}$, and therefore $S - z + w$ is a basis of the morphism Φ. This completes the proof. $\square$

5 Combinatorial atlases and hyperbolic matrices

In this section we introduce combinatorial atlases and present the local–global principle which allows one to recursively establish hyperbolicity of vertices. See §17.4 for some background.

5.1 Combinatorial atlas

Let $\mathcal{P} = (\Omega, \prec)$ be a locally finite poset of bounded height.[8] Denote by $\Gamma = (\Omega, \Theta) = \mathcal{H}_{\mathcal{P}}$ be the acyclic digraph given by the Hasse diagram of $\mathcal{P}$. Let $\Omega^0 \subseteq \Omega$ be the set of maximal elements in $\mathcal{P}$, so these are *sink vertices* in Γ. Similarly, denote by $\Omega^+ := \Omega \smallsetminus \Omega^0$ the *non-sink vertices*. We write v^* for the set of out-neighbor vertices $v' \in \Omega$, such that $(v, v') \in \Theta$.

Definition 5.1. A *combinatorial atlas* $\mathbb{A} = \mathbb{A}_{\mathcal{P}}$ of dimension d is an acyclic digraph $\Gamma := (\Omega, \Theta) = \mathcal{H}_{\mathcal{P}}$ with an additional structure:

- Each vertex $v \in \Omega$ is associated with a pair $(\mathbf{M}_v, \mathbf{h}_v)$, where $\mathbf{M}_v$ is a nonnegative symmetric $d \times d$ matrix, and $\mathbf{h}_v \in \mathbb{R}^d_{\geq 0}$ is a nonnegative vector.
- Every vertex $v \in \Omega^+$ has outdegree d, and the outgoing edges of each vertex $v \in \Omega^+$ are labeled with indices $i \in [d]$. We denote the edge labeled i as $e^{\langle i \rangle} = (v, v^{\langle i \rangle})$, where $1 \leq i \leq d$.
- Each edge $e^{\langle i \rangle}$ is associated to a linear transformation $\mathbf{T}^{\langle i \rangle}_v : \mathbb{R}^d \to \mathbb{R}^d$.

[8] In our examples, the poset $\mathcal{P}$ can be both finite and infinite.

Whenever clear, we drop the subscript v to avoid cluttering. We call $\mathbf{M} = (\mathrm{M}_{ij})_{i,j\in[d]}$ the *associated matrix* of v, and $\mathbf{h} = (\mathrm{h}_i)_{i\in[d]}$ the *associated vector* of v. In notation above, we have $v^{\langle i\rangle} \in v^*$, for all $1 \le i \le d$.

5.2 Local-global principle

As in the introduction (see §1.20), matrix $\mathbf{M}$ is called *hyperbolic*, if

$$\langle \mathbf{v}, \mathbf{M}\mathbf{w}\rangle^2 \ \geq\ \langle \mathbf{v}, \mathbf{M}\mathbf{v}\rangle\langle \mathbf{w}, \mathbf{M}\mathbf{w}\rangle \quad \text{for every} \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^d, \quad \text{such that} \quad \langle \mathbf{w}, \mathbf{M}\mathbf{w}\rangle > 0. \tag{Hyp}$$

For the atlas $\mathbb{A}$, we say that $v \in \Omega$ is *hyperbolic*, if the associated matrix $\mathbf{M}_v$ is hyperbolic, i.e. satisfies (Hyp). We say that atlas $\mathbb{A}$ satisfies *hyperbolic property* if every $v \in \Omega$ is hyperbolic.

Note that property (Hyp) depends only on the support of $\mathbf{M}$, i.e. it continues to hold after adding or removing zero rows or columns. This simple observation will be used repeatedly through the paper.

We say that atlas $\mathbb{A}$ satisfies *inheritance property* if for every non-sink vertex $v \in \Omega^+$, we have:

$$(\mathbf{M}\mathbf{v})_i \ = \ \left\langle \mathbf{T}^{\langle i\rangle}\,\mathbf{v},\, \mathbf{M}^{\langle i\rangle}\,\mathbf{T}^{\langle i\rangle}\,\mathbf{h}\right\rangle \quad \text{for every} \quad i \in \operatorname{supp}(\mathbf{M}) \quad \text{and} \quad \mathbf{v} \in \mathbb{R}^d, \tag{Inh}$$

where $\mathbf{T}^{\langle i\rangle} = \mathbf{T}_v^{\langle i\rangle}$, $\mathbf{h} = \mathbf{h}_v$ and $\mathbf{M}^{\langle i\rangle} := \mathbf{M}_{v^{\langle i\rangle}}$ is the matrix associated with $v^{\langle i\rangle}$.

Similarly, we say that atlas $\mathbb{A}$ satisfies the *pullback property* if for every non-sink vertex $v \in \Omega^+$, we have:

$$\sum_{i\in\operatorname{supp}(\mathbf{M})} \mathrm{h}_i \left\langle \mathbf{T}^{\langle i\rangle}\,\mathbf{v},\, \mathbf{M}^{\langle i\rangle}\,\mathbf{T}^{\langle i\rangle}\,\mathbf{v}\right\rangle \ \geq\ \langle \mathbf{v}, \mathbf{M}\mathbf{v}\rangle \qquad \text{for every} \ \mathbf{v} \in \mathbb{R}^d. \tag{Pull}$$

We say that a non-sink vertex $v \in \Omega^+$ is *regular* if the following positivity conditions are satisfied:

The associated matrix $\mathbf{M}_v$ restricted to its support is irreducible. (Irr)

The associated vector $\mathbf{h}_v$ restricted to the support of $\mathbf{M}_v$ is strictly positive. (h-Pos)

Note that a matrix is *irreducible* if if it is not similar via a permutation to a block upper triangular matrix that has more than one block of positive size.

We now present the first main result of this section, which is a local-global principle for (Hyp).

Theorem 5.2 (*local–global principle*). *Let $\mathbb{A}$ be a combinatorial atlas that satisfies properties* (Inh) *and* (Pull)*, and let $v \in \Omega^+$ be a non-sink regular vertex of Γ. Suppose every out-neighbor of v is hyperbolic. Then v is also hyperbolic.*

Theorem 5.2 reduces checking the property (Hyp) to sink vertices $v \in \Omega^0$. In our applications, the pullback property (Pull) is more complicated condition to check than the inheritance property (Inh). In the next Section 6, we present conditions implying (Pull) that are easier to check.

5.3 Eigenvalue interpretation of hyperbolicity

The following lemma that gives an equivalent condition to (Hyp) that is often easier to check. A symmetric matrix $\mathbf{M}$ satisfies (OPE) if

$$\mathbf{M} \text{ has at most } \textit{one positive eigenvalue} \text{ (counting multiplicity).} \tag{OPE}$$

The equivalence between (Hyp) and (OPE) is well-known in the literature, see e.g., [Gre81], [COSW04, Thm 5.3], [SvH19, Lem. 2.9] and [BH20, Lem. 2.5]. We present a short proof for completeness.

Lemma 5.3. *Let* $\mathbf{M}$ *be a self-adjoint operator on* $\mathbb{R}^d$ *for an inner product* $\langle\cdot,\cdot\rangle$. *Then* $\mathbf{M}$ *satisfies* (Hyp) *if and only if* $\mathbf{M}$ *satisfies* (OPE).

Proof. For the (Hyp) $\Rightarrow$ (OPE) direction, suppose to the contrary that $\mathbf{M}$ has eigenvalues $\lambda_1, \lambda_2 > 0$ (not necessarily distinct). Let $\mathbf{v}$ and $\mathbf{w}$ be orthonormal eigenvectors of $\mathbf{M}$ for λ_1 and λ_2, respectively. It then follows that

$$0 \ = \ \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle \quad \text{and} \quad \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \, \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle \ = \ \lambda_1 \lambda_2 \, ,$$

which contradicts (Hyp).

For the (OPE) $\Rightarrow$ (Hyp) direction, let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ be such that $\langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle > 0$. Let λ be the largest eigenvalue of $\mathbf{M}$, and let $\mathbf{h}$ be a corresponding eigenvector. Since $\langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle > 0$, this implies that λ is a positive eigenvalue. Since $\mathbf{M}$ has at most one positive eigenvalue (counting multiplicity), it follows that λ is the unique positive eigenvalue of $\mathbf{M}$, and is a simple eigenvalue. In particular, this implies that

$$\langle \mathbf{w}, \mathbf{M}\mathbf{h} \rangle \neq 0,$$

as otherwise, we would have $\langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle \leq 0$. Let $\mathbf{z} \in \mathbb{R}^d$ be the vector

$$\mathbf{z} \ := \ \mathbf{v} \ - \ \frac{\langle \mathbf{v}, \mathbf{M}\mathbf{h} \rangle}{\langle \mathbf{w}, \mathbf{M}\mathbf{h} \rangle} \, \mathbf{w} \, .$$

It follows that $\langle \mathbf{z}, \mathbf{M}\mathbf{h} \rangle = 0$. Since λ is the only positive eigenvalue of $\mathbf{M}$, we then have

$$\langle \mathbf{z}, \mathbf{M}\mathbf{z} \rangle \ \leq \ 0. \tag{5.1}$$

On the other hand, we have

$$\begin{aligned} \langle \mathbf{z}, \mathbf{M}\mathbf{z} \rangle \ &= \ \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \ - \ 2 \, \frac{\langle \mathbf{v}, \mathbf{M}\mathbf{h} \rangle \, \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{M}\mathbf{h} \rangle} \ + \ \frac{\langle \mathbf{v}, \mathbf{M}\mathbf{h} \rangle^2 \, \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{M}\mathbf{h} \rangle^2} \\ &\geq \ \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \ - \ \frac{\langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle^2}{\langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle} \, , \end{aligned}$$

where the last inequality is due to the AM–GM inequality. Combining this inequality with (5.1), we get

$$\langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle^2 \ \geq \ \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \, \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle \, ,$$

which proves (Hyp). $\square$

5.4 Proof of Theorem 5.2

Let $\mathbf{M} := \mathbf{M}_v$ and $\mathbf{h} := \mathbf{h}_v$ be the associated matrix and the associated vector of v, respectively. Since (Hyp) is a property that is invariant under restricting to the support of $\mathbf{M}$, it follows from (Irr) that we can assume that $\mathbf{M}$ is irreducible.

Let $\mathbf{D} := (\mathrm{D}_{ij})$ be the $d \times d$ diagonal matrix given by

$$\mathrm{D}_{ii} \ := \ \frac{(\mathbf{Mh})_i}{\mathrm{h}_i} \qquad \text{for every } \ 1 \leq i \leq d\,.$$

Note that $\mathbf{D}$ is well defined and $\mathrm{D}_{ii} > 0$, by (h-Pos) and the assumption that $\mathbf{M}$ is irreducible. Define a new inner product $\langle \cdot, \cdot \rangle_{\mathbf{D}}$ on $\mathbb{R}^d$ by $\langle \mathbf{v}, \mathbf{w} \rangle_{\mathbf{D}} \ := \ \langle \mathbf{v}, \mathbf{Dw} \rangle$.

Let $\mathbf{N} := \mathbf{D}^{-1}\mathbf{M}$. Note that $\langle \mathbf{v}, \mathbf{Nw} \rangle_{\mathbf{D}} = \langle \mathbf{v}, \mathbf{Mw} \rangle$ for every $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$. Since $\mathbf{M}$ is a symmetric matrix, this implies that $\mathbf{N}$ is a self-adjoint operator on $\mathbb{R}^d$ for the inner product $\langle \cdot, \cdot \rangle_{\mathbf{D}}$. A direct calculation shows that $\mathbf{h}$ is an eigenvector of $\mathbf{N}$ for eigenvalue $\lambda = 1$. Since $\mathbf{M}$ is irreducible matrix and $\mathbf{h}$ is a strictly positive vector, it then follows from the Perron–Frobenius theorem that $\lambda = 1$ is the largest real eigenvalue of $\mathbf{N}$, and that it has multiplicity one.

Claim: $\lambda = 1$ *is the only positive eigenvalue of* $\mathbf{N}$ (*counting multiplicity*).

By applying Lemma 5.3 to the matrix $\mathbf{N}$ and the inner product $\langle \cdot, \cdot \rangle_{\mathbf{D}}$, it then follows that

$$\langle \mathbf{v}, \mathbf{Nw} \rangle_{\mathbf{D}}^2 \ \geq \ \langle \mathbf{v}, \mathbf{Nv} \rangle_{\mathbf{D}} \, \langle \mathbf{w}, \mathbf{Nw} \rangle_{\mathbf{D}} \quad \text{for every } \ \mathbf{v}, \mathbf{w} \in \mathbb{R}^d\,.$$

Since $\langle \mathbf{v}, \mathbf{Nw} \rangle_{\mathbf{D}} = \langle \mathbf{v}, \mathbf{Mw} \rangle$, this implies (Hyp) for v, and completes the proof of the theorem. $\square$

Proof of the Claim. Let $i \in [d]$ and $\mathbf{v} \in \mathbb{R}^d$. It follows from (Inh) that

$$\big((\mathbf{Mv})_i\big)^2 \ = \ \big\langle \mathbf{T}^{\langle i \rangle}\mathbf{v}, \, \mathbf{M}^{\langle i \rangle}\, \mathbf{T}^{\langle i \rangle}\mathbf{h} \big\rangle^2. \tag{5.2}$$

Since $\mathbf{M}^{\langle i \rangle}$ satisfies (Hyp) by the assumption of the theorem, applying (Hyp) to the RHS of (5.2) gives:

$$\big((\mathbf{Mv})_i\big)^2 \ \geq \ \big\langle \mathbf{T}^{\langle i \rangle}\mathbf{v}, \, \mathbf{M}^{\langle i \rangle}\, \mathbf{T}^{\langle i \rangle}\mathbf{v} \big\rangle \big\langle \mathbf{T}^{\langle i \rangle}\mathbf{h}, \, \mathbf{M}^{\langle i \rangle}\, \mathbf{T}^{\langle i \rangle}\mathbf{h} \big\rangle, \tag{5.3}$$

Here (Hyp) can be applied since $\big\langle \mathbf{T}^{\langle i \rangle}\mathbf{h}, \, \mathbf{M}^{\langle i \rangle}\, \mathbf{T}^{\langle i \rangle}\mathbf{h} \big\rangle \ = \ (\mathbf{Mh})_i > 0$. Now note that

$$\begin{aligned} \big((\mathbf{Nv})_i\big)^2 \mathrm{D}_{ii} \ &= \ \big((\mathbf{Mv})_i\big)^2 \frac{\mathrm{h}_i}{(\mathbf{Mh})_i} \ =_{\text{(Inh)}} \ \big((\mathbf{Mv})_i\big)^2 \frac{\mathrm{h}_i}{\big\langle \mathbf{T}^{\langle i \rangle}\mathbf{h}, \, \mathbf{M}^{\langle i \rangle}\, \mathbf{T}^{\langle i \rangle}\mathbf{h} \big\rangle} \\ &\geq_{(5.3)} \ \mathrm{h}_i \big\langle \mathbf{T}^{\langle i \rangle}\mathbf{v}, \, \mathbf{M}^{\langle i \rangle}\, \mathbf{T}^{\langle i \rangle}\mathbf{v} \big\rangle. \end{aligned}$$

Summing this inequality over all $i \in [d]$, gives:

$$\langle \mathbf{Nv}, \mathbf{Nv} \rangle_{\mathbf{D}} \ \geq \ \sum_{i=1}^{r} \mathrm{h}_i \big\langle \mathbf{T}^{\langle i \rangle}\mathbf{v}, \, \mathbf{M}^{\langle i \rangle}\, \mathbf{T}^{\langle i \rangle}\mathbf{v} \big\rangle \ \geq_{\text{(Pull)}} \ \langle \mathbf{v}, \mathbf{Mv} \rangle \ = \ \langle \mathbf{v}, \mathbf{Nv} \rangle_{\mathbf{D}}\,. \tag{5.4}$$

Now, let λ be an arbitrary eigenvalue of $\mathbf{N}$, and let $\mathbf{g}$ be an eigenvector of λ. We have:

$$\lambda^2 \langle \mathbf{g}, \mathbf{g} \rangle_{\mathbf{D}} \ = \ \langle \mathbf{Ng}, \mathbf{Ng} \rangle_{\mathbf{D}} \ \geq_{(5.4)} \ \langle \mathbf{g}, \mathbf{Ng} \rangle_{\mathbf{D}} \ = \ \lambda \, \langle \mathbf{g}, \mathbf{g} \rangle_{\mathbf{D}}\,.$$

This implies that $\lambda \geq 1$ or $\lambda \leq 0$. Since $\lambda = 1$ is the largest eigenvalue of $\mathbf{N}$ and is simple, we obtain the result. $\square$

Remark 5.4. In the proof above, neither the Claim nor the proof of the Claim are new, but a minor revision of Theorem 5.2 in [SvH19]. We include the proof for completeness and to help the reader get through our somewhat cumbersome notation.

6 Pullback property

In this section we present sufficient conditions for (Pull) that are easier to verify, together with a construction of the maps $\mathbf{T}^{\langle i \rangle}$.

6.1 Three new properties

Let $\mathbb{A}$ be a combinatorial atlas. We say that $\mathbb{A}$ satisfies the *projective property*, if for every non-sink vertex $v \in \Omega^+$ and every $i \in \operatorname{supp}(\mathbf{M})$, we have:

$$\big(\mathbf{T}^{\langle i \rangle}\mathbf{v}\big)_j \ = \ \begin{cases} \mathbf{v}_j & \text{if} \quad j \in \operatorname{supp}\big(\mathbf{M}^{\langle i \rangle}\big) \cap \operatorname{supp}(\mathbf{M}), \\ \mathbf{v}_i & \text{if} \quad j \in \operatorname{supp}\big(\mathbf{M}^{\langle i \rangle}\big) \setminus \operatorname{supp}(\mathbf{M}). \end{cases} \tag{Proj}$$

We say that $\mathbb{A}$ satisfies the *transposition-invariant property*, if for every non-sink vertex $v \in \Omega^+$, we have:

$$\mathrm{M}^{\langle i \rangle}_{jk} \ = \ \mathrm{M}^{\langle j \rangle}_{ki} \ = \ \mathrm{M}^{\langle k \rangle}_{ij} \qquad \text{for every distinct } i,j,k \in \operatorname{supp}(\mathbf{M}). \tag{T-Inv}$$

Now, let $v \in \Omega^+$ be a non-sink vertex of Γ, and let $i \in \operatorname{supp}(\mathbf{M})$. We partition the support of matrix $\mathbf{M}^{\langle i \rangle}$ associated with vertex $v^{\langle i \rangle}$, into two parts:

$$\mathrm{Aunt}^{\langle i \rangle} \ := \ \operatorname{supp}\big(\mathbf{M}^{\langle i \rangle}\big) \cap \big(\operatorname{supp}(\mathbf{M}) - i\big), \qquad \mathrm{Fam}^{\langle i \rangle} \ := \ \operatorname{supp}\big(\mathbf{M}^{\langle i \rangle}\big) \setminus \big(\operatorname{supp}(\mathbf{M}) - i\big). \tag{6.1}$$

In other words, $\mathrm{Aunt}^{\langle i \rangle}$ consists of elements in the support of $\mathbf{M}$ that do not include i,[9] while $\mathrm{Fam}^{\langle i \rangle}$ consists of i together with elements that initially are not in the support of $\mathbf{M}$, but is then included in the support of $\mathbf{M}^{\langle i \rangle}$.[10] For every distinct $i,j \in \operatorname{supp}(\mathbf{M})$, let

$$\mathrm{K}_{ij} \ := \ \mathrm{h}_j \mathrm{M}^{\langle i \rangle}_{jj} \ - \ \mathrm{h}_j \sum_{k \in \mathrm{Fam}^{\langle j \rangle}} \mathrm{M}^{\langle j \rangle}_{ik}. \tag{6.2}$$

Let us emphasize that $\mathrm{Aunt}^{\langle i \rangle}$, $\mathrm{Fam}^{\langle i \rangle}$, and K_{ij} all depend on non-sink vertex v of Γ, even though v does not appear in these notation.

We say that $\mathbb{A}$ satisfies the K-*nonnegative property*, if for every non-sink vertex $v \in \Omega^+$,

$$\mathrm{K}_{ij} \ \geq \ 0 \quad \text{ for every distinct } \ i,j \ \in \ \operatorname{supp}(\mathbf{M}). \tag{K-Non}$$

The main result of this subsection is the following sufficient condition for (Pull).

Theorem 6.1. *Let $\mathbb{A}$ be a combinatorial atlas that satisfies* (Inh)*,* (Proj)*,* (T-Inv) *and* (K-Non)*. Then $\mathbb{A}$ also satisfies* (Pull)*.*

---

[9] The name "aunt" here is referring to the siblings of the parent.

[10] The name "family" here is referring to both the parents and their children.

6.2 Symmetry lemma

To prove Theorem 6.1, we need the following:

Lemma 6.2. *Let $\mathbb{A}$ be a combinatorial atlas that satisfies* (Inh)*,* (Proj)*, and* (T-Inv)*. Then, for every non-sink vertex $v \in \Omega^+$, we have:*

$$\mathrm{K}_{ij} \ = \ \mathrm{K}_{ji} \quad \textit{for every distinct} \ \ i,j \, \in \, \operatorname{supp}(\mathbf{M}).$$

Proof. Let $\mathbf{e}_1,\ldots,\mathbf{e}_d$ be the standard basis for $\mathbb{R}^d$. It follows from (Inh) that:

$$\begin{aligned}
\mathrm{M}_{ij} \ &= \ \big(\mathbf{M}\mathbf{e}_j\big)_i \ =_{\text{(Inh)}} \ \big\langle \mathbf{T}^{\langle i \rangle}\mathbf{e}_j, \mathbf{M}^{\langle i \rangle}\mathbf{T}^{\langle i \rangle}\mathbf{h} \big\rangle \ = \ \sum_{k=1}^{d} \mathrm{M}_{jk}^{\langle i \rangle}\big(\mathbf{T}^{\langle i \rangle}\mathbf{h}\big)_k \\
&= \ \sum_{k\in \mathrm{Fam}^{\langle i \rangle}} \mathrm{M}_{jk}^{\langle i \rangle}\big(\mathbf{T}^{\langle i \rangle}\mathbf{h}\big)_k \ + \ \sum_{k\in \mathrm{Aunt}^{\langle i \rangle}} \mathrm{M}_{jk}^{\langle i \rangle}\big(\mathbf{T}^{\langle i \rangle}\mathbf{h}\big)_k \\
&= \ \mathrm{M}_{jj}^{\langle i \rangle}\big(\mathbf{T}^{\langle i \rangle}\mathbf{h}\big)_j \ + \ \sum_{k\in \mathrm{Fam}^{\langle i \rangle}} \mathrm{M}_{jk}^{\langle i \rangle}\big(\mathbf{T}^{\langle i \rangle}\mathbf{h}\big)_k \ + \ \sum_{k\in \operatorname{supp}(\mathbf{M})\setminus\{i,j\}} \mathrm{M}_{jk}^{\langle i \rangle}\big(\mathbf{T}^{\langle i \rangle}\mathbf{h}\big)_k \,.
\end{aligned}$$

Applying (Proj) to the equation above, we get:

$$\mathrm{M}_{ij} \ = \ \mathrm{M}_{jj}^{\langle i \rangle}\,\mathrm{h}_j \ + \sum_{k\in \mathrm{Fam}^{\langle i \rangle}} \mathrm{M}_{jk}^{\langle i \rangle}\,\mathrm{h}_i \ + \sum_{k\in \operatorname{supp}(\mathbf{M})\setminus\{i,j\}} \mathrm{M}_{jk}^{\langle i \rangle}\,\mathrm{h}_k \,. \tag{6.3}$$

By the same reasoning, we also get:

$$\mathrm{M}_{ji} \ = \ \mathrm{M}_{ii}^{\langle j \rangle}\,\mathrm{h}_i \ + \sum_{k\in \mathrm{Fam}^{\langle j \rangle}} \mathrm{M}_{ik}^{\langle j \rangle}\,\mathrm{h}_j \ + \sum_{k\in \operatorname{supp}(\mathbf{M})\setminus\{i,j\}} \mathrm{M}_{ik}^{\langle j \rangle}\,\mathrm{h}_k \,. \tag{6.4}$$

By (T-Inv), the rightmost sums in (6.3) and (6.4) are equal. On the other hand, the left side of (6.3) and (6.4) are equal since $\mathbf{M}$ is a symmetric matrix. Equating (6.3) and (6.4), we obtain:

$$\mathrm{M}_{jj}^{\langle i \rangle}\,\mathrm{h}_j \ + \sum_{k\in \mathrm{Fam}^{\langle i \rangle}} \mathrm{M}_{jk}^{\langle i \rangle}\,\mathrm{h}_i \ = \ \mathrm{M}_{ii}^{\langle j \rangle}\,\mathrm{h}_i \ + \sum_{k\in \mathrm{Fam}^{\langle j \rangle}} \mathrm{M}_{ik}^{\langle j \rangle}\,\mathrm{h}_j \,,$$

which is equivalent to

$$\mathrm{M}_{jj}^{\langle i \rangle}\,\mathrm{h}_j \ - \sum_{k\in \mathrm{Fam}^{\langle j \rangle}} \mathrm{M}_{ik}^{\langle j \rangle}\,\mathrm{h}_j \ = \ \mathrm{M}_{ii}^{\langle j \rangle}\,\mathrm{h}_i \ - \sum_{k\in \mathrm{Fam}^{\langle i \rangle}} \mathrm{M}_{jk}^{\langle i \rangle}\,\mathrm{h}_i \,.$$

The lemma now follows by noting that the LHS of the equation above is equal to K_{ij}, while the RHS is equal to K_{ji}. ◻

6.3 Proof of Theorem 6.1

Let v be a non-sink vertex of Γ, and let $\mathbf{v} \in \mathbb{R}^d$. The left side of (Pull) is equal to

$$\sum_{i \in \operatorname{supp}(\mathbf{M})} \mathrm{h}_i \left\langle \mathbf{T}^{\langle i \rangle}\mathbf{v}, \mathbf{M}^{\langle i \rangle}\mathbf{T}^{\langle i \rangle}\mathbf{v} \right\rangle \ = \sum_{i \in \operatorname{supp}(\mathbf{M})} \ \sum_{j,k \in \operatorname{supp}(\mathbf{M}^{\langle i \rangle})} \mathrm{h}_i \left(\mathbf{T}^{\langle i \rangle}\mathbf{v}\right)_j \left(\mathbf{T}^{\langle i \rangle}\mathbf{v}\right)_k \mathrm{M}^{\langle i \rangle}_{jk} \,. \tag{6.5}$$

<u>First</u>, this sum can be partitioned into the sum over the following five families:

(1) The triples (i,j,k), where $i \in \operatorname{supp}(\mathbf{M})$, and $j,k \in \mathrm{Aunt}^{\langle i \rangle}$ are distinct. By (Proj), the term in (6.5) is equal to
$$\mathrm{h}_i \, \mathrm{v}_j \, \mathrm{v}_k \, \mathrm{M}^{\langle i \rangle}_{jk} \,.$$

(2) The triples (i,j,k), where $i \in \operatorname{supp}(\mathbf{M})$, and $j,k \in \mathrm{Fam}^{\langle i \rangle}$ (not necessarily distinct). By (Proj), the term in (6.5) is equal to
$$\mathrm{h}_i \, \mathrm{v}_i^2 \, \mathrm{M}^{\langle i \rangle}_{jk} \,.$$

(3) The triples (i,j,k), where $i \in \operatorname{supp}(\mathbf{M})$, $j \in \mathrm{Aunt}^{\langle i \rangle}$, and $k \in \mathrm{Fam}^{\langle i \rangle}$. By (Proj), the term in (6.5) is equal to
$$\mathrm{h}_i \, \mathrm{v}_i \, \mathrm{v}_j \, \mathrm{M}^{\langle i \rangle}_{jk} \,.$$

(4) The triples (i,j,k), where $i \in \operatorname{supp}(\mathbf{M})$, $j \in \mathrm{Fam}^{\langle i \rangle}$, and $k \in \mathrm{Aunt}^{\langle i \rangle}$. By (Proj), the term in (6.5) is equal to
$$\mathrm{h}_i \, \mathrm{v}_i \, \mathrm{v}_k \, \mathrm{M}^{\langle i \rangle}_{jk} \,.$$

(5) The triples (i,j,k), where $i \in \operatorname{supp}(\mathbf{M})$, and $j = k \in \mathrm{Aunt}^{\langle i \rangle}$. By (Proj), the term in (6.5) is equal to
$$\mathrm{h}_i \, \mathrm{v}_j^2 \, \mathrm{M}^{\langle i \rangle}_{jj} \ = \ \frac{\mathrm{h}_i}{\mathrm{h}_j} \, \mathrm{v}_j^2 \, \mathrm{K}_{ij} \ + \sum_{k \in \mathrm{Fam}^{\langle j \rangle}} \mathrm{h}_i \, \mathrm{v}_j^2 \, \mathrm{M}^{\langle j \rangle}_{ik} \,.$$

Thus the sum over this family can be partitioned further into the sum over the following two families:

(5a) The pair (i,j), where $i,j \in \operatorname{supp}(\mathbf{M})$ are distinct, with the term
$$\frac{\mathrm{h}_i}{\mathrm{h}_j} \, \mathrm{v}_j^2 \, \mathrm{K}_{ij} \,.$$

(5b) The triples (i,j,k), where $i,j \in \operatorname{supp}(\mathbf{M})$ are distinct, and $k \in \mathrm{Fam}^{\langle j \rangle}$, with the term
$$\mathrm{h}_i \, \mathrm{v}_j^2 \, \mathrm{M}^{\langle j \rangle}_{ik} \,.$$

Second, the right side of (Pull) is equal to

$$\begin{aligned}\langle \mathbf{v}, \mathbf{M}\mathbf{v}\rangle \ &= \sum_{i' \in \operatorname{supp}(\mathbf{M})} \mathrm{v}_{i'} \, (\mathbf{M}\mathbf{v})_{i'} \ =_{(\text{Inh})} \sum_{i' \in \operatorname{supp}(\mathbf{M})} \mathrm{v}_{i'} \left\langle \mathbf{T}^{\langle i'\rangle}\mathbf{v}, \, \mathbf{M}^{\langle i'\rangle}\, \mathbf{T}^{\langle i'\rangle}\mathbf{h}\right\rangle \\ &= \sum_{i' \in \operatorname{supp}(\mathbf{M})} \ \sum_{j',k' \in \operatorname{supp}(\mathbf{M}^{\langle i'\rangle})} \mathrm{v}_{i'} \left(\mathbf{T}^{\langle i'\rangle}\mathbf{v}\right)_{j'} \left(\mathbf{T}^{\langle i'\rangle}\mathbf{h}\right)_{k'} \mathrm{M}^{\langle i'\rangle}_{j'k'}. \end{aligned} \tag{6.6}$$

This sum can be partitioned into the sum over the following five families:

(1′) The triples (i', j', k'), where $i' \in \operatorname{supp}(\mathbf{M})$, and $j', k' \in \mathrm{Aunt}^{\langle i'\rangle}$ are distinct. By (Proj), the term in (6.6) is equal to

$$\mathrm{h}_{k'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle i'\rangle}_{j'k'}\,.$$

(2′) The triples (i', j', k'), where $i' \in \operatorname{supp}(\mathbf{M})$, and $j', k' \in \mathrm{Fam}^{\langle i'\rangle}$ (not necessarily distinct). By (Proj), the term in (6.6) is equal to

$$\mathrm{h}_{i'}\, \mathrm{v}_{i'}^2\, \mathrm{M}^{\langle i'\rangle}_{j'k'}\,.$$

(3′) The triples (i', j', k'), where $i' \in \operatorname{supp}(\mathbf{M})$, $j' \in \mathrm{Aunt}^{\langle i'\rangle}$, and $k' \in \mathrm{Fam}^{\langle i'\rangle}$. By (Proj), the term in (6.6) is equal to

$$\mathrm{h}_{i'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle i'\rangle}_{j'k'}\,.$$

(4′) The triples (i', j', k'), where $i' \in \operatorname{supp}(\mathbf{M})$, $j' \in \mathrm{Fam}^{\langle i'\rangle}$, and $k' \in \mathrm{Aunt}^{\langle i'\rangle}$. By (Proj), the term in (6.6) is equal to

$$\mathrm{h}_{k'}\, \mathrm{v}_{i'}^2\, \mathrm{M}^{\langle i'\rangle}_{j'k'}\,.$$

(5′) The triples (i', j', k'), where $i' \in \operatorname{supp}(\mathbf{M})$, and $j' = k' \in \mathrm{Aunt}^{\langle i'\rangle}$. By (Proj), the term in (6.6) is equal to

$$\mathrm{h}_{j'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle i'\rangle}_{j'j'} \ = \ \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{K}_{i'j'} \ + \sum_{k' \in \mathrm{Fam}^{\langle j'\rangle}} \mathrm{h}_{j'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle j'\rangle}_{i'k'}\,.$$

Thus the sum over this family can be partitioned further into the sum over the following two families:

(5a′) The pair (i', j'), where $i', j' \in \operatorname{supp}(\mathbf{M})$ are distinct, with the term

$$\mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{K}_{i'j'}\,.$$

(5b′) The triples (i', j', k'), where $i', j' \in \operatorname{supp}(\mathbf{M})$ are distinct, and $k' \in \mathrm{Fam}^{\langle j'\rangle}$, with the term

$$\mathrm{h}_{j'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}^{\langle j'\rangle}_{i'k'}\,.$$

Third, we show that the RHS of (6.5) is at least as large as the RHS of (6.6). We have the following six cases:

(i) The term in (1) is equal to that of (1′) by substituting $i' \leftarrow j,\ j' \leftarrow k,\ k' \leftarrow i$ (counterclockwise substitution) to (1):

$$\mathrm{h}_i\, \mathrm{v}_j\, \mathrm{v}_k\, \mathrm{M}_{jk}^{\langle i\rangle} \ =_{\text{(T-Inv)}}\ \mathrm{h}_i\, \mathrm{v}_j\, \mathrm{v}_k\, \mathrm{M}_{ki}^{\langle j\rangle} \ = \ \mathrm{h}_{k'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}_{j'k'}^{\langle i'\rangle}\,.$$

(ii) The term in (2) is equal to that of (2′) by substituting $i' \leftarrow i,\ j' \leftarrow j,\ k' \leftarrow k$ (identity substitution) to (2):

$$\mathrm{h}_i\, \mathrm{v}_i^2\, \mathrm{M}_{jk}^{\langle i\rangle} \ = \ \mathrm{h}_{i'}\, \mathrm{v}_{i'}^2\, \mathrm{M}_{j'k'}^{\langle i'\rangle}\,.$$

(iii) The term in (3) is equal to that of (3′) by substituting $i' \leftarrow i,\ j' \leftarrow j,\ k' \leftarrow k$ (identity substitution) to (3):

$$\mathrm{h}_i\, \mathrm{v}_i\, \mathrm{v}_j\, \mathrm{M}_{jk}^{\langle i\rangle} \ = \ \mathrm{h}_{i'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}_{j'k'}^{\langle i'\rangle}\,.$$

(iv) The term in (4) is equal to that of (5b′) by substituting $i' \leftarrow k,\ j' \leftarrow i,\ k' \leftarrow j$ (clockwise substitution) to (4):

$$\mathrm{h}_i\, \mathrm{v}_i\, \mathrm{v}_k\, \mathrm{M}_{jk}^{\langle i\rangle} \ = \ \mathrm{h}_{j'}\, \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{M}_{i'k'}^{\langle j'\rangle}\,.$$

(v) The term in (5a) is equal to that of (5a′) by substituting $i' \leftarrow i,\ j' \leftarrow j$ (identity substitution) to (5a):

$$\begin{aligned}\frac{\mathrm{h}_i}{\mathrm{h}_j}\, \mathrm{v}_j^2\, \mathrm{K}_{ij} \ + \ \frac{\mathrm{h}_j}{\mathrm{h}_i}\, \mathrm{v}_i^2\, \mathrm{K}_{ji} \ &\geq \ 2\, \mathrm{v}_i\, \mathrm{v}_j \sqrt{\mathrm{K}_{ij}\, \mathrm{K}_{ji}} \ =_{\text{Lem 6.2}}\ \mathrm{v}_i\, \mathrm{v}_j\, \mathrm{K}_{ij} \ + \ \mathrm{v}_j\, \mathrm{v}_i\, \mathrm{K}_{ji} \\ &= \ \mathrm{v}_{i'}\, \mathrm{v}_{j'}\, \mathrm{K}_{i'j'} \ + \ \mathrm{v}_{j'}\, \mathrm{v}_{i'}\, \mathrm{K}_{j'i'}\,,\end{aligned}$$

where the first inequality follows from (K-Non) and the AM-GM inequality.[11]

(vi) The term in (5b) is equal to that of (4′) by substituting $i' \leftarrow j,\ j' \leftarrow k,\ k' \leftarrow i$ (clockwise substitution) to (5b):

$$\mathrm{h}_i\, \mathrm{v}_j^2\, \mathrm{M}_{ik}^{\langle j\rangle} \ = \ \mathrm{h}_{k'}\, \mathrm{v}_{i'}^2\, \mathrm{M}_{j'k'}^{\langle i'\rangle}\,.$$

This completes the proof of the theorem. □

Remark 6.3. The condition (K-Non) in Theorem 6.1 can be weakened as follows. Let $v \in \Omega^+$ be a non-sink vertex, and let $\mathbf{K} := (\mathrm{K}_{ij})_{i,j \in \operatorname{supp}(\mathbf{M})}$ be the matrix defined by

$$\mathrm{K}_{ij} \ := \ \begin{cases} \mathrm{K}_{ij} \text{ as in (6.2)} & \text{if } \ i,j \in \operatorname{supp}(\mathbf{M}), \ \ i \neq j, \\ -\displaystyle\sum_{\ell \in \operatorname{supp}(\mathbf{M}) \setminus \{i\}} \frac{\mathrm{h}_i}{\mathrm{h}_\ell}\, \mathrm{K}_{i,\ell} & \text{if } \ i = j \in \operatorname{supp}(\mathbf{M}). \end{cases}$$

We claim that the condition (K-Non) in Theorem 6.1 can be replaced with

$$\text{The matrix } \ -\mathbf{K} \ \text{ is positive semidefinite,} \tag{K-PSD}$$

for every non-sink vertex v of Γ. This generalization follows from the same proof as Theorem 6.1 by a straightforward modification to step (v). Note that in this paper we never apply this (slightly more general) version of Theorem 6.1, as all interesting applications that we found satisfy the stronger condition (K-Non), which is also easier to check.

[11] Note that this is only instance of inequality in this proof.

7 Hyperbolic equality for combinatorial atlases

In this section we characterize when the equality conditions in (Hyp) hold for all non-sink vertices in a combinatorial atlas. For that, we obtain the equality variation of the local-global principle (Theorem 5.2). See §17.4 for some background.

7.1 Statement

Let $\mathbb{A}$ be a combinatorial atlas of dimension d. Recall that, for a non-sink vertex v of Γ, we denote by $\mathbf{M} = \mathbf{M}_v$ the associated matrix of v, by $\mathbf{h} = \mathbf{h}_v$ the associated vector of v, by $\mathbf{T}^{\langle i \rangle} = \mathbf{T}_v^{\langle i \rangle}$ the associated linear transformation of the edge $e^{\langle i \rangle} = (v, v^{\langle i \rangle})$, and by $\mathbf{M}^{\langle i \rangle}$ the associated matrix of the vertex $v^{\langle i \rangle}$.

A *global pair* $\mathbf{f}, \mathbf{g} \in \mathbb{R}^d$ is a pair of nonnegative vectors, such that

$$\mathbf{f} + \mathbf{g} \ \text{ is a strictly positive vector}. \tag{Glob-Pos}$$

Here $\mathbf{f}$ and $\mathbf{g}$ are global in a sense that they are the same for all vertices $v \in \Omega$.

Fix a number $\mathrm{s} > 0$. We say that a vertex $v \in \Omega$ satisfies (s-Equ), if

$$\langle \mathbf{f}, \mathbf{M}\mathbf{f} \rangle \ = \ \mathrm{s} \, \langle \mathbf{g}, \mathbf{M}\mathbf{f} \rangle \ = \ \mathrm{s}^2 \, \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle, \tag{s-Equ}$$

where $\mathbf{M} = \mathbf{M}_v$ as above. Observe that (s-Equ) implies that equality occurs in (Hyp) for substitutions $\mathbf{v} \leftarrow \mathbf{g}$ and $\mathbf{w} \leftarrow \mathbf{f}$, since

$$\langle \mathbf{g}, \mathbf{M}\mathbf{f} \rangle^2 \ = \ \mathrm{s} \ \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle \ \mathrm{s}^{-1} \ \langle \mathbf{f}, \mathbf{M}\mathbf{f} \rangle \ = \ \langle \mathbf{g}, \mathbf{M}\mathbf{g} \rangle \ \langle \mathbf{f}, \mathbf{M}\mathbf{f} \rangle \, . \tag{7.1}$$

We say that the atlas $\mathbb{A}$ satisfies s-*equality property* if (s-Equ) holds for every $v \in \Omega$.

We now present the first main result of this section, which is a *local-global principle* for (s-Equ). A vertex $v \in \Omega^+$ is called *functional source* if the following conditions are satisfied:

$$\mathrm{f}_j \ = \ \big(\mathbf{T}^{\langle i \rangle} \, \mathbf{f}\big)_j \quad \text{and} \quad \mathrm{g}_j \ = \ \big(\mathbf{T}^{\langle i \rangle} \, \mathbf{g}\big)_j \quad \text{for every } \ i \in \operatorname{supp}(\mathbf{M}), \ j \in \operatorname{supp}(\mathbf{M}^{\langle i \rangle}), \tag{Glob-Proj}$$

$$\mathbf{f} \ = \ \mathbf{h}_v \, . \tag{h-Glob}$$

Here condition (Glob-Proj) means that $\mathbf{f}, \mathbf{g}$ are fixed points of the projection $\mathbf{T}^{\langle i \rangle}$ when restricted to the support.

We say that an edge $e^{\langle i \rangle} = (v, v^{\langle i \rangle}) \in \Theta$ is *functional* if v is a functional source and $i \in \operatorname{supp}(\mathbf{M}) \cap \operatorname{supp}(\mathbf{h})$. A vertex $w \in \Omega$ is a *functional target* of v, if there exists a directed path $v \to w$ in Γ consisting of only functional edges. Note that a functional target is not necessarily a functional source.

Theorem 7.1 (*local-global equality principle*)**.** *Let $\mathbb{A}$ be a combinatorial atlas that satisfies properties* (Inh), (Pull)*. Suppose also $\mathbb{A}$ satisfies property* (Hyp) *for every vertex $v \in \Omega$. Let $\mathbf{f}, \mathbf{g}$ be a global pair of $\mathbb{A}$. Suppose a non-sink vertex $v \in \Omega^+$ satisfies* (s-Equ) *with constant $\mathrm{s} > 0$. Then every functional target of v also satisfies* (s-Equ) *with the same constant* s.

7.2 Algebraic lemma

We start with the following general algebraic result. Recall that a matrix is *hyperbolic* if it satisfies (Hyp).

Lemma 7.2. *Let* $\mathbf{M}$ *be a nonnegative symmetric hyperbolic* $\mathrm{r}\times\mathrm{r}$ *matrix. Let* $\mathbf{f},\mathbf{g}\in\mathbb{R}^{\mathrm{r}}$ *be nonnegative vectors, let* $\mathrm{s}>0$*, and let* $\mathbf{z}:=\mathbf{f}-\mathrm{s}\,\mathbf{g}$*. Then* (s-Equ) *holds* <u>*if and only if*</u> $\mathbf{Mz}=0$.

Proof. The $\Leftarrow$ direction follows from the fact that

$$\langle \mathbf{f},\mathbf{Mf}\rangle \,-\, \mathrm{s}\,\langle \mathbf{g},\mathbf{Mf}\rangle \,=\, \langle \mathbf{z},\mathbf{Mf}\rangle \,=\, \langle \mathbf{Mz},\mathbf{f}\rangle, \quad \text{and} \quad \mathrm{s}\,\langle \mathbf{g},\mathbf{Mf}\rangle \,-\, \mathrm{s}^2\,\langle \mathbf{g},\mathbf{Mg}\rangle \,=\, \mathrm{s}\,\langle \mathbf{g},\mathbf{Mz}\rangle. \tag{7.2}$$

Thus it suffices to prove the $\Rightarrow$ direction. We will assume that $\mathbf{M}$ is nonzero when restricted to the support of $\mathbf{g}+\mathbf{f}$, as otherwise every term in (s-Equ) is equal to 0 and the lemma follows immediately. Let $\mathbf{w}:=\mathbf{g}+\mathbf{f}$, and the assumption implies that $\langle \mathbf{w},\mathbf{Mw}\rangle>0$. By (Hyp), we then have that the matrix $\mathbf{M}$ is negative semidefinite on $(\mathbf{Mw})^{\perp}$. Now note that $\mathbf{z}\in(\mathbf{Mw})^{\perp}$, since $\langle \mathbf{z},\mathbf{Mw}\rangle=0$ by (7.2) and (s-Equ). Also note that

$$\langle \mathbf{z},\mathbf{Mz}\rangle \,=\, \langle \mathbf{f},\mathbf{Mf}\rangle \,-\, 2\,\mathrm{s}\,\langle \mathbf{g},\mathbf{Mf}\rangle \,+\, \mathrm{s}^2\,\langle \mathbf{g},\mathbf{Mg}\rangle \,=_{\text{(s-Equ)}}\, 0. \tag{7.3}$$

It then follows from these three observations that $\mathbf{Mz}=0$, as desired. $\square$

7.3 Proof of Theorem 7.1

By induction, it suffices to show that, for every functional edge $(v,v^{\langle i\rangle})\in v^*$, we have that $v^{\langle i\rangle}$ satisfies (s-Equ) with the same constant $\mathrm{s}>0$.

It follows from (Inh), that for every $i\in\operatorname{supp}(\mathbf{M})$ we have:

$$\big(\mathbf{Mg}\big)_i \,=\, \langle \mathbf{T}^{\langle i\rangle}\,\mathbf{g},\mathbf{M}^{\langle i\rangle}\mathbf{T}^{\langle i\rangle}\,\mathbf{h}\rangle \qquad \text{and} \qquad \big(\mathbf{Mh}\big)_i \,=\, \langle \mathbf{T}^{\langle i\rangle}\,\mathbf{h},\mathbf{M}^{\langle i\rangle}\,\mathbf{T}^{\langle i\rangle}\,\mathbf{h}\rangle\,.$$

It then follows from (Glob-Proj) and the fact that $\mathbf{f}=\mathbf{h}=\mathbf{h}_v$ by (h-Glob) that

$$\big(\mathbf{Mg}\big)_i \,=\, \langle \mathbf{g},\mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle \qquad \text{and} \qquad \big(\mathbf{Mf}\big)_i \,=\, \langle \mathbf{f},\mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle. \tag{7.4}$$

Let $\mathbf{z}:=\mathbf{f}-s\,\mathbf{g}$. It then follows from (s-Equ) and (7.3) that $\langle \mathbf{z},\mathbf{Mz}\rangle=0$. By Lemma 7.2, (s-Equ) implies that $\mathbf{Mz}=\mathbf{0}$, which is equivalent to $\mathrm{s}\,\mathbf{Mg}=\mathbf{Mf}$. Together with (7.4), this implies that

$$\mathrm{s}\,\langle \mathbf{g},\mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle \,=_{\text{(Inh)}}\, \mathrm{s}\,\big(\mathbf{Mg}\big)_i \,=\, \big(\mathbf{Mf}\big)_i \,=\, \langle \mathbf{f},\mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle. \tag{7.5}$$

On the other hand, we have

$$\langle \mathbf{f},\mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle \,=_{(7.4)}\, \big(\mathbf{Mf}\big)_i \,=_{(7.5)}\, \frac{\mathrm{s}+1}{\mathrm{s}}\,\big(\mathbf{M}(\mathbf{f}+\mathbf{g})\big)_i \,>\, 0,$$

where the positivity follows by (Glob-Pos) and the assumption that $i\in\operatorname{supp}(\mathbf{M})$. Now note that,

$$\begin{aligned}\langle \mathbf{z},\mathbf{M}^{\langle i\rangle}\,\mathbf{z}\rangle \quad &= \,\mathrm{s}^2\,\langle \mathbf{g},\mathbf{M}^{\langle i\rangle}\,\mathbf{g}\rangle \,-\, 2\mathrm{s}\,\langle \mathbf{g},\mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle \,+\, \langle \mathbf{f},\mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle \\ &=_{(7.5)}\, \mathrm{s}^2\left(\langle \mathbf{g},\mathbf{M}^{\langle i\rangle}\,\mathbf{g}\rangle \,-\, \frac{\langle \mathbf{g},\mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle^2}{\langle \mathbf{f},\mathbf{M}^{\langle i\rangle}\,\mathbf{f}\rangle}\right),\end{aligned} \tag{7.6}$$

which is nonpositive as $v^{\langle i \rangle}$ satisfies (Hyp). On the other hand, we have

$$\sum_{i \in \mathrm{supp}(\mathbf{M})} \mathrm{h}_i \langle \mathbf{z}, \mathbf{M}^{\langle i \rangle} \mathbf{z} \rangle =_{\text{(Glob-Proj)}} \sum_{i \in \mathrm{supp}(\mathbf{M})} \mathrm{h}_i \langle \mathbf{T}^{\langle i \rangle} \mathbf{z}, \mathbf{M}^{\langle i \rangle} \mathbf{T}^{\langle i \rangle} \mathbf{z} \rangle \geq_{\text{(Pull)}} \langle \mathbf{z}, \mathbf{M}\mathbf{z} \rangle =_{(7.3)} 0.$$

So the RHS of this inequality is equal to 0, while the LHS is a sum of nonpositive terms by (7.6). This implies that every term in the first sum is equal to 0, and thus $\mathrm{h}_i \langle \mathbf{z}, \mathbf{M}^{\langle i \rangle} \mathbf{z} \rangle = 0$ for every $i \in \mathrm{supp}(\mathbf{M})$. This in turn implies that $\langle \mathbf{z}, \mathbf{M}^{\langle i \rangle} \mathbf{z} \rangle = 0$ whenever $(v, v^{\langle i \rangle})$ is a functional edge. This is equivalent to saying that the left side of (7.6) is zero, and we have:

$$\langle \mathbf{g}, \mathbf{M}^{\langle i \rangle} \mathbf{g} \rangle = \frac{\langle \mathbf{g}, \mathbf{M}^{\langle i \rangle} \mathbf{f} \rangle^2}{\langle \mathbf{f}, \mathbf{M}^{\langle i \rangle} \mathbf{f} \rangle} =_{(7.5)} \frac{1}{\mathrm{s}} \langle \mathbf{g}, \mathbf{M}^{\langle i \rangle} \mathbf{f} \rangle. \tag{7.7}$$

It then follows from (7.5) and (7.7) that $v^{\langle i \rangle}$ satisfies (s-Equ) whenever $(v, v^{\langle i \rangle})$ is a functional edge, which completes the proof. $\square$

8 Log-concave inequalities for interval greedoids

In this section, we prove Theorem 1.31 by constructing a combinatorial atlas corresponding to a greedoid, and applying both local-global principle in Theorem 5.2 and sufficient conditions for hyperbolicity given in Theorem 6.1.

8.1 Combinatorial atlas for interval greedoids

Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid on $|X| = n$ elements, let $1 \leq k < \mathrm{rk}(\mathcal{G})$, and let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be the weight function in Theorem 1.31. We define a combinatorial atlas $\mathbb{A}$ corresponding to $(\mathcal{G}, k, \mathrm{q})$ as follows.

Define an acyclic graph $\Gamma := (\Omega, \Theta)$, where the set of vertices $\Omega := \Omega^0 \cup \Omega^1 \cup \ldots \cup \Omega^{k-1}$ is given by[12]

$$\begin{aligned} \Omega^m &:= \big\{ (\alpha, m, t) \ \mid \ \alpha \in X^* \text{ with } |\alpha| \leq k-1-m, \ t \in [0,1] \big\} \quad \text{for } \ m \geq 1, \\ \Omega^0 &:= \big\{ (\alpha, 0, 1) \ \mid \ \alpha \in X^* \text{ with } |\alpha| \leq k-1 \big\}. \end{aligned}$$

Here the restriction $t = 1$ in Ω^0 is crucial for a technical reason that will be apparent later in the section.

Let $\widehat{X} := X \cup \{\mathsf{null}\}$ be the set of letters X with one special element null added. The reader should think of element null as the empty letter. Let $d := |\widehat{X}| = (n+1)$ be the dimension of the atlas, so each vertex $v \in \Omega^m$, $m \geq 1$, has exactly $(n+1)$ outgoing edges we label $\big(v, v^{\langle x \rangle}\big) \in \Theta$, where $x \in \widehat{X}$ and $v^{\langle x \rangle} \in \Omega^{m-1}$ is defined as follows:

$$v^{\langle x \rangle} := \begin{cases} (\alpha x, m-1, 1) & \text{if } \ x \in X, \\ (\alpha, m-1, 1) & \text{if } \ x = \mathsf{null}. \end{cases}$$

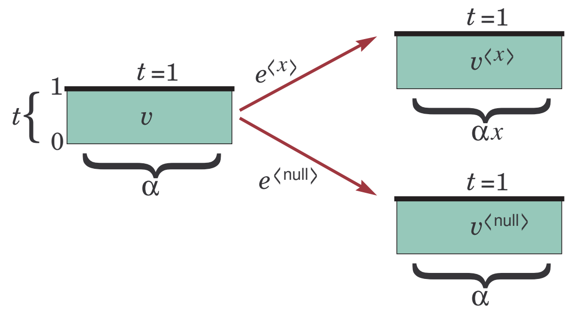


Figure 8.1: Edges of two type: $e^{\langle x\rangle} = \big(v, v^{\langle x\rangle}\big)$, $v = (\alpha, m, t)$, $v^{\langle x\rangle} = (\alpha x, m-1, 1)$, and $e^{\langle \mathsf{null}\rangle} = \big(v, v^{\langle \mathsf{null}\rangle}\big)$, $v = (\alpha, m, t)$, $v^{\langle \mathsf{null}\rangle} = (\alpha, m-1, 1)$.

Let us emphasize that this is not a typo and we indeed have the last parameter $t = 1$, for all $v^{\langle x\rangle}$ (see Figure 8.1).

For every $\alpha \in X^*$ and every $m \in \{1, \ldots, \mathrm{rk}(\mathcal{G}) - |\alpha| - 1\}$, we denote by $\mathbf{A}(\alpha, m) := (\mathrm{A}_{xy})_{x,y \in \widehat{X}}$ the symmetric $d \times d$ matrix defined as follows:[13]

$$\begin{aligned}
&\mathrm{A}_{xy} := 0 \qquad \text{for} \quad x \notin \mathrm{Cont}(\alpha) + \mathsf{null} \quad \text{or} \quad y \notin \mathrm{Cont}(\alpha) + \mathsf{null},\\
&\mathrm{A}_{xy} := \sum_{\beta \in \mathrm{Cont}_{m-1}(\alpha x y)} \mathrm{q}(\alpha x y \beta) \qquad \text{for} \quad x \neq y, \quad x, y \in \mathrm{Cont}(\alpha),\\
&\mathrm{A}_{xx} := \sum_{y \in \mathrm{Des}_\alpha(x)} \ \sum_{\beta \in \mathrm{Cont}_{m-1}(\alpha x y)} \mathrm{q}(\alpha x y \beta) \qquad \text{for} \quad x \in \mathrm{Cont}(\alpha),\\
&\mathrm{A}_{x\,\mathsf{null}} = \mathrm{A}_{\mathsf{null}\,x} := \sum_{\beta \in \mathrm{Cont}_{m-1}(\alpha x)} \mathrm{q}(\alpha x \beta) \qquad \text{for} \quad x \in \mathrm{Cont}(\alpha) \quad \text{and} \quad y = \mathsf{null},\\
&\mathrm{A}_{\mathsf{null}\,\mathsf{null}} := \sum_{\beta \in \mathrm{Cont}_{m-1}(\alpha)} \mathrm{q}(\alpha \beta).
\end{aligned}$$

For the second line, note that (ContInv) implies $\mathrm{A}_{xy} = \mathrm{A}_{yx}$. Note also that $\mathrm{A}_{x\,\mathsf{null}} > 0$, since by the exchange property the word $\alpha x \in \mathcal{L}$ can be extended to $\alpha x \beta \in \mathcal{L}$ for some $\beta \in X^*$ with $|\beta| \leq \mathrm{rk}(\mathcal{G}) - |\alpha| - 1$.

For each vertex $v = (\alpha, m, t) \in \Omega$, define the associated matrix as follows:

$$\mathbf{M} = \mathbf{M}_{(\alpha,m,t)} := t\,\mathbf{A}(\alpha, m+1) + (1-t)\,\mathbf{A}(\alpha, m).$$

Similarly, define the associated vector $\mathbf{h} = \mathbf{h}_{(\alpha,m,t)} \in \mathbb{R}^d$ with coordinates

$$\mathrm{h}_x := \begin{cases} t & \text{if } x \in X, \\ 1-t & \text{if } x = \mathsf{null}. \end{cases}$$

[12] Yes, graph Γ has uncountably many vertices.

[13] When $m = 1$ and $\alpha \in \mathcal{L}$, we have $\mathrm{Cont}_{m-1}(\alpha)$ consists of exactly one element, namely the empty word.

Finally, define the linear transformation $\mathbf{T}^{\langle x\rangle} : \mathbb{R}^d \to \mathbb{R}^d$ associated to the edge $(v, v^{\langle x\rangle})$, as follows:

$$\left(\mathbf{T}^{\langle x\rangle}\,\mathbf{v}\right)_y \;:=\; \begin{cases} \mathrm{v}_y & \text{if } \; y \in \operatorname{supp}(\mathbf{M}), \\ \mathrm{v}_x & \text{if } \; y \in \widehat{X} \setminus \operatorname{supp}(\mathbf{M}). \end{cases}$$

8.2 Properties of the atlas

We now show that our combinatorial atlas $\mathbb{A}$ satisfies the conditions in Theorem 5.2, in the following series of lemmas.

Lemma 8.1. *For every vertex* $v = (\alpha, m, t) \in \Omega$*, we have:*

(i) *the support of the associated matrix* $\mathbf{M}_v$ *is given by*

$$\operatorname{supp}(\mathbf{M}_v) \;=\; \operatorname{supp}(\mathbf{A}(\alpha, m+1)) \;=\; \operatorname{supp}(\mathbf{A}(\alpha, m)) \;=\; \begin{cases} \operatorname{Cont}(\alpha) + \mathsf{null} & \text{if } \; \alpha \in \mathcal{L}, \\ \varnothing & \text{if } \; \alpha \notin \mathcal{L}. \end{cases}$$

(ii) *vertex* v *satisfies* (Irr)*, and*

(iii) *vertex* v *satisfies* (h-Pos) *for* $t \in (0,1)$.

Proof. Part (i) follows directly from the definition of matrices $\mathbf{M}$, $\mathbf{A}(\alpha, m+1)$, and $\mathbf{A}(\alpha, m)$. Part (iii) follows from the fact that $\mathbf{h}_v$ is a strictly positive vector when $t \in (0,1)$.

We now prove part (ii). If $\alpha \notin \mathcal{L}$, then $\mathbf{M}$ is a zero matrix and v trivially satisfies (Irr). If $\alpha \in \mathcal{L}$, then it follows from the definition of $\mathbf{M} = \left(\mathrm{M}_{xy}\right)$, that $\mathrm{M}_{x\,\mathsf{null}} > 0$ for every $x \in \operatorname{Cont}(\alpha)$. Since the support of $\mathbf{M}$ is $\operatorname{Cont}(\alpha) + \mathsf{null}$, this proves (Irr), as desired. ☐

Lemma 8.2. *For every greedoid* $\mathcal{G} = (X, \mathcal{L})$*, the atlas* $\mathbb{A}$ *satisfies* (Proj).

Proof. Let $v = (\alpha, m, t) \in \Omega^m$, $m \geq 1$, be a non-sink vertex of Γ. The condition (Proj) follows directly from the definition of $\mathbf{T}^{\langle x\rangle}$. ☐

Lemma 8.3. *For every greedoid* $\mathcal{G} = (X, \mathcal{L})$*, the atlas* $\mathbb{A}$ *satisfies* (Inh).

Proof. Let $v = (\alpha, m, t) \in \Omega^m$, $m \geq 1$, be a non-sink vertex of Γ. Let $x \in \operatorname{supp}(\mathbf{M}) = \operatorname{Cont}(\alpha) \cup \{\mathsf{null}\}$. By the linearity of $\mathbf{T}^{\langle x\rangle}$, it suffices to show that for every $y \in \operatorname{Cont}(\alpha) \cup \{\mathsf{null}\}$, we have:

$$\mathrm{M}_{xy} \;=\; \left\langle \mathbf{T}^{\langle x\rangle}\,\mathbf{e}_y\,,\, \mathbf{M}^{\langle x\rangle}\,\mathbf{T}^{\langle x\rangle}\,\mathbf{h} \right\rangle,$$

where $\left\{ \mathbf{e}_y,\, y \in \widehat{X} \right\}$ is the standard basis for $\mathbb{R}^d$. We present only the proof for the case $x, y \in \operatorname{Cont}(\alpha)$, as the proof of the other cases are analogous.

First suppose that $x,y \in \mathrm{Cont}(\alpha)$ are distinct. Then:

$$\begin{aligned}
\left\langle \mathbf{T}^{\langle x\rangle} \mathbf{e}_y \,,\, \mathbf{M}^{\langle x\rangle}\, \mathbf{T}^{\langle x\rangle}\, \mathbf{h} \right\rangle \quad &= \sum_{z\in \mathrm{supp}(\mathbf{M}^{\langle x\rangle})} \mathrm{M}^{\langle x\rangle}_{yz} \left(\mathbf{T}^{\langle x\rangle}\, \mathbf{h}\right)_z \\
&= \sum_{z\in \mathrm{supp}(\mathbf{M}^{\langle x\rangle}),\, z\neq \mathsf{null}} t\, \mathbf{A}(\alpha x, m)_{yz} \,+\, (1-t)\, \mathbf{A}(\alpha x, m)_{y\,\mathsf{null}} \\
&= \sum_{z\in X} \sum_{\beta\in \mathrm{Cont}_{m-1}(\alpha xyz)} t\, \mathrm{q}(\alpha xyz\beta) \,+\, \sum_{\beta\in \mathrm{Cont}_{m-1}(\alpha xy)} (1-t)\, \mathrm{q}(\alpha xy\beta) \\
&= \sum_{\gamma\in \mathrm{Cont}_{m}(\alpha xy)} t\, \mathrm{q}(\alpha xy\gamma) \,+\, \sum_{\beta\in \mathrm{Cont}_{m-1}(\alpha xy)} (1-t)\, \mathrm{q}(\alpha xy\beta),
\end{aligned}$$

where we substitute $\gamma \leftarrow z\beta$ in the first term of the last equality. This implies that

$$\left\langle \mathbf{T}^{\langle x\rangle} \mathbf{e}_y \,,\, \mathbf{M}^{\langle x\rangle}\, \mathbf{T}^{\langle x\rangle}\, \mathbf{h} \right\rangle \,=\, t\, \mathbf{A}(\alpha, m+1)_{xy} \,+\, (1-t)\, \mathbf{A}(\alpha, m)_{xy} \,=\, \mathrm{M}_{xy}\,,$$

which proves (Inh) for this case.

Now suppose that $x = y \in \mathrm{Cont}(\alpha)$. Then:

$$\left\langle \mathbf{T}^{\langle x\rangle} \mathbf{e}_x, \mathbf{M}^{\langle x\rangle}\, \mathbf{T}^{\langle x\rangle}\, \mathbf{h} \right\rangle \,=\, \sum_{y\in \mathrm{supp}(\mathbf{M}^{\langle x\rangle})\setminus \mathrm{supp}(\mathbf{M})} \; \sum_{z\in \mathrm{supp}(\mathbf{M}^{\langle x\rangle})} \mathrm{M}^{\langle x\rangle}_{yz} \left(\mathbf{T}^{\langle x\rangle}\, \mathbf{h}\right)_z .$$

By the same argument as above, this equation becomes

$$\begin{aligned}
&\sum_{y\in \mathrm{supp}(\mathbf{M}^{\langle x\rangle})\setminus \mathrm{supp}(\mathbf{M})} \; \sum_{\gamma\in \mathrm{Cont}_m(\alpha xy)} t\, \mathrm{q}(\alpha xy\gamma) \,+\, \sum_{\beta\in \mathrm{Cont}_{m-1}(\alpha xy)} (1-t)\, \mathrm{q}(\alpha xy\beta) \\
&= \sum_{y\in \mathrm{Des}_\alpha(x)} \; \sum_{\gamma\in \mathrm{Cont}_{\alpha xy}(m)} t\, \mathrm{q}(\alpha xy\gamma) \,+\, \sum_{y\in \mathrm{Des}_\alpha(x)} \; \sum_{\beta\in \mathrm{Cont}_{\alpha xy}(m-1)} (1-t)\, \mathrm{q}(\alpha xy\beta) \\
&= t\, \mathbf{A}(\alpha, m+1)_{xx} \,+\, (1-t)\, \mathbf{A}(\alpha, m)_{xx} \,=\, \mathrm{M}_{xx}\,,
\end{aligned}$$

which proves (Inh) for this case. This completes the proof. □

Lemma 8.4. *Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid, and suppose the weight function $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ satisfies* (ContInv). *Then the atlas $\mathbb{A}$ satisfies* (T-Inv).

Proof. Let $v = (\alpha, m, t) \in \Omega^m$, $m \geq 1$, be a non-sink vertex of Γ, and let x,y,z be distinct elements of $\mathrm{supp}(\mathbf{M}) = \mathrm{Cont}(\alpha) + \mathsf{null}$. We present only the proof for the case when $x,y,z \in \mathrm{Cont}(\alpha)$, as other cases follow analogously.

First suppose that $\alpha x'y'z' \notin \mathcal{L}$ for every permutation (x',y',z') of $\{x,y,z\}$. Then

$$\mathrm{M}^{\langle x\rangle}_{yz} \,=\, \mathrm{M}^{\langle y\rangle}_{zx} \,=\, \mathrm{M}^{\langle z\rangle}_{xy} \,=\, 0,$$

and (T-Inv) is satisfied. So, without loss of generality, we assume that $\alpha xyz \in \mathcal{L}$. It then follows from the interval exchange property that $\alpha x'y'z' \in \mathcal{L}$ for every permutation (x',y',z') of $\{x,y,z\}$. This allows us to apply (ContInv) for $\alpha \in \mathcal{L}$ and any two elements from $\{x,y,z\}$.

We now have

$$\begin{aligned}\mathrm{M}^{\langle x\rangle}_{yz} \ &= \ \mathbf{A}(\alpha x,m)_{yz} \ = \ \sum_{\beta\,\in\,\mathrm{Cont}_{m-1}(\alpha xyz)} \mathrm{q}(\alpha xyz\beta) \\ &=_{(\text{ContInv})} \sum_{\beta\,\in\,\mathrm{Cont}_{m-1}(\alpha yxz)} \mathrm{q}(\alpha yxz\beta) \ = \ \mathbf{A}(\alpha y,m)_{xz} \ = \ \mathrm{M}^{\langle y\rangle}_{xz}\,.\end{aligned}$$

By an analogous argument, it follows that $\mathrm{M}^{\langle x\rangle}_{yz} = \mathrm{M}^{\langle z\rangle}_{xy}$, and thus (T-Inv) is satisfied, as desired. □

Lemma 8.5. *Let $\mathcal{G} = (X,\mathcal{L})$ be a greedoid, let $1 \le k < \mathrm{rk}(\mathcal{G})$, and suppose the weight function $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ satisfies (ContInv) and (PAMon). Then the atlas $\mathbb{A}$ satisfies (K-Non).*

Proof. Let $v = (\alpha,m,t) \in \Omega^m$, $m \ge 1$, be a non-sink vertex of Γ. We need to check the condition (K-Non) for distinct $x,y \in \mathrm{supp}(\mathbf{M}) = \mathrm{Cont}(\alpha) + \mathsf{null}$.

First suppose that x,y are distinct elements of $\mathrm{Cont}(\alpha)$. We have:

$$\mathrm{M}^{\langle x\rangle}_{yy} \ = \ \mathbf{A}(\alpha x,m)_{yy} \ = \ \sum_{z\,\in\,\mathrm{Des}_{\alpha x}(y)} \ \sum_{\beta\,\in\,\mathrm{Cont}_{m-1}(\alpha xyz)} \mathrm{q}(\alpha xyz\beta).$$

Note that $z \in X$ in the equation above is summed over the set

$$\big\{z \in X \,:\, \alpha xz \notin \mathcal{L},\, \alpha xyz \in \mathcal{L}\big\}.$$

By the interval exchange property, every element z in the set above also satisfies $\alpha z \notin \mathcal{L}$. We can then partition the set above into

$$\begin{aligned}&\big\{z : \alpha z \notin \mathcal{L},\, \alpha xz \notin \mathcal{L},\, \alpha yz \notin \mathcal{L},\, \alpha xyz \in \mathcal{L}\big\} \cup \big\{z : \alpha z \notin \mathcal{L},\, \alpha xz \notin \mathcal{L},\, \alpha yz \in \mathcal{L},\, \alpha xyz \in \mathcal{L}\big\} \\ &\quad = \ \mathrm{Pas}_\alpha(x,y) \cup \big\{z : \alpha z \notin \mathcal{L},\, \alpha xz \notin \mathcal{L},\, \alpha yz \in \mathcal{L},\, \alpha xyz \in \mathcal{L}\big\}.\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}\sum_{z\in\mathrm{Fam}^{\langle y\rangle}} \mathrm{M}^{\langle y\rangle}_{xz} \ &= \ \sum_{z\in\mathrm{Fam}^{\langle y\rangle}} \mathbf{A}(\alpha y,m)_{xz} \ = \ \sum_{z\in\mathrm{Des}_\alpha(y)} \ \sum_{\beta\in\mathrm{Cont}_{m-1}(\alpha yxz)} \mathrm{q}(\alpha yxz\beta) \\ &=_{(\text{ContInv})} \sum_{z\in\mathrm{Des}_\alpha(y)} \ \sum_{\beta\in\mathrm{Cont}_{m-1}(\alpha xyz)} \mathrm{q}(\alpha xyz\beta),\end{aligned}$$

where in the last equality we apply (ContInv) to swap x and y. Note that $z \in X$ in the equation above is summed over the set

$$\{z \in X \,:\, \alpha z \notin \mathcal{L},\, \alpha yz \in \mathcal{L},\, \alpha xyz \in \mathcal{L}\},$$

which can be partitioned into

$$\begin{aligned}&\big\{z : \alpha z \notin \mathcal{L},\, \alpha xz \in \mathcal{L},\, \alpha yz \in \mathcal{L},\, \alpha xyz \in \mathcal{L}\big\} \cup \big\{z : \alpha z \notin \mathcal{L},\, \alpha xz \notin \mathcal{L},\, \alpha yz \in \mathcal{L},\, \alpha xyz \in \mathcal{L}\big\} \\ &\quad = \ \mathrm{Act}_\alpha(x,y) \cup \big\{z : \alpha z \notin \mathcal{L},\, \alpha xz \notin \mathcal{L},\, \alpha yz \in \mathcal{L},\, \alpha xyz \in \mathcal{L}\big\}.\end{aligned}$$

It follows from the calculations above that

$$\mathrm{M}_{yy}^{\langle x\rangle} \ - \sum_{z\in \mathrm{Fam}^{\langle y\rangle}} \mathrm{M}_{xz}^{\langle y\rangle} \ = \sum_{z\in \mathrm{Pas}_\alpha(x,y)} \ \sum_{\beta\in \mathrm{Cont}_{m-1}(\alpha xyz)} \mathrm{q}(\alpha xyz\beta) \\ - \sum_{z\in \mathrm{Act}_\alpha(x,y)} \ \sum_{\beta\in \mathrm{Cont}_{m-1}(\alpha xyz)} \mathrm{q}(\alpha xyz\beta),$$

which is nonnegative by (PAMon). This proves (K-Non) in this case.

Now suppose that $x\in \mathrm{Cont}(\alpha)$ and $y=\mathsf{null}$. Without loss of generality, we assume that $\alpha\in\mathcal{L}$, as otherwise $\mathrm{M}_{yz}^{\langle x\rangle}$ is always equal to the zero matrix and (K-Non) is trivially satisfied. Then we have:

$$\mathrm{M}_{\mathsf{null}\,\mathsf{null}}^{\langle x\rangle} \ = \ \mathbf{A}(\alpha x, m)_{\mathsf{null}\,\mathsf{null}} \ = \sum_{\beta\in \mathrm{Cont}_{m-1}(\alpha x)} \mathrm{q}(\alpha x\beta).$$

On the other hand, we have $\mathrm{supp}(\mathbf{M}) = \mathrm{supp}\big(\mathbf{M}^{\langle \mathsf{null}\rangle}\big) = \mathrm{Cont}(\alpha)+\mathsf{null}$, which implies that

$$\mathrm{Fam}^{\langle \mathsf{null}\rangle} \ = \ \mathrm{supp}\big(\mathbf{M}^{\langle \mathsf{null}\rangle}\big) \setminus \big(\mathrm{supp}(\mathbf{M}) - \mathsf{null}\big) \ = \ \{\mathsf{null}\}.$$

Therefore, we have:

$$\sum_{z\in \mathrm{Fam}^{\langle \mathsf{null}\rangle}} \mathrm{M}_{xz}^{\langle \mathsf{null}\rangle} \ = \ \mathrm{M}_{x\,\mathsf{null}}^{\langle \mathsf{null}\rangle} \ = \ \mathbf{A}(\alpha,m)_{x\,\mathsf{null}} \ = \sum_{\beta\in \mathrm{Cont}_{m-1}(\alpha x)} \mathrm{q}(\alpha x\beta).$$

It thus follows from the calculations above that $\mathrm{K}_{x\,\mathsf{null}}=0$. This completes the proof of (K-Non). $\square$

8.3 Basic hyperbolicity

To prove hyperbolicity of vertices in Ω^0, we need the following straightforward linear algebra lemma. We include the proof for completeness.

Lemma 8.6. *Let $\mathbf{N}=\big(\mathrm{N}_{ij}\big)$ be a nonnegative symmetric $(n+1)\times(n+1)$ matrix, such that its nondiagonal entries are equal to* 1. *Suppose that*

$$(*) \qquad \mathrm{N}_{11},\ldots,\mathrm{N}_{nn} \leq 1 \qquad \text{and} \qquad \mathrm{N}_{n+1\,n+1} \ \geq \ \sum_{i=1}^{n} \frac{\mathrm{N}_{n+1\,n+1}-1}{1-\mathrm{N}_{ii}} \quad \text{if } \mathrm{N}_{ii}<1 \text{ for all } i\in[n].$$

Then $\mathbf{N}$ *satisfies* (Hyp).

Proof. Fix $\varepsilon>0$. Substituting $\mathrm{N}_{ii}\leftarrow \mathrm{N}_{ii}-\varepsilon$ for every $1\leq i\leq n$ if necessary, we can assume that all inequalities in $(*)$ are strict. Note that (Hyp) is preserved under taking the limit $\varepsilon\to 0$, so it suffices to prove the result in this case.

We prove that $\mathbf{N}$ satisfies (OPE) by induction on n. By Lemma 5.3 this implies (Hyp). The base case $n=0$ is trivial. Assume that the claim is true for $(n-1)$. Let $\lambda_1\geq\ldots\geq\lambda_{n+1}$ be the eigenvalues of $\mathbf{N}$, and let $\lambda_1' \geq \ldots \geq \lambda_n'$ be the eigenvalues of the matrix obtained by removing the first row and column of $\mathbf{N}$. By the *Cauchy interlacing theorem*, we have

$$\lambda_1 \ \geq \ \lambda_1' \ \geq \ \lambda_2 \ \geq \ \ldots \ \geq \ \lambda_n' \ \geq \ \lambda_{n+1} \, .$$

Note that $\lambda_2', \ldots, \lambda_n'$ are nonpositive by induction. It then follows that $\lambda_3, \ldots, \lambda_{n+1}$ are nonpositive. By the Perron–Frobenius theorem, we also have $\lambda_1 > 0$. It thus suffices to show that $\lambda_2 \leq 0$, which will follow from showing that $\det(\mathbf{N})$ has sign $(-1)^n$. Observe that $\det(\mathbf{N})$ is equal to

$$\begin{vmatrix} \mathrm{N}_{1\,1}-1 & 0 & \cdots & 0 & 1-\mathrm{N}_{n+1\,n+1} \\ 0 & \mathrm{N}_{2\,2}-1 & & 0 & 1-\mathrm{N}_{n+1\,n+1} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{N}_{n\,n}-1 & 1-\mathrm{N}_{n+1\,n+1} \\ 1 & 1 & \ldots & 1 & \mathrm{N}_{n+1\,n+1} \end{vmatrix} = \begin{vmatrix} \mathrm{N}_{1\,1}-1 & 0 & \cdots & 0 & 1-\mathrm{N}_{n+1\,n+1} \\ 0 & \mathrm{N}_{2\,2}-1 & & 0 & 1-\mathrm{N}_{n+1\,n+1} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathrm{N}_{n\,n}-1 & 1-\mathrm{N}_{n+1\,n+1} \\ 0 & 0 & \ldots & 0 & \mathrm{J} \end{vmatrix},$$

$$\text{where} \qquad \mathrm{J} \ := \ \mathrm{N}_{n+1\,+1} \ - \ \sum_{i=1}^{n} \frac{\mathrm{N}_{n+1\,n+1} \ - \ 1}{1 \ - \ \mathrm{N}_{i\,i}} \ > \ 0 \, ,$$

by the assumption $(*)$. Therefore, we have

$$\det(\mathbf{N}) \ = \ \mathrm{J} \cdot \prod_{i=1}^{n} \big(\mathrm{N}_{i\,i} \ - \ 1\big) \, ,$$

and by the assumptions on $\mathrm{N}_{i\,i}$ this determinant has sign $(-1)^n$. This completes the proof. $\square$

8.4 Proof of Theorem 1.31

We first show that every sink vertex in the combinatorial atlas $\mathbb{A}$ is hyperbolic.

Lemma 8.7. *Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid on $|X| = n$ elements, let $1 \leq k < \mathrm{rk}(\mathcal{G})$, and let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be a k-admissible weight function. Then every vertex in Ω^0 satisfies* (Hyp).

Proof. Let $v = (\alpha, 0, 1) \in \Omega^0$ be a sink vertex. It suffices to show that $\mathbf{A}(\alpha, 1)$ satisfies (Hyp). First note that if $\alpha \notin \mathcal{L}$, then $\mathbf{A}(\alpha, 1)$ is a zero matrix, and (Hyp) is trivially true. Thus, we can assume that $\alpha \in \mathcal{L}$. We write $\mathrm{A}_{x,y} := \mathbf{A}(\alpha, 1)_{xy}$ for every $x, y \in X$.

Let $\mathcal{C} \in \mathrm{Par}(\alpha)$ be a parallel class. Suppose that $|\,\mathcal{C}\,| \geq 2$, and let x, y be distinct elements of $\mathcal{C}$.

Claim: *For every $z \in \widehat{X}$, we have* $\ \omega(\alpha y)\, \mathrm{A}_{xz} \ = \ \omega(\alpha x)\, \mathrm{A}_{yz}$.

Proof of Claim. First suppose that $z \in \{x, y\}$. It then follows from (FewDes) and the fact that $\alpha x y \notin \mathcal{L}$ that $\mathrm{A}_{x,z} \ = \ \mathrm{A}_{y,z} \ = \ 0$, which implies the claim in this case.

Now suppose that $z \in X \setminus \{x, y\}$. It follows from the exchange property that $\alpha x z \in \mathcal{L}$ if and only if $\alpha y z \in \mathcal{L}$. There are now two cases. If $\alpha x z \notin \mathcal{L}$ and $\alpha y z \notin \mathcal{L}$, then again we have $\mathrm{A}_{x,z} = \mathrm{A}_{y,z} = 0$, which implies the claim. If $\alpha x z \in \mathcal{L}$ and $\alpha y z \in \mathcal{L}$, we then have:

$$\mathrm{A}_{xz} \ = \ \mathrm{q}(\alpha x z) \ =_{\text{(LogMod)}} \ c_{\ell+2} \, \frac{\omega(\alpha x)\, \omega(\alpha z)}{\omega(\alpha)} \, , \qquad \mathrm{A}_{yz} \ = \ \mathrm{q}(\alpha y z) \ =_{\text{(LogMod)}} \ c_{\ell+2} \, \frac{\omega(\alpha y)\, \omega(\alpha z)}{\omega(\alpha)} \, ,$$

where $\ell := |\alpha|$. This implies the claim in this case. Finally, let $z = \textsf{null}$. Then we have $\mathrm{A}_{xz} = c_{\ell+1}\,\omega(\alpha x)$ and $\mathrm{A}_{yz} = c_{\ell+1}\,\omega(\alpha y)$, which implies the claim. □

Deduct the y-row and y-column of $\mathbf{A}(\alpha,1)$ by $\frac{\omega(\alpha y)}{\omega(\alpha x)}$ of the x-row and x-column of $\mathbf{A}(\alpha,1)$. It then follows from the claim that the resulting matrix has y-row and y-column is equal to zero. Also, note that (Hyp) is preserved under this transformation. Applying this linear transformation repeatedly, and by restricting to the support of resulting matrix which preserves (Hyp), without loss of generality we can assume that $|\mathcal{C}| = 1$ for every parallel class $\mathcal{C} \in \mathrm{Par}(\alpha)$. Then the matrix $\mathbf{A}(\alpha,1)$ is equal to

$$\begin{pmatrix} c_{\ell+2}\,\mathrm{b}_\alpha(\mathcal{C}_1)\,\frac{\omega(\alpha x_1)^2}{\omega(\alpha)} & \mathrm{q}(\alpha x_1 x_2) & \cdots & \mathrm{q}(\alpha x_1 x_n) & \mathrm{q}(\alpha x_1) \\ \mathrm{q}(\alpha x_2 x_1) & c_{\ell+2}\,\mathrm{b}_\alpha(\mathcal{C}_2)\,\frac{\omega(\alpha x_1)^2}{\omega(\alpha)} & & \vdots & \vdots \\ \vdots & & \ddots & \mathrm{q}(\alpha x_{n-1} x_n) & \mathrm{q}(\alpha x_{n-1}) \\ \mathrm{q}(\alpha x_n x_1) & \cdots & \mathrm{q}(\alpha x_n x_{n-1}) & c_{\ell+2}\,\mathrm{b}_\alpha(\mathcal{C}_d)\,\frac{\omega(\alpha x_1)^2}{\omega(\alpha)} & \mathrm{q}(\alpha x_n) \\ \mathrm{q}(\alpha x_1) & \ldots & \mathrm{q}(\alpha x_{n-1}) & \mathrm{q}(\alpha x_n) & \mathrm{q}(\alpha) \end{pmatrix},$$

where $\mathcal{C}_i = \{x_i\}$ for $i \in [n]$, the rows and columns are indexed by $\widehat{X} = \big\{x_1, \ldots, x_n, \textsf{null}\big\}$, and $\mathrm{b}_\alpha(\mathcal{C})$ is as defined in (3.2). We now rescale the x_i-row and x_i-column by $\frac{\sqrt{\omega(\alpha)}}{\sqrt{c_{\ell+2}}\,\omega(\alpha x_i)}$, and the null-row and null-column by $\frac{\sqrt{c_{\ell+2}}}{c_{\ell+1}\sqrt{\omega(\alpha)}}$. Again, note that (Hyp) is preserved under this transformation. It then follows from (LogMod) that the matrix becomes

$$\begin{pmatrix} \mathrm{b}_\alpha(\mathcal{C}_1) & 1 & \cdots & 1 & 1 \\ 1 & \mathrm{b}_\alpha(\mathcal{C}_2) & & \vdots & \vdots \\ \vdots & & \ddots & 1 & 1 \\ 1 & \cdots & 1 & \mathrm{b}_\alpha(\mathcal{C}_n) & 1 \\ 1 & \ldots & 1 & 1 & \frac{c_{\ell+2}\,c_\ell}{c_{\ell+1}^2} \end{pmatrix}.$$

It follows from (SynMon) and (ScaleMon) that this matrix satisfies conditions $(*)$ in Lemma 8.6. Hence, by the lemma, this matrix satisfies (Hyp). We conclude that v satisfies (Hyp), as desired. □

We can now prove that every vertex in Γ is hyperbolic.

Proposition 8.8. *Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid on $|X| = n$ elements, let $1 \le k < \mathrm{rk}(\mathcal{G})$, and let $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ be a k-admissible weight function. Then every vertex in Ω satisfies* (Hyp)*.*

Proof. We will show that every vertex in Ω^m for $m \le k-1$ satisfies (Hyp) by induction on m. The claim is true for $m = 0$ by Lemma 8.7. Suppose that the claim is true for Ω^{m-1}. Note that the atlas $\mathbb{A}$ satisfies the assumptions of Theorem 5.2 by Lemmas 8.2, 8.3, 8.4, and 8.5. It then follows that every regular vertex in Ω^m satisfies (Hyp).

On the other hand, by Lemma 8.1, the regular vertices of Ω^m are those of the form $v = (\alpha, m, t)$ with $t \in (0,1)$. Since (Hyp) is preserved under taking the limits $t \to 0$ and $t \to 1$, it then follows that every vertex in Ω^m satisfies (Hyp), and the proof is complete. □

Proof of Theorem 1.31. Let $\mathbf{M} = \mathbf{M}_v$ be the matrix associated with the vertex $v = (\varnothing, k-1, 1)$. Let $\mathbf{v}$ and $\mathbf{w}$ be the characteristic vectors of X and $\{\textsf{null}\}$, respectively. Then:

$$\mathrm{L}_{\mathrm{q}}(k+1) \ = \ \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle, \qquad \mathrm{L}_{\mathrm{q}}(k) \ = \ \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle, \qquad \mathrm{L}_{\mathrm{q}}(k-1) \ = \ \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle. \tag{8.1}$$

By Proposition 8.8, vertex v satisfies (Hyp). Substituting (8.1) into (Hyp), gives the log-concave inequality (1.29) in the theorem. □

9 Proof of equality conditions for interval greedoids

In this section we prove Theorem 3.3. The implication (GE-b) $\Rightarrow$ (GE-a) is obvious. We now prove the other implications.

9.1 Proof of (GE-a) $\Rightarrow$ (GE-$c1$) & (GE-$c2$)

Let $\mathbb{A}$ be the combinatorial atlas defined in §8.1, that corresponds to $(\mathcal{G}, k, \mathrm{q})$. Recall that every vertex of Γ satisfies (Hyp) by Proposition 8.8.

As at the end of previous section, let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ be the characteristic vectors of X and $\{\textsf{null}\}$, respectively. It is straightforward to verify that $\mathbf{v}, \mathbf{w}$ is a global pair of Γ, i.e. they satisfy (Glob-Pos).

Let $v = (\varnothing, k-1, 1) \in \Omega$ and let $\mathbf{M} = \mathbf{M}_v$ be the matrix associated with v. Recall that $\mathbf{M} = \mathbf{A}(\varnothing, k)$ and we have equalities (8.1) again:

$$\mathrm{L}_{\mathrm{q}}(k+1) \ = \ \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle, \qquad \mathrm{L}_{\mathrm{q}}(k) \ = \ \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle, \qquad \mathrm{L}_{\mathrm{q}}(k-1) \ = \ \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle. \tag{9.1}$$

Note also that $\mathrm{L}_{\mathrm{q}}(k+1), \mathrm{L}_{\mathrm{q}}(k), \mathrm{L}_{\mathrm{q}}(k-1) > 0$ since $k < \mathrm{rk}(\mathcal{G})$. It then follows from (GE-a), that v satisfies (s-Equ) for some $\mathrm{s} > 0$.

Let us show that, for every $\alpha \in \mathcal{L}$ of length $(k-1)$, we have:

$$\langle \mathbf{v}, \mathbf{A}(\alpha,1)\, \mathbf{v} \rangle \ = \ \mathrm{s}\, \langle \mathbf{w}, \mathbf{A}(\alpha,1)\, \mathbf{v} \rangle \ = \ \mathrm{s}^2\, \langle \mathbf{w}, \mathbf{A}(\alpha,1)\, \mathbf{w} \rangle. \tag{9.2}$$

First, suppose that $k = 1$. It then follows that $\alpha = \varnothing$ and $v = (\varnothing, 0, 1)$. Equation (9.2) now follows from the fact that v satisfies (s-Equ).

Now suppose that $k > 1$. Then it is straightforward to verify that v is a functional source, i.e. satisfies (Glob-Proj) and (h-Glob), where we apply the substitution $\mathbf{f} \leftarrow \mathbf{v}$ for (h-Glob). By Theorem 7.1, every functional target of v also satisfies (s-Equ) with the same $\mathrm{s} > 0$. On the other hand, it is straightforward to verify that the functional targets of v are those of the form $(\alpha, 0, 1)$. Combining these observations, we conclude (9.2).

Let $\mathbf{z} := \mathbf{v} - \mathrm{s}\,\mathbf{w}$. It follows from (9.2) that $\langle \mathbf{z}, \mathbf{A}(\alpha,1)\, \mathbf{z} \rangle = 0$. It then follows from Lemma 7.2 that $\mathbf{A}(\alpha,1)\, \mathbf{z} = \mathbf{0}$, which is equivalent to $\mathrm{s}\, \mathbf{A}(\alpha,1)\, \mathbf{w} = \mathbf{A}(\alpha,1)\, \mathbf{v}$. This implies that

$$\mathrm{s}\, \mathrm{q}(\alpha) \ = \ \mathrm{s}\, \big(\mathbf{A}(\alpha,1)\, \mathbf{w}\big)_{\textsf{null}} \ = \ \big(\mathbf{A}(\alpha,1)\, \mathbf{v}\big)_{\textsf{null}} \ = \ \sum_{x \in \mathrm{Cont}(\alpha)} \mathrm{q}(\alpha x),$$

which proves (GE-c1) for $\mathrm{s}(k-1)=\mathrm{s}$.

Let $x \in \mathrm{Cont}(\alpha)$ be an arbitrary continuation. By the same reasoning as above, we have:

$$\mathrm{s}\,\mathrm{q}(\alpha x) \;=\; \mathrm{s}\,\big(\mathbf{A}(\alpha,1)\,\mathbf{w}\big)_x \;=\; \big(\mathbf{A}(\alpha,1)\,\mathbf{v}\big)_x\,.$$

On the other hand, we also have:

$$\big(\mathbf{A}(\alpha,1)\,\mathbf{v}\big)_x \;=\; \sum_{y\in\mathrm{Des}_\alpha(x)} \mathrm{q}(\alpha xy) \;+\; \sum_{\substack{y\in\mathrm{Cont}(\alpha)\\ y\not\sim_\alpha x}} \mathrm{q}(\alpha xy)\,.$$

It then follows that:

$$\mathrm{s}\,\mathrm{q}(\alpha x) \;=\; \sum_{y\in\mathrm{Des}_\alpha(x)} \mathrm{q}(\alpha xy) \;+\; \sum_{\substack{y\in\mathrm{Cont}(\alpha)\\ y\not\sim_\alpha x}} \mathrm{q}(\alpha xy)\,. \tag{9.3}$$

Let $\mathcal{C}$ be the parallel class in Par_α containing x. We now show that (9.3) is equivalent to (GE-c2).

Applying (LogMod) to (9.3) and dividing both sides by $\omega(\alpha)$, we get:

$$\mathrm{s}\,c_k\,\frac{\omega(\alpha x)}{\omega(\alpha)} \;=\; \sum_{y\in\mathrm{Des}_\alpha(x)} c_{k+1}\,\frac{\omega(\alpha xy)}{\omega(\alpha)} \;+\; \sum_{\substack{y\in\mathrm{Cont}(\alpha)\\ y\not\sim_\alpha x}} c_{k+1}\,\frac{\omega(\alpha x)\,\omega(\alpha y)}{\omega(\alpha)^2}\,. \tag{9.4}$$

Now note that, (GE-c1) gives:

$$\sum_{\substack{y\in\mathrm{Cont}(\alpha)\\ y\not\sim_\alpha x}} \frac{\omega(\alpha y)}{\omega(\alpha)} \;=\; \mathrm{s}\,\frac{c_{k-1}}{c_k} \;-\; \mathrm{a}_\alpha(\mathcal{C}), \tag{9.5}$$

where

$$\mathrm{a}_\alpha(\mathcal{C}) \;:=\; \sum_{y\in\mathcal{C}} \frac{\omega(\alpha y)}{\omega(\alpha)}.$$

Now note that, when $|\,\mathcal{C}\,|\geq 2$,

$$\sum_{y\in\mathrm{Des}_\alpha(x)} \frac{\omega(\alpha xy)}{\omega(\alpha)} \;=_{(\mathrm{FewDes})}\; 0 \;=\; \frac{\omega(\alpha x)}{\omega(\alpha)}\,\mathrm{a}_\alpha(\mathcal{C})\,\mathrm{b}_\alpha(\mathcal{C}),$$

where the last equality is because $\mathrm{b}_\alpha(\mathcal{C})=0$ when $|\,\mathcal{C}\,|\geq 2$. On the other hand, when $\mathcal{C}=\{x\}$,

$$\sum_{y\in\mathrm{Des}_\alpha(x)} \frac{\omega(\alpha xy)}{\omega(\alpha)} \;=\; \frac{\omega(\alpha x)^2}{\omega(\alpha)^2}\,\mathrm{b}_\alpha(\mathcal{C}) \;=\; \frac{\omega(\alpha x)}{\omega(\alpha)}\,\mathrm{a}_\alpha(\mathcal{C})\,\mathrm{b}_\alpha(\mathcal{C}),$$

where the last equality is because $\mathrm{a}_\alpha(\mathcal{C})=\frac{\omega(\alpha x)}{\omega(\alpha)}$ when $\mathcal{C}=\{x\}$. This allows us to conclude that

$$\sum_{y\in\mathrm{Des}_\alpha(x)} \frac{\omega(\alpha xy)}{\omega(\alpha)} \;=\; \frac{\omega(\alpha x)}{\omega(\alpha)}\,\mathrm{a}_\alpha(\mathcal{C})\,\mathrm{b}_\alpha(\mathcal{C}), \tag{9.6}$$

Substituting (9.5) and (9.6) into (9.4), we obtain:

$$\mathrm{s}\, c_k \, \frac{\omega(\alpha x)}{\omega(\alpha)} \ = \ c_{k+1} \, \frac{\omega(\alpha x)}{\omega(\alpha)} \, \mathrm{a}_\alpha(\mathcal{C}) \, \mathrm{b}_\alpha(\mathcal{C}) \ + \ c_{k+1} \, \frac{\omega(\alpha x)}{\omega(\alpha)} \left(\mathrm{s} \, \frac{c_{k-1}}{c_k} \ - \ \mathrm{a}_\alpha(\mathcal{C}) \right).$$

This is equivalent to

$$\mathrm{s}\left(\frac{c_{k-1}}{c_k} - \frac{c_k}{c_{k+1}} \right) \ = \ \mathrm{a}_\alpha(\mathcal{C}) \left(1 - \mathrm{b}_\alpha(\mathcal{C})\right),$$

which is (GE-c2). This completes the proof. □

9.2 Proof of (GE-c1) & (GE-c2) ⇒ (GE-b)

Write $\mathrm{s} := \mathrm{s}(k-1)$. We have $\mathrm{L}_{\mathrm{q},\alpha}(0) = \mathrm{q}(\alpha)$ by definition, and

$$\mathrm{L}_{\mathrm{q},\alpha}(1) \ = \sum_{x \in \mathrm{Cont}(\alpha)} \mathrm{q}(\alpha x) \ =_{(\text{GE-}c\,1)} \ \mathrm{s}\,\mathrm{q}(\alpha) \ = \ \mathrm{s}\,\mathrm{L}_{\mathrm{q},\alpha}(0),$$

which proves the first part of (GE-b). For the second part of (GE-b), we have:

$$\mathrm{L}_{\mathrm{q},\alpha}(2) \ = \sum_{x \in \mathrm{Cont}(\alpha)} \left(\sum_{y \in \mathrm{Des}_\alpha(x)} \mathrm{q}(\alpha x y) \ + \sum_{\substack{y \in \mathrm{Cont}(\alpha) \\ y \not\leftarrow_\alpha x}} \mathrm{q}(\alpha x y) \right). \tag{9.7}$$

On the other hand, we showed in the proof above (see §9.1), that (GE-c2) is equivalent to (9.3). Therefore, for every $x \in \mathrm{Cont}(\alpha)$, we have:

$$\mathrm{s}\,\mathrm{q}(\alpha x) \ = \sum_{y \in \mathrm{Des}_\alpha(x)} \mathrm{q}(\alpha x y) \ + \sum_{\substack{y \in \mathrm{Cont}(\alpha) \\ y \not\leftarrow_\alpha x}} \mathrm{q}(\alpha x y).$$

Substituting this equation into (9.7), we conclude:

$$\mathrm{L}_{\mathrm{q},\alpha}(2) \ = \sum_{x \in \mathrm{Cont}(\alpha)} \mathrm{s}\,\mathrm{q}(\alpha x) \ =_{(\text{GE-}c\,1)} \ \mathrm{s}^2\,\mathrm{q}(\alpha) \ = \ \mathrm{s}^2\,\mathrm{L}_{\mathrm{q},\alpha}(0).$$

This proves the second part of (GE-b), and completes the proof. □

10 Proof of matroid inequalities and equality conditions

In this section we give proofs of Theorem 1.6, Theorem 1.9, Theorem 1.10, Proposition 1.11 and give further extension of graphical matroid results. We conclude with two explicit examples of small combinatorial atlases.

10.1 Proof of Theorem 1.6

We deduce the result from Theorem 1.31. Let $\mathcal{G} = (X, \mathcal{L})$ be the interval greedoid constructed in §4.3, and corresponding to matroid $\mathcal{M} = (X, \mathcal{I})$. Let $1 \leq k < \operatorname{rk}(\mathcal{M})$ and let $\omega : X \to \mathbb{R}_{>0}$ be as in the theorem. Define the weight function $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ by the product formula:

$$\mathrm{q}(\alpha) \ := \ c_\ell \prod_{x \in \alpha} \omega(x), \tag{10.1}$$

where $\ell := |\alpha|$, and c_ℓ is given by

$$c_\ell \ := \ \begin{cases} 1 & \text{for } \ \ell \neq k+1\,, \\ 1 + \dfrac{1}{\mathrm{p}(k-1)-1} & \text{for } \ \ell = k+1\,. \end{cases} \tag{10.2}$$

Since every permutation of an independent set gives rise to a feasible word, we then have:

$$\mathrm{L}_\mathrm{q}(k-1) \ = \ (k-1)! \cdot \mathrm{I}_\omega(k-1), \qquad \mathrm{L}_\mathrm{q}(k) \ = \ k! \cdot \mathrm{I}_\omega(k), \quad \text{and}$$
$$\mathrm{L}_\mathrm{q}(k+1) \ = \ (k+1)!\left(1 + \frac{1}{\mathrm{p}(k-1)-1}\right) \cdot \mathrm{I}_\omega(k+1)\,.$$

This reduces (1.8) to (1.29).

By Theorem 1.31, it remains to show that q is a k-admissible weight function. First note that the weight function q is multiplicative and thus satisfies (ContInv) and (LogMod). By Proposition 4.6, greedoid $\mathcal{G}$ satisfies (WeakLoc), which in turn implies (PAMon). By the same proposition, greedoid $\mathcal{G}$ is interval and satisfies (FewDes). Further, property (4.4) implies that $\mathrm{Des}_\alpha(x) = \varnothing$ for every $\alpha \in \mathcal{L}$ and $x \in X$, which in turn trivially implies (SynMon).

To verify (ScaleMon), first suppose that $\ell < k-1$. Then $c_\ell = c_{\ell+1} = c_{\ell+2} = 1$, which implies that the LHS of (ScaleMon) is equal to 0 while the RHS of (ScaleMon) is equal to 1, as desired. Now suppose that $\ell = k-1$. Note that $\mathrm{b}_\alpha(\mathcal{C}) = 0$ for every $\alpha \in \mathcal{L}$ and $\mathcal{C} \in \mathrm{Par}(\alpha)$, since $\mathrm{Des}_\alpha(x) = \varnothing$. Then, for every $\alpha \in \mathcal{L}$ of length $k-1$, we have:

$$\left(1 - \frac{c_k^2}{c_{k-1}\, c_{k+1}}\right) \sum_{\mathcal{C} \in \mathrm{Par}(\alpha)} \frac{1}{1 - \mathrm{b}_\alpha(\mathcal{C})} \ = \ \left(1 - \frac{c_k^2}{c_{k-1}\, c_{k+1}}\right) \left|\mathrm{Par}_\alpha\right| \ = \ \frac{\left|\mathrm{Par}_\alpha\right|}{\mathrm{p}(k-1)} \ \leq \ 1.$$

This finishes the proof of (ScaleMon).

In summary, greedoid $\mathcal{G} = (X, \mathcal{L})$ satisfies (ContInv), (PAMon), (LogMod), (FewDes), (SynMon) and (ScaleMon). By Definition 3.2, we conclude that weight function q is k-admissible, which completes the proof of the theorem. □

10.2 Proof of Theorem 1.10

We will prove the theorem as a consequence of Theorem 3.3. From the proof of Theorem 1.6 given above, it suffices to show that (GE-c1) and (GE-c2) are equivalent to (ME1) and (ME2) for the greedoid $\mathcal{G}$.

Let $\alpha \in \mathcal{L}$ of length $|\alpha| = k-1$. We denote by $\mathrm{s}_{\mathcal{M}}(k-1)$ the constant that appears in (ME2), and $\mathrm{s}_{\mathcal{G}}(k-1)$ the constant that appears in (GE-c2). Recall that $\mathrm{b}_\alpha(\mathcal{C}) = 0$ for every $\alpha \in \mathcal{L}$ and $\mathcal{C} \in \mathrm{Par}(\alpha)$. Note that

$$\sum_{x\in\mathcal{C}} \frac{\mathrm{q}(\alpha x)}{\mathrm{q}(\alpha)} = \sum_{x\in\mathcal{C}} \omega(x) \quad \text{and} \quad 1 - \frac{c_k^2}{c_{k-1}\, c_{k+1}} = \frac{1}{\mathrm{p}(k-1)},$$

where the first equality follows from the product formula (10.1) and the second equality is because of the choice of constants c_ℓ in (10.2). This implies that (GE-c2) and (ME2) are equivalent under the substitution $\mathrm{s}_{\mathcal{M}}(k-1) := \mathrm{s}_{\mathcal{G}}(k-1)/\mathrm{p}(k-1)$.

Now, let $S = \{x_1, \ldots, x_{k-1}\}$ be an arbitrary independent set of size $(k-1)$, and let $\alpha := x_1 \cdots x_{k-1}$. We have:

$$\sum_{x\in\mathrm{Cont}(\alpha)} \frac{\mathrm{q}(\alpha x)}{\mathrm{q}(\alpha)} = \sum_{x\in\mathrm{Cont}(\alpha)} \omega(x) =_{(\mathrm{ME2})} \big|\mathrm{Par}_S\big| \cdot \mathrm{s}_{\mathcal{M}}(k-1).$$

This implies that (GE-c1) and (ME1) are equivalent, and completes the proof of the theorem. □

10.3 Proof of Theorem 1.9

The direction $\Leftarrow$ is trivial, so it suffices to prove the $\Rightarrow$ direction.

Let S be an arbitrary independent set of size $k-1$. Recall that $\mathrm{p}(k-1) \le n-k+1$. From the equality (1.10) and inequality (1.8), it follows that $\mathrm{p}(k-1) = n-k+1$. On the other hand, it follows from equation (ME1) in Theorem 1.10, that $|\mathrm{Par}_S| = \mathrm{p}(k-1)$. Combining these two observations, we obtain:

$$S \cup \{x,y\} \ \text{ is an independent set for every distinct } x,y \in X \setminus S. \tag{10.3}$$

Let us show that every $(k+1)$-subset of X is independent. Fix an independent set U of size $k+1$, and take an arbitrary $(k+1)$-subset T of X. If $T = U$ then we are done, so suppose that $T \ne U$. Let $x \in T \setminus U$ and let $y \in U \setminus T$. Let U' be the $(k+1)$-subset given by $U' := U + x - y$. It follows from (10.3) that U' is an independent set. Observe that the size of the intersection has increased: $|T \cap U'| > |T \cap U|$, Letting $U \leftarrow U'$, we can iterate this argument until we eventually get $U' = T$, as desired.

We can now prove that the weight function $\omega : X \to \mathbb{R}_{>0}$ is uniform. Let $x,y \in X$ be distinct elements. Let S be a $(k-1)$-subset of X that contains neither x nor y. It follows from the argument in the previous paragraph that S is an independent set of the matroid $\mathcal{M}$, and every parallel class of S has cardinality 1. By applying (ME2) to the parallel class $\mathcal{C}_1 = \{x\}$ and $\mathcal{C}_2 = \{y\}$, we conclude that $\omega(x) = \omega(y)$. This completes the proof. □

10.4 Proof of Proposition 1.11

The inequality (1.12) in the proposition is a restatement of (1.7). Thus, we need to show that equality in (1.7) holds if and only if G is an N-cycle. The $\Leftarrow$ direction follows from a direct calculation, so it suffices to prove the $\Rightarrow$ direction.

We first show that $\deg(v) \ge 2$ for every $v \in V$. Suppose to the contrary, that there exists $v \in V$ such that $\deg(v) = 1$. Let e be the unique edge adjacent to v, and let $S \subset E$ be a forest with $\mathrm{N}-3$ edges not containing e. Then v is a leaf vertex in the contraction graph G/S. On the other hand, the graph G/S is the complete graph K_3 by (ME1), a contradiction.

We now show that $\deg(v) \leq 2$ for every $v \in V$. Suppose to the contrary, that $\deg(v) \geq 3$ for some $v \in V$. Let $e, f, g \in E$ be three distinct edges adjacent to v. Then there exists a spanning tree T in G that contains e, f, g. Let $S = T - \{e, f\}$, and let x and y be the other endpoint of e and f, respectively. Note that S is a forest with $\mathrm{N}-3$ edges, so it follows from (ME2) that there exists $\mathrm{s}(\mathrm{N}-3)$ many edges connecting the component of G/S containing x, and the component of G/S containing y. Now let $U := T - \{f, g\} = S + e - g$, which is another forest with $\mathrm{N}-3$ edges. Note that there are at least $\mathrm{s}(\mathrm{N}-3)+1$ edges connecting the component of G/U containing $\{v, x\}$, and the component of G/S containing y, namely the edge f and the other $\mathrm{s}(\mathrm{N}-3)$ many edges connecting the component of G/S containing x and the component of G/S containing y. This contradicts (ME2).

Finally, observe that the N-cycle is the only connected graph for which every vertex has degree two. This completes the proof. $\square$

10.5 More graphical matroids

The following result is a counterpart to the Proposition 1.11 proved above.

Theorem 10.1. *Let $G = (V, E)$ be a simple connected graph on $|V| = \mathrm{N}$ vertices, and let $\mathrm{I}(k)$ be the number of spanning forests with k edges. Then*

$$\frac{\mathrm{I}(k)^2}{\mathrm{I}(k+1) \cdot \mathrm{I}(k-1)} \;\geq\; \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\binom{\mathrm{N}-k+1}{2} - 1}\right), \tag{10.4}$$

and the inequality is always strict if $1 < k < \mathrm{N}-2$.

Proof. The inequality (10.4) follows immediately from (1.6) and the fact that $\mathrm{p}(k-1) \leq \binom{\mathrm{N}-k+1}{2}$.

For the second part, suppose to the contrary, that we have equality in (10.4) for some simple connected graph G. It then follows from (ME1) and (ME2), that there exists $\mathrm{s} > 0$ such that G satisfies the following *clique-partition property*:

> Let $A_1, \ldots, A_{\mathrm{N}-k+1}$ be a partition of V, such that each $A_i \subset V$ spans a connected subgraph of G. Then the graph obtained by contracting each A_i to one vertex (loops are removed but multiple edges remain) is the complete graph $K_{\mathrm{N}-k+1}$, with the multiplicity of every edge equal to s.

Now, start with an arbitrary partition $A_1, \ldots, A_{\mathrm{N}-k+1}$ of V such that each $A_i \subset V$ is nonempty and spans a connected subgraph of G. We get our contradiction if this partition does not satisfy the clique-partition property above. Since $k > 1$, without loss of generality, we assume that A_1 has at least two vertices.

Let x be a vertex in A_1 that is adjacent to a vertex in A_2. If x is adjacent to any other vertex in A_i, $i \geq 3$, then by moving x to A_2 we create a new partition $A'_1, \ldots, A'_{\mathrm{N}-k+1}$ and note that now there are $\mathrm{s}+1$ edges connecting A'_2 and A'_i, contradicting the clique-partition property. Thus, x is not adjacent to $A_3, \ldots, A_{\mathrm{N}-k+1}$, and we can then move x to A_2 to create a new partition. By iteratively moving elements to A_2 until only one element y remains in A_1, and applying the clique-partition property to the resulting partition, we conclude that y is adjacent to $A_3,\ldots,A_{\mathrm{N}-k+1}$.

We now return to the original partition $A_1,\ldots,A_{\mathrm{N}-k+1}$, and we move y to A_3 to obtain a new partition $A_1'',\ldots A_{\mathrm{N}-k+1}''$. In this new partition, there are $\mathrm{s}+1$ edges connecting A_3'' and A_4'', a contradiction. Note that here that part A_4'' is nonempty since $k<\mathrm{N}-2$. This completes the proof. $\square$

Remark 10.2. The inequality (10.4) is incomparable with (1.3), and is stronger only for very dense graphs:

$$|E| \;\geq\; \binom{\mathrm{N}-k+1}{2} + k - 1.$$

To explain this, note that $\mathrm{p}(k-1)$ is usually smaller than the binomial coefficient above. This is why the inequality (10.4) is strict for $1<k<\mathrm{N}-2$. This also underscores the power of our main matroid inequality (1.6).

10.6 Proof of Corollary 1.13

The result follows immediately from Theorem 1.4 and the fact that

$$\mathrm{Par}(S) \;\leq\; q^{m-k+1}-1 \quad \text{for every} \quad S\in\mathcal{I}_{k-1}\,.$$

This is because the contraction $\mathcal{M}/S$ with parallel elements removed is a realizable matroid over $\mathbb{F}_q$ of rank $m-k+1$, which can have at most $q^{m-k+1}-1$ nonzero vectors. $\square$

10.7 Examples of combinatorial atlases

The numbers of independent sets can grow rather large, so we give two rather small matroid examples to help the reader navigate the definitions. We assume that the weight function ω is uniform in both examples.

Example 10.3 (Free matroid)**.** Let $\mathcal{M}=(X,\mathcal{I})$ be a *free matroid* on $n=4$ elements: $X=\{x_1,\ldots x_4\}$ and $\mathcal{I}=2^X$. In this case, we have $\mathrm{I}(1)=\mathrm{I}(3)=4$, $\mathrm{I}(2)=6$, and the inequality (1.3) is an equality.

Following §4.3 and the proof of Theorem 1.6, the corresponding greedoid $\mathcal{G}=(X,\mathcal{L})$ has all simple words in X^*. Let $k=2$ and $\alpha=\varnothing$. Then $\mathbf{A}(\alpha,k-1)$ and $\mathbf{A}(\alpha,k)$ are $(n+1)\times(n+1)$ matrices given by

$$\mathbf{A}(\varnothing,1) \;=\; \begin{pmatrix} 0 & 1 & 1 & 1 & 1\\ 1 & 0 & 1 & 1 & 1\\ 1 & 1 & 0 & 1 & 1\\ 1 & 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A}(\varnothing,2) \;=\; \begin{pmatrix} 0 & 3 & 3 & 3 & 3\\ 3 & 0 & 3 & 3 & 3\\ 3 & 3 & 0 & 3 & 3\\ 3 & 3 & 3 & 0 & 3\\ 3 & 3 & 3 & 3 & 4 \end{pmatrix},$$

where the rows and columns are labeled by $\{x_1,x_2,x_3,x_4,\mathsf{null}\}$. Recall that each entry of the matrices is counting the number of certain feasible words, and only words of length $k+1=3$ are weighted by $1+\frac{1}{\mathrm{p}(k-1)-1}=\frac{3}{2}$.

As in the proof of Theorem 1.31 (see §8.4), let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^5$ be the vectors given by

$$\mathbf{v} := (1,1,1,1,0)^{\intercal} \quad \text{and} \quad \mathbf{w} := (0,0,0,0,1)^{\intercal}.$$

Inequality (1.3) in this case is equivalent to (1.29), which in turn can be rewritten as:

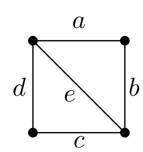
$$\langle \mathbf{v}, \mathbf{A}(\alpha,k)\, \mathbf{w}\rangle^2 \;\geq\; \langle \mathbf{v}, \mathbf{A}(\alpha,k)\, \mathbf{v}\rangle \, \langle \mathbf{w}, \mathbf{A}(\alpha,k)\, \mathbf{w}\rangle. \tag{10.5}$$

In this case the equality holds, since

$$\langle \mathbf{v}, \mathbf{A}(\varnothing,2)\, \mathbf{w}\rangle \;=\; 12, \qquad \langle \mathbf{v}, \mathbf{A}(\varnothing,2)\, \mathbf{v}\rangle \;=\; 36, \qquad \langle \mathbf{v}, \mathbf{A}(\varnothing,2)\, \mathbf{v}\rangle \;=\; 4,$$

as implied by Theorem 1.8.

Example 10.4 (Graphical matroid)**.** Let $G = (V,E)$ be a graph as in the figure below, where $\textsc{n} := |V| = 4$ and $E = \{a,b,c,d,e\}$. Let $\mathcal{M} = (E,\mathcal{L})$ be the corresponding graphical matroid (see Example 1.5). In this case $n = |E| = 5$ and $\mathrm{rk}(\mathcal{M}) = \textsc{n} - 1 = 3$.



Let $\alpha = \varnothing$ and $k = 2$. Then $\mathbf{A}(\alpha, k-1)$ and $\mathbf{A}(\alpha,k)$ are $(n+1)\times(n+1)$ matrices given by

$$\mathbf{A}(\varnothing,1) \;=\; \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & 1 & 1 & 1 & 1\\ 1 & 1 & 0 & 1 & 1 & 1\\ 1 & 1 & 1 & 0 & 1 & 1\\ 1 & 1 & 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A}(\varnothing,2) \;=\; \begin{pmatrix} 0 & 3 & 4.5 & 4.5 & 3 & 4\\ 3 & 0 & 4.5 & 4.5 & 3 & 4\\ 4.5 & 4.5 & 0 & 3 & 3 & 4\\ 4.5 & 4.5 & 3 & 0 & 3 & 4\\ 3 & 3 & 3 & 3 & 0 & 4\\ 4 & 4 & 4 & 4 & 4 & 5 \end{pmatrix},$$

where the rows and columns are labeled by $\{a,b,c,d,e,\mathsf{null}\}$. As in the previous example, each entry of the matrices is counting the number of certain feasible words, and only words of length $k+1 = 3$ is weighted by $1 + \frac{1}{\mathrm{p}(k-1)-1} = \frac{3}{2}$.

As above, let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^6$ be the vectors given by

$$\mathbf{v} := (1,1,1,1,1,0)^{\intercal} \quad \text{and} \quad \mathbf{w} := (0,0,0,0,0,1)^{\intercal}.$$

Inequality (1.3) in this case is equivalent to (1.29), which in turn can be rewritten as (10.5). Note that in this case we have

$$\langle \mathbf{v}, \mathbf{A}(\alpha,k)\, \mathbf{w}\rangle \;=\; 72, \qquad \langle \mathbf{v}, \mathbf{A}(\alpha,k)\, \mathbf{v}\rangle \;=\; 20, \qquad \langle \mathbf{v}, \mathbf{A}(\alpha,k)\, \mathbf{v}\rangle \;=\; 5,$$

and indeed we have a strict inequality $20^2 > 72 \times 5$, as implied by Theorem 1.8.

11 Proof of discrete polymatroid inequalities and equality conditions

In this section we give proofs of Theorem 1.21, Theorem 1.23 and Theorem 1.24.

11.1 Proof of Theorem 1.21

We deduce the result from Theorem 1.31. This proof is similar to the argument in the proof of Theorem 1.6 in the previous section, so we will emphasize the differences.

Let $\mathcal{G} = (X, \mathcal{L})$ be the interval greedoid constructed in §4.4, and corresponding to discrete polymatroid $\mathcal{D} = ([n], \mathcal{J})$. Let $1 \leq k < \mathrm{rk}(\mathcal{D})$, let $0 < t \leq 1$, and let $\omega : X \to \mathbb{R}_{>0}$ be as in the theorem.

Let $\boldsymbol{a}_\alpha = (\mathrm{a}_1, \ldots, \mathrm{a}_n) \in \mathbb{N}^n$ be the vector corresponding to the word $\alpha \in \mathcal{L}$. We define the weight function $\mathrm{q} : \mathcal{L} \to \mathbb{R}_{>0}$ by the product formula

$$\mathrm{q}(\alpha) \;:=\; c_\ell \, t^{\pi(\boldsymbol{a}_\alpha)} \, \omega(\boldsymbol{a}_\alpha),$$

where $\ell := |\alpha| = |\mathrm{b}_\alpha|$, and c_ℓ is given by

$$c_\ell \;:=\; \begin{cases} 1 & \text{for } \ \ell \neq k+1\,, \\ 1 + \dfrac{1-t}{\mathrm{p}(k-1) - 1 + t} & \text{for } \ \ell = k+1\,. \end{cases} \tag{11.1}$$

Using this weight function, we obtain:

$$\mathrm{L}_\mathrm{q}(k) \;=\; \sum_{\alpha \in \mathcal{L}_k} t^{\pi(\boldsymbol{a}_\alpha)} \, \omega(\boldsymbol{a}_\alpha) \;=\; \sum_{\mathrm{b} \in \mathcal{J}_k} t^{\pi(\boldsymbol{a}_\alpha)} \, \omega(\boldsymbol{a}) \, \frac{k!}{\mathrm{a}_1! \cdots \mathrm{a}_n!} \;=\; k! \cdot \mathrm{J}_{\omega,t}(k),$$

where the third equality follows from every permutation of a feasible word that is well-ordered is again a feasible word. By the same calculation, we have

$$\mathrm{L}_\mathrm{q}(k-1) \;=\; (k-1)! \cdot \mathrm{J}_{\omega,t}(k-1)\,, \quad \mathrm{L}(k+1) \;=\; (k+1)! \left(1 + \frac{1-t}{\mathrm{p}(k-1) - 1 + t}\right) \cdot \mathrm{J}_{\omega,t}(k+1).$$

This reduces (1.22) to (1.29).

By Theorem 1.31, it remains to show that q is a k-admissible weight function. First note that the weight function q is multiplicative and thus satisfies (ContInv) and (LogMod). By Proposition 4.7, greedoid $\mathcal{G}$ satisfies (WeakLoc), which in turn implies (PAMon). By the same proposition, greedoid $\mathcal{G}$ is interval and satisfies (FewDes).

We now verify (SynMon), which is no longer similar to the matroid case. It follows from (4.5) that the right side of (SynMon) is 0, unless $x = x_{ij}$ and $\mathrm{Des}_\alpha(x) = x_{i\,j+1}$. In the latter case, we have

$$\frac{\omega(\alpha x)}{\omega(\alpha)} \;=\; t^{j-1} \, \omega(i)\,, \qquad \sum_{y \in \mathrm{Des}_\alpha(x)} \frac{\omega(\alpha x y)}{\omega(\alpha x)} \;=\; t^j \, \omega(i), \tag{11.2}$$

and (SynMon) follows from $t \leq 1$.

For (ScaleMon), the same argument as for matroids works for $\ell < k-1$. Now suppose that $\ell = k-1$. Note that, for every $\alpha \in \mathcal{L}$ and $\mathcal{C} \in \mathrm{Par}(\alpha)$, we have $\mathrm{b}_\alpha(\mathcal{C}) \leq t$ by (11.2). Hence, for every $\alpha \in \mathcal{L}_{k-1}$, we have:

$$\left(1 - \frac{c_k^2}{c_{k-1}\, c_{k+1}}\right) \sum_{\mathcal{C} \in \mathrm{Par}(\alpha)} \frac{1}{1 - \mathrm{b}_\alpha(\mathcal{C})} \leq \left(1 - \frac{c_k^2}{c_{k-1}\, c_{k+1}}\right) \sum_{\mathcal{C} \in \mathrm{Par}(\alpha)} \frac{1}{1-t}$$
$$= \left(1 - \frac{c_k^2}{c_{k-1}\, c_{k+1}}\right) \frac{\left|\mathrm{Par}_\alpha\right|}{1-t} = \frac{\left|\mathrm{Par}_\alpha\right|}{\mathrm{p}(k-1)} \leq 1,$$

which proves (ScaleMon).

In summary, greedoid $\mathcal{G} = (X, \mathcal{L})$ satisfies (ContInv), (PAMon), (LogMod), (FewDes), (SynMon) and (ScaleMon). By Definition 3.2, we conclude that weight function q is k-admissible, which completes the proof of the theorem. □

11.2 Proof of Theorem 1.23

We deduce the result from Theorem 3.3. The proof below only assumes that $0 < t \leq 1$. For the $\Rightarrow$ direction, let $\alpha \in \mathcal{L}$ with $|\alpha| = k-1$. Note that, since $c_k = c_{k-1} = 1$, we have

$$\sum_{x \in \mathcal{C}} \frac{\mathrm{q}(\alpha x)}{\mathrm{q}(\alpha)} = \sum_{x \in \mathcal{C}} \frac{\omega(\alpha x)}{\omega(\alpha)} = \mathrm{a}_\alpha(\mathcal{C}).$$

By (GE-c2), there exists $\mathrm{s} > 0$, s.t. for every $\mathcal{C} \in \mathrm{Par}(\alpha)$ we have:

$$\mathrm{a}_\alpha(\mathcal{C})\left(1 - \mathrm{b}_\alpha(\mathcal{C})\right) = \mathrm{s}\left(1 - \frac{c_k^2}{c_{k-1}\, c_{k+1}}\right) = \mathrm{s}\, \frac{1-t}{\mathrm{p}(k-1)}. \tag{11.3}$$

Summing over all $\mathcal{C} \in \mathrm{Par}(\alpha)$, we get:

$$\sum_{\mathcal{C} \in \mathrm{Par}_\alpha} \mathrm{a}_\alpha(\mathcal{C})\left(1 - \mathrm{b}_\alpha(\mathcal{C})\right) = \mathrm{s}\,(1-t)\, \frac{\left|\mathrm{Par}_\alpha\right|}{\mathrm{p}(k-1)}.$$

On the other hand, the equality (GE-c1) gives:

$$\mathrm{s} = \sum_{\mathcal{C} \in \mathrm{Par}_\alpha} \mathrm{a}_\alpha(\mathcal{C}).$$

Combining these equations, we obtain:

$$\sum_{\mathcal{C} \in \mathrm{Par}_\alpha} \mathrm{a}_\alpha(\mathcal{C})\left(1 - \mathrm{b}_\alpha(\mathcal{C})\right) = \sum_{\mathcal{C} \in \mathrm{Par}_\alpha} \mathrm{a}_\alpha(\mathcal{C})\,(1-t)\, \frac{\left|\mathrm{Par}_\alpha\right|}{\mathrm{p}(k-1)},$$

which is equivalent to

$$\sum_{\mathcal{C} \in \mathrm{Par}_\alpha} \mathrm{a}_\alpha(\mathcal{C})\left(1 - \mathrm{b}_\alpha(\mathcal{C}) - (1-t)\, \frac{\left|\mathrm{Par}_\alpha\right|}{\mathrm{p}(k-1)}\right) = 0.$$

Now note that, the LHS of the equation above is always nonnegative since $\mathrm{b}_\alpha(\mathcal{C}) \leq t$ by (11.2), and $\left|\mathrm{Par}_\alpha\right| \leq \mathrm{p}(k-1)$ by definition. Therefore, the equality hold for both inequalities, so in particular we have:

$$\mathrm{b}_\alpha(\mathcal{C}) = t \quad \text{for every} \quad \alpha \in \mathcal{L}_{k-1} \quad \text{and} \quad \mathcal{C} \in \mathrm{Par}(\alpha).$$

Since $t > 0$ by assumption, it follows from (FewDes) that

$$|\mathcal{C}| = 1 \;\text{ and }\; \mathrm{b}_\alpha(\mathcal{C}) = t > 0 \quad \text{for every} \quad \alpha \in \mathcal{L} \quad \text{with} \quad |\alpha| = k-1 \quad \text{and} \quad \mathcal{C} \in \mathrm{Par}(\alpha). \tag{11.4}$$

Restating this equation in the language of polymatroids, we conclude: for every $\boldsymbol{a} \in \mathcal{J}_{k-1}$, and every $i, j \in [n]$ (not necessarily distinct), we have:

$$\boldsymbol{a} + \mathbf{e}_i, \boldsymbol{a} + \mathbf{e}_j \in \mathcal{J} \quad \Longrightarrow \quad \boldsymbol{a} + \mathbf{e}_i + \mathbf{e}_j \in \mathcal{J}. \tag{11.5}$$

We can now show that every $\mathbf{n} = (\mathrm{n}_1, \ldots, \mathrm{n}_n) \in \mathbb{N}^n$ with $|\mathbf{n}| = k+1$ is contained in $\mathcal{J}$. We follow the corresponding argument int the matroid case. Let $\boldsymbol{a} \in \mathcal{J}$ with $|\boldsymbol{a}| = k+1$. If $\boldsymbol{a} = \mathbf{n}$, we are done, so suppose that $\boldsymbol{a} \neq \mathbf{n}$. Then there exists $i, j \in [n]$, such that $\mathrm{a}_i > \mathrm{n}_i$ and $\mathrm{a}_j < \mathrm{n}_j$. By the polymatroid hereditary property, we have $\boldsymbol{a} - \mathbf{e}_i \in \mathcal{J}$. Since $\mathbf{e}_j \in \mathcal{J}$ by the assumption that the polymatroid is normal, we can then apply the exchange property to $\mathbf{e}_j$ and $\boldsymbol{a} - \mathbf{e}_i$ to conclude that $\boldsymbol{a} - \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_h \in \mathcal{J}$ for some $h \in [n]$. Let $\mathbf{u} := \boldsymbol{a} - \mathbf{e}_i - \mathbf{e}_h$. Note that $\mathbf{u} \in \mathcal{J}_{k-1}$ by hereditary property, and

$$\mathbf{u} + \mathbf{e}_j = \boldsymbol{a} - \mathbf{e}_i + \mathbf{e}_j - \mathbf{e}_h \in \mathcal{J}, \quad \text{and} \quad \mathbf{u} + \mathbf{e}_h = \boldsymbol{a} - \mathbf{e}_i \in \mathcal{J}.$$

It then follows from (11.5) that

$$\boldsymbol{a} - \mathbf{e}_i + \mathbf{e}_j = \mathbf{u} + \mathbf{e}_j + \mathbf{e}_h \in \mathcal{J}.$$

Make substitution $\boldsymbol{a} \leftarrow \boldsymbol{a} - \mathbf{e}_i + \mathbf{e}_j$ and iterate this argument until eventually $\mathbf{n} = \boldsymbol{a}$, as desired. This proves the $\Rightarrow$ direction.

For the $\Leftarrow$ direction, assume now that $t = 1$. The equality now follows from a direct calculation, since that

$$1 + \frac{1-t}{\mathrm{p}(k-1) - 1 + t} = 1, \quad \text{and} \quad \mathrm{J}_\omega(\ell) = \frac{\big(\omega(1) + \ldots + \omega(n)\big)^\ell}{\ell!} \quad \text{for every} \quad \ell \leq k+1.$$

This completes the proof. $\square$

11.3 Proof of Theorem 1.24

Assume now that $0 < t < 1$. From the proof above, it remains to show that $k = 1$ and that the weight function ω is uniform.

Let $i, j \in [n]$ be distinct elements, let $\alpha := x_{i1} \cdots x_{ik-1}$, let $x := x_{ik}$, and let $y := x_{j1}$. By (11.4), we have $\mathcal{C}_1 = \{x\}$ and $\mathcal{C}_2 = \{y\}$ are both parallel classes of α. It then follows from (11.3) and (11.4), that

$$\mathrm{a}_\alpha(\mathcal{C}_1) = \mathrm{a}_\alpha(\mathcal{C}_2).$$

On the other hand, we have

$$\mathrm{a}_\alpha(\mathcal{C}_1) \;=\; t^k\,\omega(i), \qquad \mathrm{a}_\alpha(\mathcal{C}_2) \;=\; t\,\omega(j),$$

so $t^{k-1} = \omega(j)/\omega(i)$. Since the choice of i and j was arbitrary, we can switch i and j to obtain $t^{k-1} = \omega(j)/\omega(i)$. This implies that $\omega(i) = \omega(j)$ and $k = 1$, which proves the $\Rightarrow$ direction.

For the $\Leftarrow$ direction, assume now that $k = 1$. From the proof above, $\omega(i) = C$ for every $i \in [n]$ and some $C > 0$. It then follows from a direct calculation that

$$1 + \frac{1-t}{\mathrm{p}(k-1) - 1 + t} \;=\; \frac{n}{n-1+t}\,,$$

and

$$\mathrm{J}_{\omega,t}(0) \;=\; 1\,, \qquad \mathrm{J}_{\omega,t}(1) \;=\; C\,t\,n\,, \qquad \mathrm{J}_{\omega,t}(2) \;=\; C^2\left(\frac{t^3\,n}{2} + \frac{t^2\,n\,(n-1)}{2}\right).$$

Thus, the equality (1.24) holds in this. This completes the proof. □

12 Proof of poset antimatroid inequalities and equality conditions

In this section we prove Theorem 1.26 and Theorem 1.28.

12.1 Proof of Theorem 1.26

As in the previous sections, we deduce the result from Theorem 1.31. Let $\mathcal{P} = (X, \prec)$ be a poset on $|X| = n$ elements and let $\mathcal{A} = (X, \mathcal{L})$ be the corresponding poset antimatroid which is an interval greedoid by the argument in §4.2. In the notation of Section 3, let $c_\ell = 1$ for all $\ell \geq 1$, and let $\mathrm{q}(\alpha) := \omega(\alpha)$, so that (1.26) coincides with (1.29) in this case.

It remains to show that ω is a k-admissible weight function. First note that ω satisfies (ContInv) and (LogMod) since the weight function is multiplicative. The condition (ScaleMon) is also trivially satisfied. By Proposition 4.5, both (WeakLoc) and (FewDes) are satisfied, and the former implies (PAMon).

For (SynMon), mote that the poset ideal greedoid $\mathcal{G}$ satisfies

$$\mathrm{Des}_\alpha(x) \;\subseteq\; \big\{y \in X \,:\, x \prec\!\!\cdot\, y\big\}, \tag{12.1}$$

for every $\alpha \in \mathcal{L}$ and every $x \in \mathrm{Cont}(\alpha)$. It then follows that

$$\frac{\omega(\alpha x)}{\omega(\alpha)} \;=\; \omega(x) \;\geq_{\text{(CM)}} \sum_{y\,:\,x \prec\!\cdot\, y} \omega(y) \;\geq_{\text{(12.1)}} \sum_{y \in \mathrm{Des}_\alpha(x)} \omega(y) \;=\; \sum_{y \in \mathrm{Des}_\alpha(x)} \frac{\omega(\alpha x y)}{\omega(\alpha x)}, \tag{12.2}$$

which proves (SynMon). Hence q is indeed a k-admissible weight function, which completes the proof. □

12.2 Proof of Theorem 1.28

We deduce the result from Theorem 3.3. From the argument above, it suffices to show that (GE-c1) and (GE-c2) are equivalent to properties (AE1)–(AE3). First note that,

$$\sum_{x\in\mathrm{Cont}(\alpha)} \frac{\mathrm{q}(\alpha x)}{\mathrm{q}(\alpha)} \ = \ \sum_{x\in\mathrm{Cont}(\alpha)} \omega(x),$$

for every $\alpha \in \mathcal{L}$. This implies that (GE-c1) is equivalent to (AE1).

Let $\alpha \in \mathcal{L}$, let $x \in \mathrm{Cont}(\alpha)$, and let $\mathcal{C} = \{x\}$ be the parallel class in $\mathrm{Par}(\alpha)$ containing x. Since $c_{k+1} = c_k = c_{k-1} = 1$, it then follows that the RHS of (GE-c2) is equal to 0, so (GE-c2) is equivalent to

$$\sum_{y\in\mathrm{Des}_\alpha(x)} \frac{\omega(\alpha)\,\omega(\alpha xy)}{\omega(\alpha x)^2} \ = \ \mathrm{b}_\alpha(\mathcal{C}) \ = \ 1.$$

This implies that (GE-c2) is equivalent to equality in (12.2), which in turn is equivalent to equality in both (CM) and (12.1). The latter is equivalent to (AE2) and (AE3), which completes the proof. □

13 Proof of morphism of matroids inequalities and equality conditions

In this section we give proofs of Theorem 1.16, Theorem 1.18 and Theorem 1.19.

13.1 Combinatorial atlas construction

Let $\mathcal{M} = (X, \mathcal{I})$ and $\mathcal{N} = (Y, \mathcal{J})$ be two matroids, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroids. Let $1 \le k < \mathrm{rk}(\mathcal{M})$ and let $\omega : X \to \mathbb{R}_{>0}$ be the weight function as in Theorem 1.16. We now define a combinatorial atlas $\mathbb{A}$ that corresponds to (Φ, k, ω).

Let $\mathcal{G} = (X, \mathcal{L})$ be the greedoid which corresponds to matroid $\mathcal{M}$, see §4.3. We extend ω to a *nonnegative weight function* $\mathrm{q} : \mathcal{L}_{\mathcal{M}} \to \mathbb{R}_{\ge 0}$ by the product formula:

$$\mathrm{q}(x_1 \cdots x_\ell) \ := \ \begin{cases} c_\ell\, \omega(x_1) \,\cdots\, \omega(x_\ell) & \text{if } \{x_1, \ldots, x_\ell\} \in \mathcal{B}_\ell\,, \\ 0 & \text{otherwise}, \end{cases}$$

where c_ℓ is defined in (10.2). Let $\Gamma := (\Omega, \Theta)$ be the acyclic graph and let $\mathbb{A}$ be the combinatorial atlas defined in §10.1 that corresponds to the greedoid $\mathcal{G}$. Note that Γ depends only on the matroid $\mathcal{M}$, but $\mathbb{A}$ depends also on the morphism Φ. Note also that, unlike the weight function in §1.14 and §10.1, here the weight q is *not* strictly positive, so Theorem 1.31 does not apply in this case.

In this section, we rework the combined proofs of Theorem 1.31 and Theorem 1.6 to apply for morphisms of matroids. Recall the properties we need to establish as summarized in the key results:

$$\text{Theorem 5.2:} \quad \left\{ \begin{array}{l} \text{(Inh), (Pull) hold for } \mathbb{A} \\ v \in \Omega^+ \ \text{satisfies (Irr), (h-Pos)} \\ \text{(Hyp)} \ \ \text{holds for all } \ v^{\langle i \rangle} \in v^* \end{array} \right\} \quad \Rightarrow \quad \text{(Hyp) holds for } v.$$

$$\text{Theorem 6.1:} \quad \Big\{\text{(Inh), (Proj), (T-Inv), (K-Non)}\Big\} \quad \Rightarrow \quad \text{(Pull)}.$$

Now, observe that (Inh), (Proj), (T-Inv), and (K-Non) are closed properties, i.e. preserved under taking limits. Thus, they follow from the arguments in §8.2. On the other hand, properties (Irr) and (h-Pos) need to be verified separately, a the arguments in §8.2 use the strict positivity of q.

Lemma 13.1. *Let $v = (\alpha, m, t) \in \Omega^m$ be a non-sink vertex of the acyclic graph Γ defined above, where $\alpha \in X^*$, $|\alpha| \leq k-1-m$, $0 < m \leq k-1$, and $0 < t < 1$. Then v satisfies* (Irr) *and* (h-Pos).

Proof. For the second part, it follows from the definition of $\mathbf{h}_v$ that the vector is strictly positive for all $t \in (0,1)$. Thus, vertex v satisfies (h-Pos), for all $t \in (0,1)$.

For the first part, let $\mathbf{M}_v$ be the associated matrix of v. Without loss of generality, we can assume that $\alpha \in \mathcal{L}$, as otherwise $\mathbf{M}_v = 0$ and (Irr) holds trivially.

Claim: *Every $x, y \in X$ in the support of $\mathbf{M}_v$ belong to the same irreducible component of $\mathbf{M}_v$.*

When null is not in the support of $\mathbf{M}_v$, property (Irr) follows from the claim. Now assume that null is in the support of $\mathbf{M}_v$. By the Claim, it remains to show that null belong to the same irreducible component of *some* $x \in X$ in the support of $\mathbf{M}_v$.

Let $\alpha = x_1 \cdots x_\ell$, where $\ell \leq k-m-1$. By the assumption, there exits a subset $S \in \mathcal{B}_\Phi$ such that $\{x_1, \ldots, x_\ell\} \subset S$ and $|S| \in \{\ell+m-1, \ell+m, \ell+m+1\}$. To see this, observe that if null is in the support of $\mathbf{M}_v$, then either $\mathrm{A}(\alpha,m)_{\mathsf{null}\,\mathsf{null}} \neq 0$, or $\mathrm{A}(\alpha,m)_{\mathsf{null}\,x} \neq 0$, or $\mathrm{A}(\alpha,m+1)_{\mathsf{null}\,x} \neq 0$, for some $x \in X$.

By adding extra elements to S if necessary, without loss of generality we can assume that $|S| = \ell+m+1$. Let $x \in S \setminus \{x_1, \ldots, x_\ell\}$. Then we have $\mathrm{A}(\alpha, m+1)_{\mathsf{null}\,x} \neq 0$. This implies that x and null belong to the same irreducible component of $\mathbf{A}(\alpha, m+1)$. Since $0 < t < 1$, this implies that x and null belong to the same irreducible component of $\mathbf{M}_v$, and completes the proof of the lemma. □

Proof of Claim. Let $\ell = |\alpha|$, as above. Since x is contained in the support of $\mathbf{M}_v$, this implies that there exits $S \in \mathcal{B}_\Phi$ such that $\{x_1, \ldots, x_\ell, x\} \subset S$ and $|S| \in \{\ell+m, \ell+m+1, \ell+m+2\}$. Similarly, there exists $T \in \mathcal{B}_\Phi$ such that $\{x_1, \ldots, x_\ell, y\} \subset T$ and $|T| \in \{\ell+m, \ell+m+1, \ell+m+2\}$. By adding extra elements to S and T if necessary, without loss of generality we can assume that $|S| = |T| = \ell+m+2$.

For $S = T$, the claim follows immediately from the definition of $\mathbf{M}_v$ and q, since $\mathrm{A}(\alpha, m+1)_{xy} \neq 0$ in this case. So assume that $S \neq T$. By the exchange property for morphism of matroids (Proposition 4.8), there exists $z \in S \setminus T$ and $w \in T \setminus S$ such that $S - z + w \in \mathcal{B}_\Phi$.

Let $S' := S - z + w$. Note that $|S' \setminus \{x_1, \ldots, x_\ell, x, w\}| = m \geq 1$, and let $x' \in S' \setminus \{x_1, \ldots, x_\ell, x, w\}$. Note that $x' \in S \setminus \{x_1, \ldots, x_\ell\}$, which implies that $\mathrm{A}(\alpha, m+1)_{xx'} \neq 0$ in this case. Therefore, elements x and x' belongs to the same irreducible component of $\mathbf{A}(\alpha, m+1)$, and thus the same irreducible component of $\mathbf{M}_v$ since $0 < t < 1$. Note also that we have $|S' \cap T| > |S \cap T|$ by the construction of S'. Substitute $x \leftarrow x'$ and $S \leftarrow S'$, and iteratively apply the same argument, until the set S will eventually becomes T. This implies that x and y are contained in the same irreducible component of $\mathbf{M}_v$, as desired. □

13.2 All atlas vertices are hyperbolic

We first show that every sink vertex in $\mathbb{A}$ satisfies (Hyp). We then use Theorem 5.2 to obtain the result.

Let $\mathcal{G} = (X, \mathcal{L})$ be the greedoid corresponding to matroid $\mathcal{M} = (X, \mathcal{I})$. Let $\alpha = x_1 \cdots x_\ell \in \mathcal{L}$ of length $\ell := |\alpha| \leq k-1$, let $S := \{x_1, \ldots, x_\ell\}$, and let $\mathbf{A}(\alpha, 1)$ be the matrix defined in §8.1 for $\mathcal{G}$. Recall that

$$\omega(S) \ = \ \omega(x_1) \cdots \omega(x_\ell) \ = \ \frac{\mathrm{q}(x_1 \ldots x_\ell)}{c_\ell}.$$

For each $x \in X$, divide the x-row and x-column of $\mathbf{A}(\alpha, 1)$ by $\sqrt{c_{\ell+2}\ \omega(S)}\,\omega(x)$. Multiply the null-row and the null-column by $\frac{1}{c_{\ell+1}} \sqrt{\frac{c_{\ell+2}}{\omega(S)}}$. Denote by $\mathbf{B}$ the resulting matrix. Note that (Hyp) is preserved under this transformation, so it suffices to show that $\mathbf{B}$ satisfies (Hyp). Observe that $\mathbf{B} = \big(\mathrm{B}_{xy}\big)_{x,\, y \in \widehat{X}}$ is given by

$$\mathrm{B}_{xy} \ = \ \begin{cases} 1 & \text{if } \ S+x+y \in \mathcal{B} \\ 0 & \text{if } \ S+x+y \notin \mathcal{B} \end{cases} \qquad \text{for distinct } \ x, y \in X,$$

$$\mathrm{B}_{x\,\mathsf{null}} \ = \ \begin{cases} 1 & \text{if } \ S+x \in \mathcal{B} \\ 0 & \text{if } \ S+x \notin \mathcal{B} \end{cases} \qquad \text{for } \ x \in X\,,$$

$$\mathrm{B}_{\mathsf{null}\,\mathsf{null}} \ = \ \begin{cases} \dfrac{c_\ell\, c_{\ell+2}}{c_{\ell+1}^2} & \text{if } \ S \in \mathcal{B} \\ \qquad 0 & \text{if } \ S \notin \mathcal{B} \end{cases}$$

$$\mathrm{B}_{xx} \ = \ 0 \qquad \text{for } \ x \in X.$$

We now split the proof into three cases, each discussed as a separate lemma.

Lemma 13.2. *Suppose that* $g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N})$. *Then the matrix* $\mathbf{B}$ *satisfies* (Hyp).

Proof. By the assumption of the lemma, every independent set of $\mathcal{M}$ containing $\{x_1, \ldots, x_\ell\}$ is also a basis of Φ. It then follows that $\mathbf{B} = \big(\mathrm{B}_{xy}\big)$ is equal to

$$\mathrm{B}_{xy} \ = \ \mathrm{B}_{yx} \ = \ \begin{cases} 1 & \text{if } \ x, y \in \mathrm{Cont}(S) \ \text{ and } \ x \not\sim_S y, \\ 1 & \text{if } \ x \in \mathrm{Cont}(S) \ \text{ and } \ y = \mathsf{null}, \\ \frac{c_\ell\, c_{\ell+2}}{c_{\ell+1}^2} & \text{if } \ x = y = \mathsf{null}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $x \sim_\alpha y$, then x-row and x-column of $\mathbf{B}$ is equal to y-row and y-column of $\mathbf{B}$. Now, choose a representative element x_i for each equivalence class $\mathcal{C}_i$ in $\mathrm{Par}(\alpha)$. For every other y in $\mathcal{C}_i$, we subtract from the y-row and y-column of $\mathbf{A}$ the x_i-row and x_i-column of $\mathbf{A}$, respectively. Note that (Hyp) is preserved under these transformations. Restricting to the support, we obtain:

$$\mathbf{N} := \begin{pmatrix} 0 & 1 & \ldots & 1 & 1 \\ 1 & 0 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 0 & 1 \\ 1 & 1 & \ldots & 1 & \frac{c_\ell c_{\ell+2}}{c_{\ell+1}^2} \end{pmatrix},$$

where the rows and columns are indexed by $\{x_1,\ldots,x_m,\mathsf{null}\}$, with $m := |\operatorname{Par}(\alpha)|$. Now note that

$$\frac{c_\ell c_{\ell+2}}{c_{\ell+1}^2} \ = \ \begin{cases} 1 & \text{for } \ell < k-1, \\ 1 \, + \, \dfrac{1}{\mathrm{p}(k-1)\, - \, 1} & \text{for } \ell = k-1\,. \end{cases}$$

In both cases, we have:

$$1 \ \leq \ \frac{c_\ell c_{\ell+2}}{c_{\ell+1}^2} \ \leq \ 1 + \frac{1}{|\operatorname{Par}(\alpha)| - 1} \ = \ 1 + \frac{1}{m-1}\,. \tag{13.1}$$

This implies that matrix $\mathbf{N}$ satisfies the conditions in Lemma 8.6. Thus $\mathbf{N}$ is hyperbolic, as desired. □

Lemma 13.3. *Suppose that* $g\big(\Phi(S)\big) = \operatorname{rk}(\mathcal{N}) - 1$. *Then the matrix* $\mathbf{B}$ *satisfies* (Hyp).

Proof. By assumptions of the lemma, we can partition $\operatorname{Cont}(\alpha) := X_1 \cup X_2$ into two subsets:

$$\begin{aligned} X_1 \ &:= \ \big\{x \in \operatorname{Cont}(\alpha) \, : \, g\big(\Phi(S+x)\big) \ = \ \operatorname{rk}(\mathcal{N})\big\}, \\ X_2 \ &:= \ \big\{x \in \operatorname{Cont}(\alpha) \, : \, g\big(\Phi(S+x)\big) \ = \ \operatorname{rk}(\mathcal{N}) - 1\big\}. \end{aligned}$$

We now make the observations in three possible cases of $x, y \in X$:

(1) For every $x, y \in X_1$, we have $S+x+y \in \mathcal{B}$ if and only if $x \not\sim_S y$. This is because $\Phi(S+x+y) \supseteq \Phi(S+x)$, which implies that $\Phi(S+x+y)$ contains a basis of $\mathcal{N}$, and because $S+x+y \in \mathcal{I}$ if and only if $x \not\sim_S y$.

(2) For every $x \in X_1$ and $y \in X_2$, we have $S+x+y$ is a basis of Φ. This is because $\Phi(S+x+y) \supseteq \Phi(S+x)$, which implies that $\Phi(S+x+y)$ contains a basis of $\mathcal{N}$, and because

$$f(S+x+y) \, - \, f(S+y) \geq g\big(\Phi(S+x+y)\big) \, - \, g\big(\Phi(S+y)\big) \, = \operatorname{rk}(\mathcal{N}) - \big(\operatorname{rk}(\mathcal{N}) - 1\big) = 1,$$

which implies that $S+x+y \in \mathcal{I}$.

(3) For every $x, y \in X_2$, we have $S+x+y$ is not a basis of Φ. This is because $g\big(\Phi(S+x)\big) = g\big(\Phi(S+y)\big) \, = \, g\big(\Phi(S)\big) \, = \, \operatorname{rk}(\mathcal{N}) - 1$, which implies that $g\big(\Phi(S+x+y)\big) \, = \, \operatorname{rk}(\mathcal{N}) - 1$.

It follows from the observations above that

$$\mathrm{B}_{xy} \ = \ \mathrm{B}_{yx} \ = \ \begin{cases} 1 & \text{if } x, y \in X_1 \text{ and } x \not\sim_S y, \\ 1 & \text{if } x \in X_1 \text{ and } y \in X_2, \\ 1 & \text{if } x \in X_1 \text{ and } y = \mathsf{null}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for $x, y \in X_1$ and $x \sim_S y$, we have x-row (x-column) of $\mathbf{B}$ equal to y-row (y-column) of $\mathbf{B}$. Similarly, for $x, y \in X_2$, we have x-row (x-column) of $\mathbf{B}$ is equal to y-row (y-column) of $\mathbf{B}$.

Now let $x_1,\ldots,x_m$ be representatives of the equivalence classes under the relation "$\sim_S$" on X_1, and let y be a representative element of X_2. For every other $z \in X_1$ in the same equivalence class of x_i, we subtract from the z-row (z-column) of $\mathbf{B}$ the x_i-row (x_i-column) of $\mathbf{B}$. For every other $w \in X_2$, subtract from the w-row (w-column) of $\mathbf{B}$ the y-row (y-column) of $\mathbf{B}$. Recall that (Hyp) is preserved under these transformations.

By applying these transformations and restricting to the support, we obtain the following matrix:

$$\begin{pmatrix} 0 & 1 & \ldots & 1 & 1 \\ 1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & 1 \\ 1 & \ldots & 1 & 0 & 0 \\ 1 & \ldots & 1 & 0 & 0 \end{pmatrix},$$

where the rows and columns are indexed by $\{x_1,\ldots,x_m,y,\mathsf{null}\}$. The eigenvalues of this matrix are $\lambda_1 = m$, $\lambda_2 = 0$, $\lambda_3 = \ldots = \lambda_{m+2} = -1$. This implies that the matrix satisfies (OPE), and by Lemma 5.3 also (Hyp), as desired. □

Lemma 13.4. *Suppose that* $g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N}) - 2$. *Then the matrix* $\mathbf{B}$ *satisfies* (Hyp).

Proof. Let $H \subseteq X$ be given in (4.1), and let "$\sim_H$" be an equivalence relation defined by (4.2). Let us show that for every $x,y \in H$, we have:

$$S + x + y \in \mathcal{B} \quad \Longleftrightarrow \quad x \not\sim_H y. \tag{13.2}$$

The $\Rightarrow$ direction is clear, so it suffices to prove the $\Leftarrow$ direction. Let $x,y \in H$ such that $x \not\sim_H y$. Then we have:

$$f(S+x+y) - f(S) \;\geq\; g\big(\Phi(S+x+y)\big) - g\big(\Phi(S)\big) \;=\; \mathrm{rk}(\mathcal{N}) - \big(\mathrm{rk}(\mathcal{N}) - 2\big) \;=\; 2,$$

which implies that $S+x+y \in \mathcal{I}$. Since $\Phi(S+x+y)$ is a basis of $\mathcal{N}$ by assumption, it then follows that $S+x+y$ is a basis of Φ, as desired.

It then follows from the claim above that

$$\mathrm{B}_{xy} \;=\; \mathrm{B}_{yx} \;=\; \begin{cases} 1 & \text{if } x,y \in H \text{ and } x \not\sim_H y, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, if $x,y \in H$ and $x \sim_H y$, then x-row (x-column) of $\mathbf{B}$ is equal to y-row (y-column) of $\mathbf{B}$. Also note that, the support of $\mathbf{B}$ is contained in H.

Let $x_1,\ldots,x_m$ be the representatives of the equivalence classes $\mathcal{C}_1,\ldots,\mathcal{C}_m$ of the relation "$\sim_H$". For every other $y \in \mathcal{C}_i$, we subtract from the y-row (y-column) of $\mathbf{B}$ the x_i-row (x_i-column) of $\mathbf{B}$. By applying this transformation and restricting to the support of the resulting matrix, we obtain the following matrix:

$$\begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \ldots & 1 & 0 \end{pmatrix},$$

where the rows and columns are indexed by $\{x_1,\ldots,x_m\}$. The eigenvalues of this matrix are $\lambda_1 = m-1$, $\lambda_2 = \ldots = \lambda_m = -1$. This implies that the matrix satisfies (OPE), and by Lemma 5.3 also (Hyp), as desired. □

Lemma 13.5. *Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroid, let $1 \le k < \mathrm{rk}(\mathcal{M})$, and let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function. Let $\mathbb{A}$ be a combinatorial atlas corresponding to Φ. Then every sink vertex $v = (\alpha, 0, 1) \in \Omega^0$ satisfies (Hyp).*

Proof. Let $\alpha = x_1 \cdots x_\ell$ and $S = \{x_1,\ldots,x_\ell\}$. It suffices to show that $\mathbf{A}(\alpha,1)$ satisfies (Hyp). If $\alpha \notin \mathcal{L}$ or $g(S) < \mathrm{rk}(\mathcal{N}) - 2$, then $\mathbf{A}(\alpha,1)$ is equal to a zero matrix, so (Hyp) is trivially satisfied. Now suppose that $\alpha \in \mathcal{L}_{\mathcal{M}}$ and $g(S) \ge \mathrm{rk}(\mathcal{N}) - 2$. Then it follows from Lemma 13.2, Lemma 13.3 and Lemma 13.4, that $\mathbf{A}(\alpha,1)$ satisfies (Hyp). □

Lemma 13.6. *Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of matroid, let $1 \le k < \mathrm{rk}(\mathcal{M})$, and let $\omega : X \to \mathbb{R}_{>0}$ be a positive weight function. Let $\mathbb{A}$ be a combinatorial atlas corresponding to Φ. Then every vertex $v \in \Omega$ satisfies (Hyp).*

Proof. Let $v = (\alpha, m, t) \in \Omega^m$. We prove that v satisfies (Hyp) by induction on m. The claim is true for $m = 0$ by Lemma 13.5. Suppose that the claim is true for Ω^{m-1}. It then follows from Theorem 5.2 that every regular vertex in Ω^m satisfies (Hyp). On the other hand, by Lemma 13.1, the regular vertices of Ω^m contain those of the form $v = (\alpha, m, t)$, where $t \in (0,1)$. Since (Hyp) is a property that is preserved under taking limits $t \to 0$ and $t \to 1$, we conclude that every vertex in Ω^m satisfies (Hyp). This completes the proof. □

13.3 Proof of Theorem 1.16

Let $\mathbf{M}_v$ be the associated matrix of the vertex $v := (\varnothing, k-1, 1) \in \Omega$. Let $\mathbf{v}$ and $\mathbf{w}$ be the characteristic vector of X and $\{\textsf{null}\}$, respectively. Then

$$\begin{aligned} \langle \mathbf{w}, \mathbf{M}\mathbf{w} \rangle \ &= \ (k-1)! \cdot \mathrm{B}_\omega(k-1), \qquad \langle \mathbf{v}, \mathbf{M}\mathbf{w} \rangle \ = \ k! \cdot \mathrm{B}_\omega(k), \\ \text{and} \quad \langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle \ &= \ (k+1)! \left(1 + \frac{1}{\mathrm{p}(k-1)-1}\right) \cdot \mathrm{B}_\omega(k+1). \end{aligned} \tag{13.3}$$

Since v satisfies (Hyp) by Lemma 13.6, it then follows from the equations above that

$$\mathrm{B}_\omega(k)^2 \ \ge \ \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{\mathrm{p}(k-1)-1}\right) \cdot \mathrm{B}_\omega(k+1)\, \mathrm{B}_\omega(k-1),$$

as desired. □

13.4 Proof of Theorem 1.19

We first prove the $\Leftarrow$ direction. It follows from (MME3) that

$$\mathrm{B}_\omega(k+1) = \mathrm{I}_\omega(k+1), \quad \mathrm{B}_\omega(k) = \mathrm{I}_\omega(k), \quad \text{and} \quad \mathrm{B}_\omega(k-1) = \mathrm{I}_\omega(k-1), \tag{13.4}$$

where

$$\mathrm{I}_\omega(r) := \sum_{S \in \mathcal{I}_r} \omega(S).$$

Similarly, $\mathrm{p}(k-1)$ coincide for Φ and $\mathcal{M}$. Thus (MME1) is equivalent to (ME1) for $\mathcal{M}$, and (MME2) is equivalent to (ME2) for $\mathcal{M}$. It then follows from Theorem 1.6 that

$$\mathrm{I}_\omega(k)^2 = \left(1+\frac{1}{k}\right)\left(1+\frac{1}{\mathrm{p}(k-1)-1}\right)\mathrm{I}_\omega(k+1)\,\mathrm{I}_\omega(k-1),$$

which together with (13.4) proves the $\Leftarrow$ direction.

We now prove the $\Rightarrow$ direction. It follows from the same argument as in the $\Leftarrow$ direction, that it suffices to show that (MME3) is satisfied. Let $\mathbb{A}$ be the combinatorial atlas that corresponds to (Φ, k, ω) from §13.1. In particular, every vertex of Γ satisfies (Hyp) by Lemma 13.6.

As in §13.3, let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$ be the characteristic vector of X and $\{\mathsf{null}\}$, respectively. It is straightforward to verify that $\mathbf{v}, \mathbf{w}$ is a global pair for Γ, i.e. they satisfy (Glob-Pos).

Let $v = (\varnothing, k-1, 1) \in \Omega$ and let $\mathbf{M} = \mathbf{M}_v$ be the associated matrix. Note that $\mathrm{B}_\omega(k+1)$, $\mathrm{B}_\omega(k)$ and $\mathrm{B}_\omega(k-1) > 0$ by the assumption of the theorem. It then follows from (13.4) that v satisfies (s-Equ) for some $\mathrm{s} > 0$.

We now show that, for every $\alpha \in \mathcal{L}_{k-1}$ such that $\langle \mathbf{v}, \mathbf{A}(\alpha,1)\,\mathbf{v}\rangle > 0$, we have:

$$\langle \mathbf{v}, \mathbf{A}(\alpha,1)\,\mathbf{v}\rangle = \mathrm{s}\,\langle \mathbf{w}, \mathbf{A}(\alpha,1)\,\mathbf{v}\rangle = \mathrm{s}^2\,\langle \mathbf{w}, \mathbf{A}(\alpha,1)\,\mathbf{w}\rangle > 0. \tag{13.5}$$

First suppose that $k=1$. This implies that $\alpha = \varnothing$ and $v = (\varnothing, 0, 1)$. Thus, (13.5) follows from the fact that v satisfies (s-Equ).

Suppose now that $k > 1$. It is easy to see that v is a functional source in this case, i.e. it satisfies (Glob-Proj) and (h-Glob), where we apply the substitution $\mathbf{f} \leftarrow \mathbf{v}$ for (h-Glob). By Theorem 7.1, every functional target of v in Γ also satisfies (s-Equ) with the same $\mathrm{s} > 0$. On the other hand, observe that the functional targets of v in Ω^0 contain those of the form $(\alpha, 0, 1)$, with $\alpha \in \mathcal{L}_{k-1}$ satisfying $\langle \mathbf{v}, \mathbf{A}(\alpha,1)\,\mathbf{v}\rangle > 0$. Combining these two observations, we obtain (13.5).

Claim: *For every $T \in \mathcal{I}_{k-1}$, we have $g\big(\Phi(T)\big) \neq \mathrm{rk}(\mathcal{N}) - 1$.*

Proof. Let $T = \{y_1, \ldots, y_{k-1}\}$ and let $\beta = y_1 \cdots y_{k-1} \in \mathcal{L}$. For every $\ell \geq 0$, let

$$\mathrm{B}_{\omega,T}(\ell) := \sum_{\substack{S \in \mathcal{B}_{|T|+\ell} \\ S \supseteq T}} \omega(S).$$

Then:

$$\begin{aligned}\langle \mathbf{w}, \mathbf{A}(\beta,1)\,\mathbf{w}\rangle \ &= \ \mathrm{B}_{\omega,T}(0), \qquad \langle \mathbf{w}, \mathbf{A}(\beta,1)\,\mathbf{v}\rangle \ = \ \mathrm{B}_{\omega,T}(1),\\ \langle \mathbf{v}, \mathbf{A}(\beta,1)\,\mathbf{v}\rangle \ &= \ 2\left(1+\frac{1}{\mathrm{p}(0)-1}\right)\mathrm{B}_{\omega,T}(2).\end{aligned} \tag{13.6}$$

Now suppose to the contrary that $g\big(\Phi(T)\big) = \mathrm{rk}(\mathcal{N}) - 1$. Since $\mathrm{B}_\omega(k+1) > 0$, there is a basis $S \in \mathcal{B}_{k+1}$. Applying the exchange property for Φ, it follows that there exist $x, y \in S \setminus T$, such that $T \cup \{x,y\} \in \mathcal{B}_{k+1}$. This implies that $\mathrm{B}_{\omega,T}(2) > 0$, which in turn implies that $\langle \mathbf{v}, \mathbf{A}(\beta,1)\,\mathbf{v}\rangle > 0$ by (13.6). Hence (13.5) applies to β, which implies that $\langle \mathbf{w}, \mathbf{A}(\beta,1)\,\mathbf{w}\rangle > 0$. Again, by (13.6) we conclude that $\mathrm{B}_{\omega,T}(0) > 0$. This contradicts the assumption that $g\big(\Phi(T)\big) = \mathrm{rk}(\mathcal{N}) - 1$. This completes the proof of the claim. □

It remains to prove (MME3), i.e. that every $T \in \mathcal{I}_{k-1}$ satisfies $g\big(\Phi(T)\big) = \mathrm{rk}(\mathcal{N})$. Suppose to the contrary that $g\big(\Phi(T)\big) < \mathrm{rk}(\mathcal{N})$. Since $\mathrm{B}_\omega(k-1) > 0$, there is at least one basis $S \in \mathcal{B}_{k-1}$. By the exchange property of the matroid $\mathcal{M}$ the basis exchange graph is connected, i.e. there exist a sequence of bases $T_1, \ldots, T_m \in \mathcal{I}_{k-1}$, such that $\big|T_{i+1} \smallsetminus T_i\big| = 1$, $T_1 = T$, and $T_m = S$. Since $g\big(\Phi(T)\big) < \mathrm{rk}(\mathcal{N})$ and $g\big(\Phi(S)\big) = \mathrm{rk}(\mathcal{N})$, there exists $i \in [m]$ such that $g\big(\Phi(T_i)\big) = \mathrm{rk}(\mathcal{N}) - 1$. This contradicts the claim above, and completes the proof of (MME3). □

13.5 Proof of Theorem 1.18

The $\Leftarrow$ direction is straightforward. For the $\Rightarrow$ direction, it follows from (MME3) in Theorem 1.19, that for every $S \subseteq X$, $|S| = k-1$, the image $\Phi(S)$ contains a basis of $\mathcal{N}$. This implies that (13.4) holds. It then follows from Theorem 1.8, that every subset of X of size $k+1$ is independent, and the weight $\omega : X \to \mathbb{R}_{>0}$ is uniform. This completes the proof. □

14 Proof of log-concavity for linear extensions

In this section we give proofs of Theorem 1.35 and some variations of the results for posets with belts (§14.8). We also give an example of a combinatorial atlas in this case (§14.7).

14.1 New notation

In the next two sections we fix a ground set X and an element $z \in X$. Let $\mathcal{P} = (X_P, \prec)$ be a poset for which the ground set X_P is a subset of X. Let $k \in \{2, \ldots, |X_P| - 1\}$, and let $\omega : X \to \mathbb{R}_{>0}$ be the order-reversing weight function, see §1.16. We define a combinatorial atlas $\mathbb{A}(\mathcal{P}, k) := \mathbb{A}(\mathcal{P}, z, k, \omega)$ as follows.

Recall that $\mathcal{E}(\mathcal{P})$ denotes the set of linear extensions of $\mathcal{P}$. By a slight abuse of notation, in the next two sections a linear extension α of $\mathcal{P}$ is a simple word $\alpha := x_1 \ldots x_{|\alpha|} \in X_P^*$ of length $|X_P|$ such that $x_i \prec x_j$ in $\mathcal{P}$ implies that $i \leq j$. Denote by $\mathcal{E}(P,k)$ the set of linear extensions $\alpha \in \mathcal{E}(\mathcal{P})$ such that $\alpha_k = z$.

For a simple word $\alpha \in X^*$, we write $x \triangleleft z$ if x appears to the left of z in α. Following (1.31), for a word $\alpha \in X^*$, let

$$\omega(\alpha) \ := \ \prod_{x \triangleleft z} \omega(x),$$

and $\omega(S) := \sum_{\alpha \in S} \omega(\alpha)$ for every $S \subseteq X^*$.

Let $Z_{\text{down}} := X - z$ and denote every element in Z_{down} as x_{down} instead of x. Similarly, let $Z_{\text{up}} := X - z$, and denote every element in Z_{up} as x_{up} instead of x. Since Z_{down} Z_{up} are two copies of the same set, labels "down" and "up" are used to distinguish them. We write $Z := Z_{\text{down}} \cup Z_{\text{up}}$. Note that $d := |Z| = 2n - 2$ since Z_{down} and Z_{up} do not intersect because of the labeling. We will sometimes drop the "down" and "up" labels from x_{down} and x_{up} when the labels are either clear from the context or are irrelevant to the discussion. We denote by $\min(\mathcal{P}, \text{down}) \subseteq Z_{\text{down}}$ the set of elements of Z_{down} that correspond to minimal elements of $\mathcal{P}$, and by $\max(\mathcal{P}, \text{up}) \subseteq Z_{\text{up}}$ the set of elements of Z_{up} that correspond to maximal elements of $\mathcal{P}$. More generally, for a subset $S \subseteq X - z$, we denote by $S_{\text{down}} \subseteq Z_{\text{down}}$ the subset in Z_{down} that corresponds to S, and by $S_{\text{up}} \subseteq Z_{\text{up}}$ the subset in Z_{up} that corresponds to S.

Let $\mathcal{P}^{\text{op}} := (X, \prec^{\text{op}})$ denote the *opposite poset* of $\mathcal{P}$, defined by $x \prec^{\text{op}} y$ if and only if $y \prec x$.[14] For every $\alpha = x_1 \ldots x_\ell \in X^*$, we denote by $\alpha^{\text{op}} := x_\ell \ldots x_1$. Let $\mathcal{E}^{\text{op}}$ denote the set of linear extensions of $\mathcal{P}^{\text{op}}$, and note that $|\mathcal{E}^{\text{op}}| = |\mathcal{E}| = e(\mathcal{P})$. Denote by $\omega^{\text{op}} : X \to \mathbb{R}_{>0}$ the weight function defined by $\omega^{\text{op}}(x) := \omega(x)^{-1}$. Note that ω^{op} is an order-reversing weight function for $\mathcal{P}^{\text{op}}$. It then follows that

$$\mathrm{N}_{\omega}(\mathcal{P}, k) \ = \ \mathrm{N}_{\omega^{\text{op}}}(\mathcal{P}^{\text{op}}, |X_P| - k + 1) \prod_{x \in X - z} \omega(x). \tag{14.1}$$

In the subsequent two sections, we shall frequently utilize this technique of interchanging between $\mathcal{P}$ and $\mathcal{P}^{\text{op}}$ to streamline certain parts of the proofs.

14.2 Combinatorial atlas construction

We denote by $\mathbf{C}(\mathcal{P}, k) := \mathbf{C}(\mathcal{P}, k, \omega) := \big(\mathrm{C}_{xy}\big)_{x,y \in Z}$ the symmetric $d \times d$ matrix where,[15]

$$\begin{aligned}
\mathrm{C}_{xy} \ &:= \ \begin{cases} \omega(x)\, \omega(\mathcal{E}(\mathcal{P} - x - y, k-1)) & \text{if } x \in \min(\mathcal{P}, \text{down}),\ y \in \max(\mathcal{P}, \text{up}), \\ \omega(x)\, \omega(y)\, \omega(\mathcal{E}(\mathcal{P} - x - y, k-1)) & \text{if } x, y \in \min(\mathcal{P}, \text{down}),\ x \neq y, \\ \omega(\mathcal{E}(\mathcal{P} - x - y, k-1)) & \text{if } x, y \in \max(\mathcal{P}, \text{up}),\ x \neq y, \\ 0 & \text{otherwise}, \end{cases} \\
\mathrm{C}_{xx} \ &:= \sum_{y \in \min(\mathcal{P}-x, \text{down}),\ y \succ x} \omega(x) \omega(y)\, \omega(\mathcal{E}(\mathcal{P} - x - y, k-1)) \quad \text{for } x \in \min(\mathcal{P}, \text{down}), \\
\mathrm{C}_{xx} \ &:= \sum_{y \in \max(\mathcal{P}-x, \text{up}),\ y \prec x} \omega(\mathcal{E}(\mathcal{P} - x - y, k-1)) \quad \text{for } x \in \max(\mathcal{P}, \text{up}), \\
\mathrm{C}_{xx} \ &:= \ 0 \quad \text{for } x \notin \min(\mathcal{P}, \text{down}) \cup \max(\mathcal{P}, \text{up}).
\end{aligned} \tag{DefC-1}$$

[14] Sometimes, $\mathcal{P}^{\text{op}}$ is also called *dual* or *reverse poset*.

[15] Here $\omega(\mathcal{E}(\mathcal{P} - x - y, k-1))$ is the sum of ω-weight of all linear extensions α of $\mathcal{P} - x - y$ for which $\alpha_{k-1} = z$.

Equivalently, $\mathbf{C}(\mathcal{P},k)$ is given by[16]

$$\mathrm{C}_{xy} \;:=\; \mathrm{C}_{yx} \;:=\; \begin{cases} \omega\big(\{x\beta y \in \mathcal{E}(P,k)\}\big) & \text{if } \ x\in\min(\mathcal{P},\mathrm{down}),\ y\in\min(\mathcal{P},\mathrm{up}),\\ \omega\big(\{xy\beta \in \mathcal{E}(P,k+1)\}\big) & \text{if } \ x,y\in\min(\mathcal{P},\mathrm{down}),\, x\neq y,\\ \omega\big(\{\beta xy \in \mathcal{E}(P,k-1)\}\big) & \text{if } \ x,y\in\max(\mathcal{P},\mathrm{up}),\, x\neq y,\\ 0 & \text{if } \{x,y\}\nsubseteq \min(\mathcal{P},\mathrm{down})\cup\max(\mathcal{P},\mathrm{up}), \end{cases}$$
$$\mathrm{C}_{xx} \;:=\; \begin{cases} \omega\big(\{xy\beta \in \mathcal{E}(\mathcal{P},k+1) \mid y\succ x\}\big) & \text{if } \ x\in\min(\mathcal{P},\mathrm{down}),\\ \omega\big(\{\beta yx \in \mathcal{E}(\mathcal{P},k-1) \mid y\prec x\}\big) & \text{if } \ x\in\max(\mathcal{P},\mathrm{up}). \end{cases} \tag{DefC-2}$$

Note that both definitions will be frequently employed throughout the next two sections, chosen based on their suitability. Also note that it follows from the definition that $\mathbf{C}$ is a nonnegative symmetric matrix.

Note that, it follows from (DefC-2) that, for every $x\in\min(\mathcal{P},\mathrm{down})$,

$$\begin{aligned} \sum_{y\in Z_{\mathrm{down}}} \mathrm{C}_{xy} \;&=\; \omega\big(\{x\beta \mid \mathcal{E}(\mathcal{P},k+1)\}\big) \;=\; \omega(x)\,\omega\big(\mathcal{P}-x,k\big),\\ \sum_{y\in Z_{\mathrm{up}}} \mathrm{C}_{xy} \;&=\; \omega\big(\{x\beta \mid \mathcal{E}(\mathcal{P},k)\}\big) \;=\; \omega(x)\,\omega\big(\mathcal{P}-x,k-1\big). \end{aligned} \tag{14.2}$$

Similarly, for every $x\in\max(\mathcal{P},\mathrm{up})$,

$$\begin{aligned} \sum_{y\in Z_{\mathrm{down}}} \mathrm{C}_{xy} \;&=\; \omega\big(\{\beta\, x \mid \mathcal{E}(\mathcal{P},k)\}\big) \;=\; \omega\big(\mathcal{P}-x,k\big),\\ \sum_{y\in Z_{\mathrm{up}}} \mathrm{C}_{xy} \;&=\; \omega\big(\{\beta\, x \mid \mathcal{E}(\mathcal{P},k-1)\}\big) \;=\; \omega\big(\mathcal{P}-x,k-1\big). \end{aligned} \tag{14.3}$$

Let $\mathbf{f},\mathbf{g}\in\mathbb{R}^d$ be the indicator vector of Z_{down} and Z_{up}, respectively. It follows from (14.2) and (14.3) that

$$\begin{aligned} &\langle \mathbf{f}, \mathbf{C}(\mathcal{P},k)\,\mathbf{g}\rangle \;=\; \mathrm{N}_\omega(\mathcal{P},k), \qquad \langle \mathbf{f}, \mathbf{C}(\mathcal{P},k)\,\mathbf{f}\rangle \;=\; \mathrm{N}_\omega(\mathcal{P},k+1),\\ &\langle \mathbf{g}, \mathbf{C}(\mathcal{P},k)\,\mathbf{g}\rangle \;=\; \mathrm{N}_\omega(\mathcal{P},k-1), \end{aligned} \tag{Cfg}$$

where recall that $\mathrm{N}_\omega(\mathcal{P},k)$ is the sum of ω-weight of linear extensions of $\mathcal{P}$ such that z is the k-th smallest element.

Let $\Gamma := \Gamma(\mathcal{P},k) := (\Omega,\Theta)$ be the acyclic graph with $\Omega = \Omega^0\cup\Omega^1$, where

$$\Omega^1 \;:=\; \{t\in\mathbb{R} \mid 0\le t\le 1\}, \qquad \Omega^0 \;:=\; Z.$$

For a non-sink vertex $v=t\in\Omega^1$ and $x\in Z$, the corresponding outneighbor in Ω^0 is $v^{\langle x\rangle} := x$.

Define the combinatorial atlas $\mathbb{A}(\mathcal{P},k)$ of dimension d corresponding to poset $\mathcal{P}$, and $k\in\{3,\ldots,|X_P|-1\}$ by the acyclic graph Γ and the linear algebraic data defined as follows. For each vertex $v=x\in\Omega^0$, the associated matrix is

$$\mathbf{M}_v \;:=\; \begin{cases} \omega(x)\,\mathbf{C}(\mathcal{P}-x,k-1) & \text{if } x\in\min(\mathcal{P},\mathrm{down}),\\ \mathbf{C}(\mathcal{P}-x,k-1) & \text{if } x\in\max(\mathcal{P},\mathrm{up}), \end{cases}$$

[16] Here $\{x\beta y\in\mathcal{E}(P,k)\}$ is the set of linear extensions $\alpha\in\mathcal{E}(P,k)$ such that $\alpha_1 = x$ and $\alpha_{|X_{\mathcal{P}}|} = y$. The word $\beta\in X^*$ here denotes $\alpha_2\cdots\alpha_{|X_{\mathcal{P}}|-1}$.

and is equal to the zero matrix otherwise. For each vertex $v = t \in \Omega^1$, the associated matrix is

$$\mathbf{M} \,:=\, \mathbf{M}_v \,:=\, t\,\mathbf{C}(\mathcal{P},k) \,+\, (1-t)\,\mathbf{C}(\mathcal{P},k-1),$$

and the associated vector $\mathbf{h} := \mathbf{h}_v \in \mathbb{R}^d$ is defined to have coordinates

$$\mathrm{h}_x \,:=\, \begin{cases} t & \text{if } \ x \in Z_{\text{down}}\,, \\ 1-t & \text{if } \ x \in Z_{\text{up}}\,. \end{cases}$$

Finally, let the linear transformation $\mathbf{T}^{\langle x \rangle} : \mathbb{R}^d \to \mathbb{R}^d$ associated to the edge $(v, v^{\langle x \rangle})$, be

$$\big(\mathbf{T}^{\langle x \rangle}\,\mathbf{v}\big)_y \,:=\, \begin{cases} \mathrm{v}_y & \text{if } \ y \in \operatorname{supp}(\mathbf{M}), \\ \mathrm{v}_x & \text{if } \ y \in Z \setminus \operatorname{supp}(\mathbf{M}). \end{cases}$$

14.3 Properties of the matrix $\mathbf{C}(\mathcal{P},k)$

In this subsection we gather properties of the matrix $\mathbf{C}(\mathcal{P},k)$ that will be used in this paper.

Lemma 14.1. *Let $\mathcal{P}$ be a poset, and let and let $k \in \{2,\ldots,|X_P|-1\}$ such that* $\mathrm{N}(\mathcal{P},k) > 0$. *Then*

- *The support of $\boldsymbol{C}(\mathcal{P},k)$ is equal to* $\min(\mathcal{P},\mathrm{down}) \cup \max(\mathcal{P},\mathrm{up})$*, and*
- *The matrix $\boldsymbol{C}(\mathcal{P},k)$ is irreducible when restricted to the support.*

Proof. Let $n := |X_P|$. It follows from (DefC-1) that the support of $\mathbf{C}(\mathcal{P},k)$ is a subset of $\min(\mathcal{P},\mathrm{down}) \cup \max(\mathcal{P},\mathrm{up})$. Now note that, since $\mathrm{N}(\mathcal{P},k) > 0$, there exists a linear extension $\alpha = x_1 \cdots x_n \in \mathcal{E}(P,k)$, and note that $x_1 = (x_1)_{\text{down}} \in \min(\mathcal{P},\mathrm{down})$ and $x_n = (x_n)_{\text{up}} \in \max(\mathcal{P},\mathrm{up})$. Now, let y be an arbitrary element of $\min(\mathcal{P},\mathrm{down}) \cup \max(\mathcal{P},\mathrm{up})$. For the first claim it suffices to show that $y \in \operatorname{supp}(\mathbf{C}(\mathcal{P},k))$, and for the second claim it suffices to show that that y is contained in the same irreducible component (of the matrix $\mathbf{C}(\mathcal{P},k)$) as $(x_1)_{\text{down}}$ and $(x_n)_{\text{up}}$.

By switching to the dual poset in (14.1) if necessary, we will without loss of generality assume that $y = y_{\text{down}} \in \min(\mathcal{P},\mathrm{down})$. Let α' be the linear extension obtained from α by *demoting* y to be the smallest element, i.e.

$$\alpha' := y x_1 \cdots x_{i-1} x_{i+1} \cdots x_n, \qquad \text{where} \quad \alpha =: x_1 \cdots x_{i-1}\, y\, x_{i+1} \cdots x_n.$$

Note that α' is still a linear extension of $\mathcal{P}$ since y is a minimal element of $\mathcal{P}$. Now note that either $\alpha' \in \mathcal{E}(\mathcal{P},k)$ or $\alpha' \in \mathcal{E}(\mathcal{P},k+1)$. In the first case we then have $(\mathbf{C}(\mathcal{P},k))_{yx_n} > 0$, so y is contained in the support of $\mathbf{C}(\mathcal{P},k)$ and is in the same irreducible component as x_n. In the second case we then have $(\mathbf{C}(\mathcal{P},k))_{yx_1} > 0$, so y is contained in the support of $\mathbf{C}(\mathcal{P},k)$ and is in the same irreducible component at x_1. This completes the proof. $\square$

Lemma 14.2. *Let $\mathcal{P}$ be a poset, and let $k \in \{3, \ldots, |X_P| - 1\}$ such that $\mathrm{N}(\mathcal{P}, k) > 0$ and $\mathrm{N}(\mathcal{P}, k-1) > 0$. Then, for every $x \in \min(\mathcal{P}, \mathrm{down}) \cup \max(\mathcal{P}, \mathrm{up})$,*

$$\mathrm{N}(\mathcal{P} - x, k-1) > 0.$$

Proof. By switching to the dual poset in (14.1) if necessary, we will without loss of generality assume that $x = x_{\mathrm{down}} \in \min(\mathcal{P}, \mathrm{down})$. By assumption there exists linear extensions $\alpha \in \mathcal{E}(\mathcal{P}, k-1)$ and $\beta \in \mathcal{E}(\mathcal{P}, k)$. Let α' and β' be the linear extension of $\mathcal{P}$ obtained from α and β by demoting x to be the smallest element, respectively. It then follows that $\alpha' \in \mathcal{E}(\mathcal{P}, k-1) \cup \mathcal{E}(\mathcal{P}, k)$ and $\beta' \in \mathcal{E}(\mathcal{P}, k) \cup \mathcal{E}(\mathcal{P}, k+1)$. If $\alpha' \in \mathcal{E}(\mathcal{P}, k)$ then we are done, as removing the smallest element from α' (which is x) will give us a linear extension in $\mathcal{E}(\mathcal{P} - x, k-1)$. If $\beta' \in \mathcal{E}(\mathcal{P}, k)$ then we are also done, as removing the smallest element from β' (which is x) will give us a linear extension in $\mathcal{E}(\mathcal{P} - x, k-1)$. So we assume that $\alpha' \in \mathcal{E}(\mathcal{P}, k-1)$ and $\beta' \in \mathcal{E}(\mathcal{P}, k+1)$.

This assumption implies that there exists $y \in X_P$ which appears to the right of z in α', but appears to the left of z in β'. This in turn implies that y is incomparable to z in $\mathcal{P}$. Now, let j be the smallest integer in the set

$$\big\{ i : \; x'_i \parallel z \; \text{ in } \mathcal{P}, \, k \leq i \leq |X_P| \, \big\},$$

where $x'_1 \cdots x'_n := \alpha'$. Note that this set is non-empty by the preceding argument. Let γ be the linear extension of $\mathcal{P}$ obtained from α' by demoting x'_j to the $k-1$-th position. Then $\gamma \in \mathcal{E}(\mathcal{P}, k)$ and furthermore x is the smallest element in γ. Then, removing the smallest element of γ gives us a linear extension in $\mathcal{E}(\mathcal{P} - x, k-1)$, and the proof is complete. $\square$

Remark 14.3. The arguments in Lemma 14.1 and Lemma 14.2 are variations of the maximality argument that appears in the proof of Thm 8.9 in [CPP21]. We refer to [CP23, §12.3, §14.2] for a detailed survey.

14.4 Properties of the combinatorial atlas

We now show that the atlas $\mathbb{A}(\mathcal{P}, k)$ defined above, satisfies all four conditions in Theorem 6.1, namely properties (Inh), (Proj), (T-Inv) and (K-Non). We prove these properties one by one, in the following series of lemmas. For every lemma in this subsection we assume that $\mathcal{P} = (X_P, \prec)$ is a poset, and $k \in \{3, \ldots, |X_P| - 1\}$ such that $\mathrm{N}(\mathcal{P}, k) > 0$ and $\mathrm{N}(\mathcal{P}, k-1) > 0$.

Lemma 14.4. *The atlas $\mathbb{A}(\mathcal{P}, k)$ satisfies (Inh) and (Proj).*

Proof. Let $v = t \in \Omega^+$ be a non-sink vertex of Γ. The property (Proj) follows directly from the definition of $\mathbf{T}^{\langle x \rangle}$. For (Inh), let $x \in \mathrm{supp}(\mathbf{M})$. By linearity of $\mathbf{T}^{\langle x \rangle}$, it suffices to show that, for every $y \in Z$, we have:

$$\mathrm{M}_{xy} \;=\; \big\langle \mathbf{T}^{\langle x \rangle} \mathbf{e}_y, \mathbf{M}^{\langle x \rangle} \, \mathbf{T}^{\langle x \rangle} \, \mathbf{h} \big\rangle, \tag{14.4}$$

where $(\mathbf{e}_y)_{y \in Z}$ is the standard basis for $\mathbb{R}^d$. Note that we can assume $y \in \mathrm{supp}(\mathbf{M})$, as otherwise $\mathbf{M}\mathbf{e}_y = \mathbf{T}^{\langle x \rangle} \mathbf{e}_y = \mathbf{0}$, and (14.4) then follows trivially. It then follows from Lemma 14.1 that $x, y \in$

$\min(\mathcal{P},\mathrm{down}) \cup \max(\mathcal{P},\mathrm{up})$. Without loss of generality, assume that $x = x_{\mathrm{down}} \in Z_{\mathrm{down}}$ and $y = y_{\mathrm{down}} \in Z_{\mathrm{down}}$, as the proofs of the other cases are analogous.

We split the proof of (14.4) into two cases. First suppose that x and y are distinct. It then follows that $\mathbf{T}^{\langle x \rangle}\,\mathbf{e}_y \;=\; \mathbf{e}_y$, and

$$\begin{aligned}
\left\langle \mathbf{T}^{\langle x \rangle}\,\mathbf{e}_y, \mathbf{M}^{\langle x \rangle}\,\mathbf{T}^{\langle x \rangle}\,\mathbf{h} \right\rangle \;&=\; \sum_{u\in Z} \mathrm{M}^{\langle x \rangle}_{uy}\,\big(\mathbf{T}^{\langle x \rangle}\,\mathbf{h}\big)_u \\
&=\; \sum_{u\in Z_{\mathrm{down}}} \big(\mathbf{C}(\mathcal{P}-x,k-1)\big)_{uy}\,t \;+\; \sum_{u\in Z_{\mathrm{up}}} \big(\mathbf{C}(\mathcal{P}-x,k-1)\big)_{uy}\,(1-t) \\
&=_{(14.2)}\; \omega(x)\,\omega(y)\,\omega(\mathcal{E}(\mathcal{P}-x-y,k-1))\,t \;+\; \omega(x)\,\omega(y)\,\omega(\mathcal{E}(\mathcal{P}-x-y,k-2))\,(1-t) \\
&=_{(\text{DefC-1})}\; (\mathbf{C}(\mathcal{P},k))_{xy}\,t \;+\; (\mathbf{C}(\mathcal{P},k-1))_{xy}\,(1-t) \quad=\quad \mathrm{M}_{xy},
\end{aligned}$$

as desired.

Now suppose that $x = y$. Then

$$\begin{aligned}
\left\langle \mathbf{T}^{\langle x \rangle}\,\mathbf{e}_x, \mathbf{M}^{\langle x \rangle}\,\mathbf{T}^{\langle x \rangle}\,\mathbf{h} \right\rangle \;&=\; \sum_{\substack{w\in Z_{\mathrm{down}}\\ w\succ x}} \sum_{u\in Z} \mathrm{M}^{\langle x \rangle}_{uw}\,\big(\mathbf{T}^{\langle x \rangle}\,\mathbf{h}\big)_u \\
&=\; \sum_{\substack{w\in Z_{\mathrm{down}}\\ w\succ x}} \left(\sum_{u\in Z_{\mathrm{down}}} \big(\mathbf{C}(\mathcal{P}-x,k-1)\big)_{uw}\,t + \sum_{u\in Z_{\mathrm{up}}} \big(\mathbf{C}(\mathcal{P}-x,k-1)\big)_{uw}\,(1-t) \right) \\
&=_{(14.2)}\; \sum_{\substack{w\in Z_{\mathrm{down}}\\ w\succ x}} \omega(x)\,\omega(w)\,\omega(\mathcal{E}(\mathcal{P}-x-w,k-1))\,t \;+\; \omega(x)\,\omega(w)\,\omega(\mathcal{E}(\mathcal{P}-x-w,k-2))\,(1-t) \\
&=_{(\text{DefC-1})}\; \big(\mathbf{C}(\mathcal{P},k)\big)_{xx}\,t + \big(\mathbf{C}(\mathcal{P},k-1)\big)_{x,x}\,(1-t) \quad=\quad \mathrm{M}_{xx}\,,
\end{aligned}$$

which completes the proof. $\square$

Lemma 14.5. *The atlas* $\mathbb{A}(\mathcal{P},k)$ *satisfies* (T-Inv).

Proof. Let $v = t \in \Omega^+$ be a non-sink vertex, and let a,b,c be distinct elements of $\mathrm{supp}(\mathbf{M})$. It follows from Lemma 14.1 that $a,b,c \in \min(\mathcal{P},\mathrm{down}) \cup \max(\mathcal{P},\mathrm{up})$. Without loss of generaltiy assume that $a,b,c \in \min(\mathcal{P}, low)$. It then follows from (DefC-1) that

$$\mathrm{M}^{\langle a \rangle}_{bc} \;=\; \mathrm{M}^{\langle b \rangle}_{ca} \;=\; \mathrm{M}^{\langle c \rangle}_{ab} \;=\; \omega(a)\,\omega(b)\,\omega(c)\,\omega(\mathcal{E}(\mathcal{P}-a-b-c,k-2)),$$

and the lemma follows. $\square$

Lemma 14.6. *The atlas* $\mathbb{A}(\mathcal{P},k)$ *satisfies* (K-Non).

Proof. Let $v = t \in \Omega^+$ be a non-sink vertex. We need to check the condition (K-Non) for distinct $x,y \in \mathrm{supp}(\mathbf{M})$. It follows from Lemma 14.1 that $x,y \in \min(\mathcal{P},\mathrm{down}) \cup \max(\mathcal{P},\mathrm{up})$. We will without loss of generality assume that $x = x_{\mathrm{down}} \in \min(\mathcal{P},\mathrm{down})$ and $y = y_{\mathrm{down}} \in \min(\mathcal{P},\mathrm{down})$, as the proof of other cases are analogous.

It follows from (DefC-1) that

$$\mathrm{M}_{yy}^{\langle x\rangle} \;=\; \sum_{w\in\min(\mathcal{P}-x-y,\text{down}),\, w\succ y} \omega(x)\,\omega(y)\,\omega(w)\,\omega(\mathcal{E}(\mathcal{P}-x-y-w,k-2)).$$

Now note that, it follows from Lemma 14.1 that

$$\begin{aligned}\operatorname{supp}(\mathbf{M})-y \;&=\; (\min(\mathcal{P},\text{down})-y)\cup\max(\mathcal{P},\text{up}),\\ \operatorname{supp}\big(\mathbf{M}^{\langle y\rangle}\big) \;&=\; \min(\mathcal{P}-y,\text{down})\cup\max(\mathcal{P}-y,\text{up}),\end{aligned}$$

Then, the set $\mathrm{Fam}^{\langle y\rangle}$ defined in (6.1), in this case is equal to

$$\begin{aligned}\mathrm{Fam}^{\langle y\rangle} \;&=\; \operatorname{supp}\big(\mathbf{M}^{\langle y\rangle}\big)\setminus\big(\operatorname{supp}(\mathbf{M})-y\big) \;=\; \min(\mathcal{P}-y,\text{down})\setminus\min(\mathcal{P},\text{down})\\ &=\; \{w\in\min(\mathcal{P}-y,\text{down}) \mid w\succ y\}.\end{aligned}$$

This implies that

$$\begin{aligned}\sum_{w\in\mathrm{Fam}^{\langle y\rangle}} \mathrm{M}_{xw}^{\langle y\rangle} \;&=\; \sum_{w\in\mathrm{Fam}^{\langle y\rangle}} \omega(x)\,\omega(y)\,\omega(w)\,\omega(\mathcal{E}(\mathcal{P}-x--yw,k-2))\\ &=\; \sum_{w\in\min(\mathcal{P}-y,\text{down}),\, w\succ y} \omega(x)\,\omega(y)\,\omega(w)\omega(\mathcal{E}(\mathcal{P}-x-y-w,k-2)).\end{aligned}$$

Taking the difference of the two equations above, we get:

$$\mathrm{M}_{yy}^{\langle x\rangle} - \sum_{w\in\mathrm{Fam}^{\langle y\rangle}} \mathrm{M}_{xw}^{\langle y\rangle} \;=\; \sum_{w\in\min(\mathcal{P}-x-y,\text{down}),\, w\succ y,\, w\succ x} \omega(x)\,\omega(y)\,\omega(w)\,\omega(\mathcal{E}(\mathcal{P}-x-y-w,k-2)).$$

This is clearly nonnegative, and thus (K-Non) holds, as desired. □

Lemma 14.7. *Every $v\in\Omega^+$ satisfies (Irr). Furthermore, every $v=t\in\Omega^+$ satisfies (h-Pos), for all $0<t<1$.*

Proof. Property (Irr) follows directly from Lemma 14.1, and Property (h-Pos) follows from the observation that $\mathbf{h}_v$ is a positive vector when $t\in(0,1)$. □

14.5 Sink vertices are hyperbolic

Before we can apply the local-global principle, we need the following result:

Lemma 14.8. *Let $\mathcal{P}=(X_P,\prec)$ be a finite poset with $|X_P|=3$, let $\omega: X\to\mathbb{R}_{>0}$ be an order-reversing weight function. Then the matrix $\boldsymbol{C}(\mathcal{P},2)$ satisfies (Hyp).*

Proof. Let $\{x,y,z\} := X_P$. We index the rows and columns of $\mathbf{C}(\mathcal{P},2)$ with $\{x_{\text{down}}, y_{\text{down}}, x_{\text{up}}, y_{\text{up}}\}$.

We now split the proof of the lemma into seven cases, depending on the relative order of $\{x,y,z\}$. First, suppose that x,y,z are incomparable to each other. Then

$$\mathbf{C}(\mathcal{P},2) \quad = \quad \begin{pmatrix} 0 & \omega(x)\,\omega(y) & 0 & \omega(x) \\ \omega(x)\,\omega(y) & 0 & \omega(y) & 0 \\ 0 & \omega(y) & 0 & 1 \\ \omega(x) & 0 & 1 & 0 \end{pmatrix}. \tag{C1}$$

We now divide x_{down}-row and x_{down}-column by $\omega(x)$, and the y_{down}-row and the y_{down}-column by $\omega(y)$. Recall that (Hyp) is preserved under this transformation. Then the matrix becomes

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\{2,0,0,-2\}$. This implies that the matrix satisfies (OPE). By Lemma 5.3 we also have (Hyp), as desired.

Second, suppose that $x \prec y$, and z are incomparable to both elements. Then

$$\mathbf{C}(\mathcal{P},2) \quad = \quad \begin{pmatrix} \omega(x)\,\omega(y) & 0 & 0 & \omega(x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega(x) & 0 & 0 & 1 \end{pmatrix}. \tag{C2}$$

Restricting the rows and columns to the support $\{x_{\text{down}}, y_{\text{up}}\}$, we get

$$\begin{pmatrix} \omega(x)\,\omega(y) & \omega(x) \\ \omega(x) & 1 \end{pmatrix}.$$

This matrix has determinant

$$\omega(x)\,\big(\omega(y) - \omega(x)\big) \ \leq \ 0,$$

where the inequality follows from ω being order-reversing. This implies that the matrix satisfies (OPE), and thus also (Hyp), as desired.

In the remaining cases, element z is comparable to either x or y, or both. By symmetry, without loss of generality, we assume that $x \prec z$. Third, suppose that $x \prec z$, $x \prec y$, and $y \,||\, z$. Then:

$$\mathbf{C}(\mathcal{P},2) \quad = \quad \begin{pmatrix} \omega(x)\,\omega(y) & 0 & 0 & \omega(x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega(x) & 0 & 0 & 0 \end{pmatrix}. \tag{C3}$$

Restricting the rows and columns to the support $\{x_{\text{down}}, y_{\text{up}}\}$, we get

$$\begin{pmatrix} \omega(x)\,\omega(y) & \omega(x) \\ \omega(x) & 0 \end{pmatrix}.$$

This matrix has a negative determinant, so it satisfies (OPE). Thus, it also satisfies (Hyp), as desired.

Fourth, suppose that $x \prec z$, $x \,||\, y$, and $y \,||\, z$. Then:

$$\mathbf{C}(\mathcal{P},2) \quad = \quad \begin{pmatrix} 0 & \omega(x)\,\omega(y) & 0 & \omega(x) \\ \omega(x)\,\omega(y) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega(x) & 0 & 0 & 0 \end{pmatrix}. \tag{C4}$$

By restricting the rows and columns to the support $\{x_{\text{down}}, y_{\text{down}}, y_{\text{up}}\}$, followed by dividing the x_{down}-row and x_{down}-column by $\omega(x)$, and the y_{down}-row and the y_{down}-column by $\omega(y)$, we get

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of this matrix are $\{\sqrt{2}, 0, -\sqrt{2}\}$, so it satisfies (OPE). Thus it also satisfies (Hyp), as desired.

Fifth, suppose that $x \prec z$, $y \prec z$, and $x \,||\, y$. Then:

$$\mathbf{C}(\mathcal{P},2) \quad = \quad \begin{pmatrix} 0 & \omega(x)\,\omega(y) & 0 & 0 \\ \omega(x)\,\omega(y) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C5}$$

The eigenvalues of this matrix are $\{\,\omega(x)\,\omega(y), 0, 0, -\,\omega(x)\,\omega(y)\}$, so it satisfies (OPE). Thus, it also satisfies (Hyp), as desired.

For the sixth case, suppose that $x \prec z \prec y$. Then:

$$\mathbf{C}(\mathcal{P},2) \quad = \quad \begin{pmatrix} 0 & 0 & \omega(x) & 0 \\ 0 & 0 & 0 & 0 \\ \omega(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C6}$$

The eigenvalues of this matrix are $\{\,\omega(x), 0, 0, -\,\omega(x)\}$, so it satisfies (OPE). Thus it also satisfies (Hyp), as desired.

Seventh and final case, suppose that $x \prec y \prec z$. Then:

$$\mathbf{C}(\mathcal{P},2) \quad = \quad \begin{pmatrix} \omega(x)\,\omega(y) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C7}$$

The eigenvalues of this matrix are $\{\,\omega(x)\,\omega(y), 0, 0, 0\}$, so it satisfies (OPE). Thus it also satisfies (Hyp), as desired. This completes the proof. □

14.6 Proof of Theorem 1.34

We can now prove that the matrix $\mathbf{C}(\mathcal{P}, k)$ is always hyperbolic.

Proposition 14.9. *Let $\mathcal{P} = (X_P, \prec)$ be a finite poset, let $k \in \{2, \ldots, |X_P| - 1\}$, and let $\omega : X \to \mathbb{R}_{>0}$ be an order-reversing weight function. Then the matrix $\boldsymbol{C}(\mathcal{P}, k)$ satisfies* (Hyp).

Proof. We will prove the proposition by induction on $|X_P|$. The base case $|X_P| = 3$ follows from Lemma 14.8. Suppose that the claim is true for $|X_P| - 1$.

First note that, if $\mathrm{N}(\mathcal{P}, k-1) = \mathrm{N}(\mathcal{P}, k) = \mathrm{N}(\mathcal{P}, k+1) = 0$, then $\mathbf{C}(\mathcal{P}, k)$ is the zero matrix, and (Hyp) immediately follows. So we will assume that either one of $\mathrm{N}(\mathcal{P}, k-1), \mathrm{N}(\mathcal{P}, k), \mathrm{N}(\mathcal{P}, k+1)$ is nonzero.

We split the proof into case (1) and case (2): For case(1), suppose that at least two of the three numbers are nonzero. Since the sequence $\mathrm{N}(\mathcal{P}, k-1), \mathrm{N}(\mathcal{P}, k), \mathrm{N}(\mathcal{P}, k+1)$ cannot have internal zeroes (this follows from the demotion argument in the proof of Lemma 14.2), this reduces to either $\mathrm{N}(\mathcal{P}, k-1), \mathrm{N}(\mathcal{P}, k) > 0$ or $\mathrm{N}(\mathcal{P}, k+1), \mathrm{N}(\mathcal{P}, k) > 0$. By switching to the dual poset in (14.1) if necessary, we can without loss of generality assume that $\mathrm{N}(\mathcal{P}, k-1), \mathrm{N}(\mathcal{P}, k) > 0$. We split the proof further into case (1a), case (1b), and case (1c).

For case (1a), assume that $k \geq 3$. Let $\mathbb{A}(\mathcal{P}, k)$ be the atlas defined in §14.2. It follows from Lemma 14.4, 14.5, 14.6 that this atlas satisfies the assumptions of Theorem 5.2 (note that these lemmas require $k \geq 3$). Also note that every sink vertex in Ω^0 satisfies (Hyp) by the induction assumption, as they correspond to posets with cardinality $|X_P| - 1$. It then follows from Theorem 5.2 that every regular vertex in Ω^1 satisfies (Hyp). On the other hand, it follows from Lemma 14.7 that every $v = t \in \Omega^+$ with $0 < t < 1$ is a regular vertex. This implies that, for $0 < t < 1$, the matrix $t\,\mathbf{C}(\mathcal{P}, k) + (1-t)\,\mathbf{C}(\mathcal{P}, k-1)$ satisfies (Hyp). By taking the limit $t \to 0$ and $t \to 1$, we then conclude that both $\mathbf{C}(\mathcal{P}, k)$ and $\mathbf{C}(\mathcal{P}, k-1)$ satisfies (Hyp), as desired.

For case (1b), assume that $k = 2$ and $\mathrm{N}(\mathcal{P}, k+1) > 0$. Then by applying the same argument as in case (1a) to the atlas $\mathbb{A}(\mathcal{P}, k+1)$, it follows that both $\mathbf{C}(\mathcal{P}, k+1)$ and $\mathbf{C}(\mathcal{P}, k)$ satisfies (Hyp), as desired.

For case (1c), assume that $k = 2$ and $\mathrm{N}(\mathcal{P}, k+1) = 0$. The assumptions imply that X_P can be partitioned into $\{x\} \cup \{z\} \cup \{T\}$, where x is the only element in X_P incomparable to z, and T is the upper ideal of z in $\mathcal{P}$. Also note that the support of $\mathbf{C}(\mathcal{P}, k)$ is contained in $\{x_{\text{down}}\} \cup T_{\text{up}}$. Now suppose that there exists $y \in \min(T)$ such that $x \,||\, y$. Let $\mathcal{P}' := (X_P, \prec')$ be the poset with the same ground set as $\mathcal{P}$ and with $\prec'$ being obtained from $\prec$ by removing the relation $z \prec y$. Now note that $\mathrm{N}(\mathcal{P}, k+1) > 0$ by construction, so it follows from case (1b) that $\mathbf{C}(\mathcal{P}', k)$ satisfies (Hyp). On the other hand, it follows from the construction that $\mathbf{C}(\mathcal{P}, k)$ is equal to $\mathbf{C}(\mathcal{P}', k)$ when restricted to rows and columns indexed by $\{x_{\text{down}}\} \cup T_{\text{up}}$. Since (Hyp) is preserved under restricting to principal submatrices, we have $\mathbf{C}(\mathcal{P}, k)$ also satisfies (Hyp), as desired. Now suppose that every element T is ordered to be greater than x by $\prec$. This implies that, for every $y \in \max(\mathcal{P}, up) \cap T_{\text{up}}$,

$$\omega(\mathcal{E}(\mathcal{P} - x - y, k-1)) \;=\; \omega(\mathcal{E}(\mathcal{P} - y, k-1)),$$

because x is the second smallest element in every linear extension counted by $\mathcal{E}(\mathcal{P} - y, k-1) = \mathcal{E}(\mathcal{P} - y, 1)$. This implies that

$$\begin{aligned}(\mathbf{C}(\mathcal{P}, k))_{x_{\text{down}}, y_{\text{up}}} &=_{\text{(DefC-1)}} \omega(x)\, \omega(\mathcal{E}(\mathcal{P} - x - y, k-1)) \;=\; \omega(x)\, \omega(\mathcal{E}(\mathcal{P} - y, k-1)) \\ &=_{(14.3)} \omega(x) \sum_{w \in Z_{\text{up}}} (\mathbf{C}(\mathcal{P}, k))_{w, y_{\text{up}}}.\end{aligned}$$

Let $\mathbf{D}$ be the matrix obtained by deducting the x_{down}-row by the $\omega(x)$ times the sum of the other rows, followed by deducting the the x_{down}-column by the $\omega(x)$ times sum of the other columns (note that this operation preserves (Hyp)). Then it follows from the previous equation that the entries of $\mathbf{D}$ are given by

$$(\mathbf{D})_{uv} \;=\; \begin{cases} (\mathbf{C}(\mathcal{P},k))_{uv} & \text{if } u,v \in Z-x, \\ 0 & \text{if } u \in Z-x \text{ and } v=x, \\ -\omega(x)\,\omega(\mathcal{E}(\mathcal{P}-x,k-1)) & \text{if } u=v=x. \end{cases}$$

It then follows that $\mathbf{C}(\mathcal{P},k)$ satisfies (Hyp) if and only if, the restriction of $\mathbf{D}$ to rows and columns indexed by T_{up}, satisfies (Hyp). Now let $\mathcal{P}'$ be the induced subposet of $\mathcal{P}$ on $X_P - x$. Note that $\mathbf{C}(\mathcal{P}',k)$ satisfies (Hyp) by the induction assumption. Also note that $\mathbf{C}(\mathcal{P}',k)$ is equal to $\mathbf{D}$ when restricted to rows and columns indexed by T_{up}. Since (Hyp) is preserved under restricting to principal submatrices, it follows that this submatrix of $\mathbf{D}$ also satisfies (Hyp). Combined with previous observations, we then conclude that $\mathbf{C}(\mathcal{P},k)$ satisfies (Hyp), as desired. This completes the proof of case (1).

For case (2), suppose that exactly one of $\mathrm{N}(\mathcal{P},k-1), \mathrm{N}(\mathcal{P},k), \mathrm{N}(\mathcal{P},k+1)$ is nonzero. The proof then splits into three subcases. For case (2a), let $\mathrm{N}(\mathcal{P},k)=0$ while $\mathrm{N}(\mathcal{P},k-1)=\mathrm{N}(\mathcal{P},k+1)=0$. This implies that X_P can be partitioned into $S \cup T \cup \{x\}$, where S is the lower ideal of z in $\mathcal{P}$ and $|S|=k-1$, and T is the upper ideal of x in $\mathcal{P}$. Then the entries of $\mathbf{C}(\mathcal{P},k)$ is given by

$$(\mathbf{C}(\mathcal{P},k))_{x,y} \;=\; \begin{cases} \omega(x) & \text{if } x \in \min(\mathcal{P},\text{down}) \cap S_{\text{down}},\ y \in \max(\mathcal{P},\text{up}) \cap T_{\text{up}}, \\ 0 & \text{otherwise.} \end{cases}$$

By a direct computation, the eigenvalues of this matrix are $\lambda, -\lambda, 0, \ldots, 0$, where

$$\lambda \;:=\; |\max(\mathcal{P},\text{up}) \cap T_{\text{up}}| \sum_{x \in \min(\mathcal{P},\text{down}) \cap S_{\text{down}}} \omega(x).$$

Thus this matrix satisfies (OPE), and so it satisfies (Hyp).

For case (2b), let $\mathrm{N}(\mathcal{P},k+1)>0$ while $\mathrm{N}(\mathcal{P},k-1)=\mathrm{N}(\mathcal{P},k)=0$. Let S be the lower ideal of z in $\mathcal{P}$. This implies that $|S|=k$, and the support of $\mathbf{C}(\mathcal{P},k)$ is contained in S_{down}. Now let $\mathcal{P}' := (X_{\mathcal{P}'}, \prec')$ be the poset with ground set $X_{\mathcal{P}'} := S \cup \{z\} \subseteq X$, and with relations $\prec'$ given by

$$\begin{aligned} &\forall\, x,y \in S, && x \prec' y \quad\Longleftrightarrow\quad x \prec y, \\ &\forall\, x \in S, && x \,||\, z. \end{aligned}$$

It follows from the construction that, for all $x,y \in S_{\text{down}}$,

$$\big(\mathbf{C}(\mathcal{P},k)\big)_{x_{\text{down}},y_{\text{down}}} \;=\; \mathcal{E}(\mathcal{P}-S-x)\big(\mathbf{C}(\mathcal{P}',k)\big)_{x_{\text{down}},y_{\text{down}}}.$$

Since (Hyp) is a property that is preserved by restricting to principal submatrices, we have that $\mathbf{C}(\mathcal{P},k)$ satisfies (Hyp) if $\mathbf{C}(\mathcal{P}',k)$ also satisfies (Hyp). Also note that $\mathrm{N}(\mathcal{P}',k-1)>0$, $\mathrm{N}(\mathcal{P}',k)>0$. Thus by the same argument as in case (1), it follows that $\mathbf{C}(\mathcal{P}',k)$ satisfies (Hyp), which in turn implies that $\mathbf{C}(\mathcal{P},k)$ satisfies (Hyp).

Finally, for case (2c), let $\mathrm{N}(\mathcal{P},k-1)>0$ while $\mathrm{N}(\mathcal{P},k+1)=\mathrm{N}(\mathcal{P},k)=0$. This case follows by applying the same argument as in case (2b) to the dual poset $\mathcal{P}^{\text{op}}$ in (14.1). This completes the proof. $\square$

Proof of Theorem 1.35. It follows from Proposition 14.9 that the matrix $\mathbf{C}(\mathcal{P},k)$ satisfies (Hyp), and it follows from (Cfg) that the theorem is a special case of $\mathbf{C}(\mathcal{P},k)$ satisfying (Hyp). This completes the proof. $\square$

14.7 Example of a combinatorial atlas

Let $\mathcal{P}$ be the poset on $X = \{a,b,c,d,z\}$, with the order given by $a \prec b \prec c$, $a \prec z$, $d \prec c$. Fix $z \in \mathcal{P}$ as in Stanley's inequality, and with uniform weight on all linear extensions. Let $k = 3$. Then the matrices $\mathbf{C}(\mathcal{P},k)$ and $\mathbf{C}(\mathcal{P},k+1)$ are given by

$$\mathbf{C}(\mathcal{P},3) \;=\; \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{C}(\mathcal{P},4) \;=\; \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the rows and columns are labeled by $\{a_{\min}, b_{\min}, c_{\min}, d_{\min}, a_{\max}, b_{\max}, c_{\max}, d_{\max}\}$. In this notation, we have:

$$\mathbf{f} \;=\; (1,1,1,1,0,0,0,0)^{\intercal}, \qquad \mathbf{g} \;=\; (0,0,0,0,1,1,1,1)^{\intercal}.$$

Recall that the inner products of these two vectors with the matrix $\mathbf{C}(\mathcal{P},k)$ has the following combinatorial interpretation by (Cfg):

$$\langle \mathbf{f}, \mathbf{C}(\mathcal{P},3)\,\mathbf{f}\rangle \;=\; \mathrm{N}(4), \qquad \langle \mathbf{f}, \mathbf{C}(\mathcal{P},3)\,\mathbf{g}\rangle \;=\; \mathrm{N}(3), \qquad \langle \mathbf{g}, \mathbf{C}(\mathcal{P},3)\,\mathbf{g}\rangle \;=\; \mathrm{N}(2).$$

Stanley's inequality (1.30) is equivalent to

$$\langle \mathbf{f}, \mathbf{C}(\mathcal{P},k)\,\mathbf{g}\rangle^2 \;\geq\; \langle \mathbf{f}, \mathbf{C}(\mathcal{P},k)\,\mathbf{f}\rangle \cdot \langle \mathbf{g}, \mathbf{C}(\mathcal{P},k)\,\mathbf{g}\rangle. \tag{14.5}$$

In this example, we have:

$$\mathrm{N}(4) \;=\; 3, \qquad \mathrm{N}(3) \;=\; 3, \qquad \mathrm{N}(2) \;=\; 2,$$

and the log-concavity in Stanley's inequality holds: $3^2 \geq 3 \times 2$.

14.8 Posets with belts

Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements. We say that $\mathcal{P}$ has *belt* at $z \in X$ if $\mathrm{inc}(z)$ is either empty or a chain in $\mathcal{P}$. Note that $\mathrm{width}(\mathcal{P}) = 2$ if and only if $\mathcal{P}$ has a belt at every element $z \in X$. Below we show how to strengthen Theorem 1.35 for posets with belts.

Let $\omega : X \to \mathbb{R}_{>0}$ be an order-reversing weight function defined by (Rev), and fix element $z \in X$. Rather than use multiplicative formula (1.31) to extend ω to $\mathcal{E}$, we define $\omega : \mathcal{E} \to \mathbb{R}_{>0}$ by the *tropical formula*:

$$\mathrm{q}(L) := \max\big\{\, \omega(x) \,:\, L(x) < L(z) \,\big\}. \tag{14.6}$$

Theorem 14.10 (*Tropical Stanley inequality for posets with belts*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements, and suppose $\mathcal{P}$ has a belt at $z \in X$. Let $\omega : X \to \mathbb{R}_{>0}$ be a positive order-reversing weight function. Define $\mathrm{q} : \mathcal{E} \to \mathbb{R}_{>0}$ by the tropical formula* (14.6)*. Then, for every $1 < k < n$, we have:*

$$\mathrm{N_q}(k)^2 \;\geq\; \mathrm{N_q}(k-1)\cdot \mathrm{N_q}(k+1)\,, \tag{14.7}$$

where $\mathrm{N_q}(k)$ is defined by (1.32)*.*

More generally, let $\omega : \mathrm{Low}(\mathcal{P}) \to \mathbb{R}_{>0}$ be a weight function on the set of *lower ideals* of the poset $\mathcal{P}$. Suppose ω satisfies the following (*submodular property*):

$$\omega\big(S+x+y\big)\cdot \omega(S) \;\leq\; \omega\big(S+x\big)^2, \tag{Submod}$$

for all $x, y \in \mathrm{inc}(z)$, $x \prec y$, and for all $S \subset X$ such that $S, S+x, S+x+y \in \mathrm{Low}(\mathcal{P})$. We can then define

$$\mathrm{q}(L) := \omega(A), \quad \text{where} \quad A := \big\{x \in X \,:\, L(x) \prec L(z)\big\}. \tag{14.8}$$

Theorem 14.11 (*Submodular Stanley inequality for posets with belts*)**.** *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements, and suppose $\mathcal{P}$ has a belt at $z \in X$. Let $\omega : \mathrm{Low}(\mathcal{P}) \to \mathbb{R}_{>0}$ be a positive weight function on the set of lower ideals of $\mathcal{P}$ which satisfies* (Submod)*. Define $\mathrm{q} : \mathcal{E} \to \mathbb{R}_{>0}$ by* (14.8)*. Then, for every $1 < k < n$, we have:*

$$\mathrm{N_q}(k)^2 \;\geq\; \mathrm{N_q}(k-1)\cdot \mathrm{N_q}(k+1)\,, \quad \textit{where} \quad \mathrm{N_q}(m) := \sum_{L \in \mathcal{E}_m} \mathrm{q}(L)\,, \quad \textit{for all} \;\; 1 \leq m \leq n\,. \tag{14.9}$$

Proof of Theorem 14.11. The result follows the same argument as Theorem 1.35 with two changes. First, in the proof of Lemma 14.8, the case (C1) does not need to be verified since $\mathcal{P}$ has a belt. Second, the case (C2) is instead verified through (Submod). We omit the details. □

Proof of Theorem 14.10. The result is a direct consequence of Theorem 14.11, as the tropical weight function in (14.6) clearly satisfies (Submod). The details are straightforward. □

15 Proof of equality conditions for linear extensions

In this section we extend and prove Theorem 1.40, see also §16.22.

15.1 More equality

Let $\mathcal{P} = (X_P, \prec)$ be a poset on $|X_P| = n$ elements, and let $\omega : X \to \mathbb{R}_{>0}$ be the order-reversing weight. Recall that in the notation of this section, $\mathrm{N}_\omega(\mathcal{P}, k) = \mathrm{N}_\omega(k)$, with the latter as defined in (1.32).

We add two more items to Theorem 1.40 and reformulate it in terms of words, to prove a stronger result:

Theorem 15.1 (cf. Theorem 1.40)**.** *Let $\mathcal{P} = (X_P, \prec)$ be a poset with $|X_P| = n$ elements, and let $\omega : X_P \to \mathbb{R}_{>0}$ be a positive order-reversing weight function. Fix element $z \in X_P$. Suppose that $\mathrm{N}_\omega(\mathcal{P}, k) > 0$. Then the following are equivalent:*

(a) $\mathrm{N}_\omega(\mathcal{P}, k)^2 = \mathrm{N}_\omega(\mathcal{P}, k-1) \cdot \mathrm{N}_\omega(\mathcal{P}, k+1)$,

(b) *there exists* $\mathrm{s} = \mathrm{s}(k, z) > 0$, *s.t.*

$$\mathrm{N}_\omega(\mathcal{P}, k+1) = \mathrm{s}\,\mathrm{N}_\omega(\mathcal{P}, k) = \mathrm{s}^2\,\mathrm{N}_\omega(\mathcal{P}, k-1),$$

(c) *there exists* $\mathrm{s} = \mathrm{s}(k, z) > 0$, *s.t.*

$$\mathrm{N}_\omega(\mathcal{P} - S - T, 3) = \mathrm{s}\,\mathrm{N}_\omega(\mathcal{P} - S - T, 2) = \mathrm{s}^2\,\mathrm{N}_\omega(\mathcal{P} - S - T, 1)$$

for every lower set S and upper set T of $\mathcal{P} - z$ satisfying $|S| = k-2$, $|T| = n-k-1$.

(d) *there exists* $\mathrm{s} = \mathrm{s}(k, z) > 0$, *s.t.* $\omega(x_{k-1}) = \omega(x_{k+1}) = \mathrm{s}$, *and for every* $\gamma = x_1 \cdots x_n \in \mathcal{E}_k$, *we have* $z \,||\, x_{k-1}$, $z \,||\, x_{k+1}$.

(e) *there exists* $\mathrm{s} = \mathrm{s}(k, z) > 0$, *s.t.* $\omega(x_{k-1}) = \omega(x_{k+1}) = \mathrm{s}$, $f(x) > k$ *for all* $x \succ z$, *and* $g(x) > n-k+1$ *for all* $x \prec z$,

The direction (b) $\Rightarrow$ (a) is trivial. For the direction (c) $\Rightarrow$ (b), note that we have

$$\mathrm{N}_\omega(k) = \sum_{S,T} \omega(S)\,|\mathcal{E}(S)|\,|\mathcal{E}(T)|\,\mathrm{N}_\omega(\mathcal{P} - S - T, 2), \tag{15.1}$$

summed over all lower sets S and upper sets T of $\mathcal{P} - z$ satisfying $|S| = k-2$, $|T| = n-k-1$. Note that the analogous formulas also hold for $\mathrm{N}_\omega(k \pm 1)$. Together with (c), this implies that

$$\mathrm{N}_\omega(k) \le \mathrm{s}\,\mathrm{N}_\omega(k-1), \qquad \mathrm{N}_\omega(k) \le \frac{1}{\mathrm{s}}\,\mathrm{N}_\omega(k+1). \tag{15.2}$$

This in turn implies that $\mathrm{N}_\omega(k)^2 \le \mathrm{N}_\omega(k-1)\,\mathrm{N}_\omega(k+1)$. On the other hand, by Theorem 1.35 we already know the inequality in the opposite direction: $\mathrm{N}_\omega(k)^2 \ge \mathrm{N}_\omega(k-1)\,\mathrm{N}_\omega(k+1)$. This implies the equality in (15.2), which in turn implies (b), as desired.

Below we prove (a) $\Rightarrow$ (c), (c) $\Leftrightarrow$ (d), and (d) $\Leftrightarrow$ (e), thus completing the proof of Theorem 15.1.

15.2 Proof of (a) ⇒ (c)

We start with the following preliminary result.

Lemma 15.2. *Let $\mathcal{P} = (X_P, \prec)$ be a finite poset, and let $\omega : X \to \mathbb{R}_{>0}$ be an order-reversing weight function, and let $k \in \{3, \ldots, |X_P| - 1\}$. Suppose that there exists $\mathrm{s} > 0$, such that*

$$\mathrm{N}_\omega(\mathcal{P}, k+1) \ = \ \mathrm{s}\,\mathrm{N}_\omega(\mathcal{P}, k) \ = \ \mathrm{s}^2\,\mathrm{N}_\omega(\mathcal{P}, k-1) \ > \ 0.$$

Then, for every $x \in \min(\mathcal{P})$,

$$\mathrm{N}_\omega(\mathcal{P} - x, k) \ = \ \mathrm{s}\,\mathrm{N}_\omega(\mathcal{P} - x, k-1) \ = \ \mathrm{s}^2\,\mathrm{N}_\omega(\mathcal{P} - x, k-2) \ > \ 0. \tag{15.3}$$

Proof. Let $\mathbb{A}(\mathcal{P}, k)$ be the combinatorial atlas defined in §14.2. Let $\mathbf{f}, \mathbf{g} \in \mathbb{R}^d$ be the characteristic vectors of Z_{down} and Z_{up}, respectively. Clearly, $\mathbf{f}, \mathbf{g}$ is a global pair for $\mathbb{A}$, i.e., they satisfy (Glob-Pos). This allows us to apply Theorem 7.1 in the reductions below.

Let $v = t = 1 \in \Omega^1$. It then follows from the assumptions of the lemma and (Cfg) that the vertex v satisfies (s-Equ). Also note that v satisfies (Hyp) by Proposition 14.9. On the other hand, it is straightforward to verify that v is a functional vertex of Γ, i.e. it satisfies (Glob-Proj) and (h-Glob). By Theorem 7.1, every functional target of v also satisfies (s-Equ) with the same $\mathrm{s} > 0$. On the other hand, it is easy to see that the functional targets of v include vertices of the form $x \in \Omega^0$, where $x = x_{\text{down}} \in \min(\mathcal{P}, \text{down})$. Hence x satisfies (s-Equ), which implies that

$$\langle \mathbf{f}, \mathbf{C}(\mathcal{P} - x, k-1)\,\mathbf{f} \rangle \ = \ \mathrm{s}\,\langle \mathbf{f}, \mathbf{C}(\mathcal{P} - x, k-1)\,\mathbf{g} \rangle \ = \ \mathrm{s}^2\,\langle \mathbf{g}, \mathbf{C}((\mathcal{P} - x, k-1)\,\mathbf{g} \rangle.$$

It now follows from (Cfg) that

$$\mathrm{N}_\omega(\mathcal{P} - x, k) \ = \ s\,\mathrm{N}_\omega(\mathcal{P} - x, k-1) \ = \ s^2\,\mathrm{N}_\omega(\mathcal{P} - x, k-2).$$

Also note that $\mathrm{N}_\omega(\mathcal{P} - x, k-1) > 0$ by Lemma 14.1. The proof is now complete. □

We can now prove (a) ⇒ (c). Since $\mathrm{N}_\omega(\mathcal{P}, k) > 0$, it follows from (a) that

$$\mathrm{N}_\omega(\mathcal{P}, k+1) \ = \ \mathrm{s}\,\mathrm{N}_\omega(\mathcal{P}, k) \ = \ \mathrm{s}^2\,\mathrm{N}_\omega(\mathcal{P}, k-1) \ > \ 0 \qquad \text{for} \qquad \mathrm{s} := \frac{\mathrm{N}_\omega(k+1)}{\mathrm{N}_\omega(k)} \ > \ 0.$$

By applying Lemma 15.2 for $k-2$ many times, we have

$$\mathrm{N}_\omega(\mathcal{P} - S, 3) \ = \ \mathrm{s}\,\mathrm{N}_\omega(\mathcal{P} - S, 2) \ = \ \mathrm{s}^2\,\mathrm{N}_\omega(\mathcal{P} - S, 1) \ > \ 0, \tag{15.4}$$

for every lower set S of $\mathcal{P} - z$ satisfing $|S| = k-2$. Recall that $n := |X_P|$. Now note that, by applying (14.1), we have that (15.4) is equivalent to

$$\mathrm{N}_{\omega^{\text{op}}}(\mathcal{P}^{\text{op}} - S, n-k) \ = \ \mathrm{s}\,\mathrm{N}_{\omega^{\text{op}}}(\mathcal{P}^{\text{op}} - S, n-k+1) \ = \ \mathrm{s}^2\,\mathrm{N}_{\omega^{\text{op}}}(\mathcal{P}^{\text{op}} - S, n-k+2) \ > \ 0. \tag{15.5}$$

By applying Lemma 15.2 for another $n-k-1$ many times, (15.5) is equivalent to

$$\mathrm{N}_{\omega^{\mathrm{op}}}(\mathcal{P}^{\mathrm{op}}-S-T,1) \ = \ \mathrm{s}\,\mathrm{N}_{\omega^{\mathrm{op}}}(\mathcal{P}^{\mathrm{op}}-S-T,2) \ = \ \mathrm{s}^2\,\mathrm{N}_{\omega^{\mathrm{op}}}(\mathcal{P}^{\mathrm{op}}-S-T,3) \ > \ 0, \qquad (15.6)$$

for every upper set T of $\mathcal{P}-z$ satisfing $|T|=n-k-1$. Finally, by applying (14.1) again, it follows that (15.6) is equivalent to

$$\mathrm{N}_{\omega}(\mathcal{P}-S-T,3) \ = \ \mathrm{s}\,\mathrm{N}_{\omega}(\mathcal{P}-S-T,2) \ = \ \mathrm{s}^2\,\mathrm{N}_{\omega}(\mathcal{P}-S-T,1) \ > \ 0,$$

and the proof is now complete. □

15.3 Proof of (c) ⇒ (d)

Let $S:=\{x_1,\ldots,x_{k-2}\}$ and $T:=\{x_{k+2},\ldots,x_n\}$. Let $\mathbf{f},\mathbf{g}\in\mathbb{R}^d$ be the characteristic vectors of Z_{down} and Z_{up}, respectively. It follows from (Cfg) and (c) that

$$\langle \mathbf{f}, \mathbf{C}(\mathcal{P}-S-T,2)\,\mathbf{v}\rangle \ = \ \mathrm{s}\,\langle \mathbf{v}, \mathbf{C}(\mathcal{P}-S-T,2)\,\mathbf{v}\rangle \ = \ \mathrm{s}^2\,\langle \mathbf{w}, \mathbf{C}(\mathcal{P}-S-T,2)\,\mathbf{w}\rangle. \qquad (15.7)$$

for some $\mathrm{s}>0$.

Let $\mathbf{z}:=\mathbf{f}-\mathrm{s}\,\mathbf{g}$. It follows from (15.7) that $\langle \mathbf{z}, \mathbf{C}(\mathcal{P}-S-T,2)\,\mathbf{z}\rangle=0$. By Lemma 7.2, this implies $\mathbf{C}(\mathcal{P}-S-T,2)\,\mathbf{z}=\mathbf{0}$. On the other hand, the matrix $\mathbf{C}(\mathcal{P}-S-T,2)$ is one of the seven matrices in (C1)–(C7) because $\mathcal{P}-S-T$ is a poset with three elements. From the seven matrices, only (C1) and (C2) can have $\mathbf{C}(\mathcal{P}-S-T,2)\,\mathbf{z}=\mathbf{0}$. Now note that in both cases we have $x\,||\,z$ and $y\,||\,z$. By a direct calculation, in both cases we have:

$$\mathbf{C}(\mathcal{P}-S-T,2)\,\mathbf{z}=\mathbf{0} \quad \Longleftrightarrow \quad \mathrm{s}=\omega(x)=\omega(y).$$

This proves (d), as desired. □

15.4 Proof of (d) ⇒ (c)

It follows from (d), that, for every lower set S and upper set T of $\mathcal{P}-z$ satisfying $|S|=k-2$, $|T|=n-k-1.$, we have:

$$\mathrm{N}_{\omega}(\mathcal{P}-S-T,2) \ \le \ \mathrm{s}\,\mathrm{N}_{\omega}(\mathcal{P}-S-T,1) \qquad \text{and} \qquad \mathrm{N}_{\omega}(\mathcal{P}-S-T,2) \ \le \ \frac{1}{\mathrm{s}}\,\mathrm{N}_{\omega}(\mathcal{P}-S-T,3), \quad (15.8)$$

where $\mathrm{s}>0$ is given in (d). Summing over all such S,T as in (15.1), we obtain:

$$\mathrm{N}_{\omega}(k) \ \le \ \mathrm{s}\,\mathrm{N}_{\omega}(k-1), \qquad \mathrm{N}_{\omega}(k) \ \le \ \frac{1}{\mathrm{s}}\,\mathrm{N}_{\omega}(k+1). \qquad (15.9)$$

This implies that $\mathrm{N}_{\omega}(k)^2 \le \mathrm{N}_{\omega}(k-1)\,\mathrm{N}_{\omega}(k+1)$. On the other hand, by Theorem 1.35 we already know the inequality in the opposite direction: $\mathrm{N}_{\omega}(k)^2 \ge \mathrm{N}_{\omega}(k-1)\,\mathrm{N}_{\omega}(k+1)$. This implies the equality in (15.9), which in turn implies the equality in (15.8), as desired. □

15.5 Proof of (d) $\Leftrightarrow$ (e)

Note that both items have the same weight function assumption, which reduces the claim to the following lemma of independent interest.

Lemma 15.3. *Let $\mathcal{P} = (X, \prec)$ be a poset with $|X| = n$ elements. Fix element $z \in X$ and suppose that* $\mathrm{N}(k) > 0$. *Then the following are equivalent:*

(i) *$f(y) > k$ for all $y \succ z$, and $g(y) > n-k+1$ for all $y \prec z$.*

(ii) *for every $\gamma = x_1 \cdots x_n \in \mathcal{E}_k$, we have $z \,||\, x_{k-1}$ and $z \,||\, x_{k+1}$.*

Proof. We first prove (i) $\Rightarrow$ (ii). Suppose to the contrary that z is comparable to $y := x_{k+1}$. Then $z \prec y$, and it follows from (i) that $f(y) > k$. This implies that there are at least $(k+1)$ elements in γ that appear before y, contradicting the assumption that $y = x_{k+1}$. An analogous argument shows that z is incomparable to x_{k-1}.

We now prove (ii) $\Rightarrow$ (i). Let $y \in X$ be such that $z \prec y$, and suppose to the contrary that $f(y) \le k$. Let $Q \subseteq X$ be given by

$$Q := \{x \in X : x \prec z, x \prec y\}.$$

Note that $|Q| \le f(y) - 1 \le k-1$, and that Q is a lower ideal of $\mathcal{P}$. Let $R \subseteq X$ be given by

$$R := \big\{x \in X : x \prec z \text{ or } x \,||\, z\big\}.$$

Note that R is a lower ideal of $\mathcal{P}$, that $z, y \notin R$, and that $Q \subseteq R$. Also note $|R| = n - g(z) - 1$. Since $g(z) \le n-k$ by the assumption that $\mathrm{N}(k) > 0$, it follows that $|R| \ge k-1$.

We conclude that there exists a lower ideal U of $\mathcal{P}$ such that $Q \subseteq U \subseteq R$ and $|U| = k-1$. This in turn implies that there exists a linear extension $\gamma = x_1 \cdots x_n \in \mathcal{E}$, such that

$$U = \{x_1, \ldots, x_{k-1}\}, \quad x_k = z, \quad x_{k+1} = y.$$

It then follows from (ii) that z and y are incomparable, and we get a contradiction. The same argument shows that $g(y) > n-k+1$ for all $y \prec z$. This completes the proof of the lemma. $\square$

16 Historical remarks

16.1

Unimodality is surprisingly difficult to establish even in some classical cases. For example, Sylvester in 1878 famously resolved Cayley's 1856 conjecture on unimodality of q-binomial coefficients $\binom{n}{k}_q$ using representations of $\mathrm{SL}(2,\mathbb{C})$, see [Sylv]. In 1982, a linear algebraic deconstruction was obtained by Proctor [Pro82]. The first purely combinatorial proof was obtained O'Hara's [O'H90] only in 1990, while the strict unimodality for $k, n-k \ge 8$ was proved in 2013, by the second author and Panova [PP13].

Log-concavity is an even harder property to establish. Over the years, a number of tools and techniques for log-concavity were found, across many areas of mathematics and applications, from elementary combinatorial to analytic, from Lie theoretic to topological. As Huh points out in [Huh18], sometimes there is only one known approach to the problem. We refer to surveys [Bre89, Bre94, Sta89] for an overview of classical unimodality and log-concavity results, to [Brä15] for a more recent overview emphasizing enumerative results and analytic methods, and to [SW14] for a survey on the role of log-concavity in analysis and probability.

16.2

Mason's matroid log-concavity conjectures were stated in [Mas72], motivated by the earlier work and conjectures in graph theory and combinatorial geometry. Many more similar and related conjectures were stated over the years. Some of them became famous quickly, and some were proved quickly, see e.g. a celebrated paper by Heilmann and Lieb [HL72] on log-concavity of the *matching polynomial* for a graph. On the other hand, *Rota's unimodality conjecture* was mentioned in passing in [Rota70], reiterated in [RH71, p. 209], generalized to log-concavity by Mason and Welsh, and proved only recently (Theorem 1.1). We refer to [Oxl92, §14.2] for a detailed overview of the early work on the subject.

16.3

In modern times, the algebraic approach was pioneered by Stanley, who used the *hard Lefschetz theorem* to establish the *Sperner property* of certain families of posets [Sta80b]. This easily implied the *Erdős–Moser conjecture* and laid ground for many recent developments. In fact, Stanley's approach was itself a rethinking of Sylvester's proof we mentioned above, see [Sta80a], and it was later deconstructed in [Pro82].

In the past decade, Huh and coauthors pushed the algebraic approach to resolve several conjectures which remained open for decades. They established the hard Lefschetz theorem and the Hodge–Riemann relations in a number of algebraic settings, which imply the log-concavity results. We will not attempt to review this work largely because it is thoroughly surveyed in Huh's ICM survey [Huh18]. Below is a quick recap of results used directly in this paper.

16.4

Matroids are often associated with several important sequences, including the f-vector whose components are the numbers $\mathrm{I}(k)$, and the h-vector, which can be computed by a certain linear transformation of the f-vector. Both are coefficients of specializations of the *Tutte polynomial* associated with the matroid. We refer to [Bry82, BO92] for the introduction and further references.

16.5

In their celebrated paper [AHK18], Adiprasito, Huh and Katz proved the log-concavity of the *characteristic polynomial* of a matroid, which is a generalization of the graph chromatic polynomial, and a specialization of the Tutte polynomial. They deduce the Welsh–Mason Conjecture (1.1) indirectly, via an observation by Brylawski [Bry77] (see also [Lenz13]). This culminated a series of previous papers [Huh12, Huh15, HK12] on the subject (see also [AS16]).

The inequality (1.3) is the strongest of the *Mason's conjectures* [Mas72]. This inequality was recently proved independently by Brändén and Huh [BH18, BH20], and by Anari et. al [ALOV18] in the third paper of the series. These papers use interrelated ideas, and avoid much of the algebraic technology in [AHK18]. Let us mention a notable application in [ALOV19] which proved that the base exchange random walk mixes in polynomial time. This was yet another long standing open problem in the area [FM92].

16.6

Brylawski [Bry82, §6] and Dawson [Daw84, Conj. 2.5] conjectured that matroid h-vectors are log-concave. This was resolved in [ADH20] and [BST20]. The latter paper proves a stronger version of log-concavity, while the former proves further results for the *no broken circuit* (NBC) complex, another popular matroid construction, see [Bry77].

For how log-concavity of h-vectors implies log-concavity of f-vectors, see e.g. [Bre94, Cor. 8.4], [Bry82, Prop. 6.13], and [Daw84, Prop. 2.7][17]. As we mentioned in the Introduction (see §1.4), Lenz [Lenz11] showed that log-concavity of the h-vector implies *strict log-concavity* of the f-vector. See also [DKK12] for many low-dimensional examples.

16.7

The matroid in Example 1.12 is a special case of a matroid *realizable* over $\mathbb{F}_q$, see e.g. [Oxl92, §6.5]. In Example 1.14, we consider a subclass of *paving matroids* defined as matroids $\mathcal{M}$ with $\text{girth}(\mathcal{M}) = \text{rk}(\mathcal{M})$, see [Wel76]. Our construction of Steiner matroids follows [Jer06, Kahn80]. Notably, Jerrum considers matroid corresponding to $\text{Stn}(5, 8, 24)$. We refer to [Dem68] for more on finite geometries arising in this example.

16.8

Theorem 1.15 for morphism of matroids is proved by Eur and Huh in [EH20], by extending the approach in [BH20]. The notion of the morphisms is quite elegant, and follows a long series of combinatorial papers of Las Vergnas on the subject, which includes a definition of the Tutte polynomial in this case. We refer to [EH20] for an overview and many references, and to [Chm21] for the extensive survey of generalizations of the Tutte polynomial to general topological embeddings.

16.9

Discrete polymatroids are also called *integral polymatroids* in [Edm70], and appear in the context of discrete convex sets [Mur03] and integral generalized permutohedra [Pos09]. We refer to [HH02] for their history and algebraic motivation. Note that discrete polymatroids are explicitly treated in [Mur03, §4.1] and [BH20] under the equivalent formulation of *M-convex sets*. They are a part the definition of Lorentzian polynomials, so in fact weighted polymatroids and Lorentzian polynomials are closely related notions.[18] Although Theorem 1.20 is not stated in this form, it follows easily from the results in [BH20]. Indeed, we need Theorem 3.10 combined with taking derivatives and limits in proof of Theorem 4.14, and where Theorem 2.10 is substituted with Theorem 2.30 (all in [BH20]). The details are straightforward.

16.10

We refer to [BKP20, §14] and [Pos09, §12], for the background on hypergraphical polymatroids in Example 1.22, and further references. Note that there are many notions of "hypertree" and "hyperforest" available in the literature. We refer to [GP14, §10.2] for a quick overview, and to [Ber89] for background on hypergraphs and more traditional definitions.

16.11

The notion of *weight function* originates in statistical physics and is now standard in probability and graph theory. In the context of graph polynomials it comes up in connection to the *Potts model* which is equivalent to the *random cluster model*. We refer to [Sok05] for an extensive introduction, and to [Gri06] for a thorough treatment.

[17] There is an unfortunate typo in the statement of the Dawson's proposition.

[18] The *completely log-concave polynomials* considered in [ALOV18] are not necessarily homogeneous and thus more general; they coincide with Lorentzian polynomials in the homogeneous case, see [BH20, p. 826].

16.12

The equality conditions have long emerged an important counterpart to the inequalities, see e.g. [BB65, HLP52]. They serve as a key check on the inequality: if the equality occurs rarely or never, perhaps there is a way to sharpen the inequality either directly or by introducing additional parameters. *Strict log-concavity* inequalities are especially suggestive of possible quantitative results.

For example, in his pioneer paper [Huh12], Huh proved the log-concavity of the *chromatic polynomial* of a graph, establishing several conjectures going back to [Read68]. In a followup paper [Huh15], Huh proved a strict log-concavity conjecture of Hoggar [Hog74]. There are no explicit stronger bounds implying strict log-concavity in the style of Theorem 1.16 and [BST20].

In the opposite direction, when there are many special cases when the inequality becomes an equality, the equality conditions are unlikely to be very precise. It seems, this is the case of our equality conditions for matroid log-concavity given in §1.9 (see also §1.13). In the context of this paper, the only nontrivial equality condition known prior to this work for matroid inequalities is Theorem 1.8 proved by Murai, Nagaoka and Yazawa in [MNY21] using an algebraic argument built on [BH20].

16.13

Greedoids were defined and heavily studied by Korte and Lovász as set systems on which the *greedy algorithm* provably works, thus the name. They generalize matroids, which in turn generalize graphs, where the greedy algorithm is classically defined to compute the *minimal spanning tree* (MST). For general greedoids, the reader should think of the (greedy) *Prim's algorithm* for the MST in undirected graphs, rather than *Kruskal's algorithm*, as a starting point of the generalization. The approach to greedoids in terms of languages goes back to original papers. We refer to [KLS91] for a foundational monograph on the subject, and to [BZ92] for a relatively short and digestible survey.

16.14

Antimatroids is a subclass of greedoids named after the *anti-exchange property*, which is a key axiom in their definition via set systems [KLS91, §3.1]. There are many examples of antimatroids coming from graph theory (e.g. *branching process*) and discrete geometry (e.g. *shelling process*), although poset antimatroids have a combinatorial nature they also have some geometric aspects (see e.g. [KL13]). Much of the terminology in the area is rather unfortunate and can be somewhat confusing, so we refer the reader to the top of page 335 in [BZ92], which defines classes of greedoids in terms of properties of the corresponding lattices of feasible sets. See Figure 17.1 below for the diagram of relationships between main greedoid classes (see also [KLS91, p. 301] for a larger diagram).

16.15

Standard Young tableaux (see Example 1.27) are fundamental in algebraic combinatorics. They play a key role in representation theory of S_n and $\mathrm{GL}(N,\mathbb{C})$, and the geometry of the Grassmannian, see e.g. [Ful97] and [Sta99, §7]. Numbers $f^{\lambda/\mu} = |\mathrm{SYT}(\lambda/\mu)|$ have an elegant *Aitken–Feit determinant formula* [Ait43, Feit53], see also [Sta99, Cor. 7.16.3]. For the sequence $\{b_k\}$ in Example 1.27, see e.g. [FS09, Ex. VIII.5].

16.16

Enumeration of increasing arborescences (also called *branchings* and *search trees*) in Example 1.32 without graphical constraints is common in enumerative combinatorics, see e.g. [BBL98, FS09]. Maximal arborescences (also called *directed spanning trees*) also appear in connection to the *reachability problem* in network theory, see [BP83, GJ19], and can be sampled by the *loop-erased random walk* and its relatives, see [GP14, Wil96].

16.17

Linear extensions of a finite poset $\mathcal{P}$ are in obvious bijection with maximal chains in the lattice $\mathbb{L}(\mathcal{P})$ of lower order ideals of $\mathcal{P}$. Lattice $\mathbb{L}(\mathcal{P})$ is always distributive, and by *Birkhoff's representation theorem* (see e.g. [Sta99, Thm 3.4.1]), every finite distributive lattice can be obtained that way. We refer to [BrW00, Tro95] for definitions and standard results on posets and linear extensions.

16.18

Stanley's inequality (1.30) was originally conjectured by Chung, Fishburn and Graham in [CFG80], extending an earlier unimodality conjecture by R. Rivest (unpublished). The proof in [Sta81] is a simple application of the *Alexandrov–Fenchel inequality*. Until now, no direct combinatorial proof of Stanley's inequality was known in full generality, although [CFG80] gives a simple proof for posets of width two (see also [CPP21]). Most recently, the authors and Panova obtained a q- and multivariate analogues of Stanley's inequality for posets of width two [CPP21]. These notions are specific to the width two case and are incompatible with the weighted analogue (Theorem 1.35) nor the case of posets with belts (Theorem 14.11).

16.19

The connection between linear extensions of two dimensional posets and lower order ideals of Bruhat order used in Example 1.37 has been discovered a number of times in varying degree of generality, see [BW91, FW97] (see also [DP18]). Statistics $\beta : S_n \to \mathbb{N}$ in that example seems different from other permutation statistics which appear in the context of log-concavity, see e.g. [Brä15, Bre89].

Statistic γ on the alternating permutations in Example 1.38 is more classical. Note, however, a major difference: while much of the literature studies permutation statistics as polynomials in $\mathbb{N}[q]$ whose coefficients can sometimes form a log-concave sequence, we study values of these polynomials at fixed $q \in \mathbb{R}$. For more on the Euler and Bernoulli numbers and the connection between them, see e.g. [FS09, §IV.6.1]. For log-concavity of Entringer numbers and their generalizations, see [B+19, GHMY21].

16.20

The Alexandrov–Fenchel inequality is a classical result in convex geometry which remains mysterious despite a number of different proofs, see e.g. [BuZ88, §20] and [Sch14, §7.3]. It generalizes the *Brunn–Minkowski inequality* to *mixed volumes*, and has remarkable applications to the *van der Waerden conjecture*, see e.g. [vL82]. Let us single out one of the original proofs by Alexandrov using polytopes [Ale38], the inspirational (independent) proofs by Khovanskii and Teissier using Hodge theory, see [BuZ88, §27] and [Tei82], a recent concise analytic proof by Cordero-Erausquin et al. [CKMS19], and the proof by Shenfeld and van Handel [SvH19], which partly inspired this paper.

16.21

For geometric inequalities such as the *isoperimetric inequalities*, the equality conditions are classical problems going back to antiquity (see e.g. [BuZ88, Sch14]). In many cases, the equality conditions are equally important and are substantially harder to prove than the original inequalities. For example, in the *Brunn–Minkowski inequality*, the equality conditions are crucially used in the proof of the *Minkowski theorem* on existence of a polytope with given normals and facet volumes (see e.g. §7.7 and §36.1 in [Pak19]). For poset inequalities, the equality conditions are surveyed in [Win86].

16.22

The equality conditions for Stanley's inequality for the case when ω is uniform (Theorem 1.39), were recently obtained by Shenfeld and van Handel in [SvH20, Thm 15.3]. They used a sophisticated geometric analysis to prove equality conditions of the Alexandrov–Fenchel inequality for convex polytopes. We should mention that part (d) of Theorem 15.1 is inspired by our results in [CPP21, §8] for the Kahn–Saks inequality, which in some sense are more general. Finally, the q-analogue of the equality condition was obtained in the same paper [CPP21, Thm 1.5] for posets of width two.

17 Final remarks and open problems

17.1

Unimodality is so natural, sooner or later combinatorialists start seeing it everywhere, generating a flood of conjectures. In the spirit of the "strong law of small numbers" [Guy88], many such conjectures do in fact hold in small examples but fail in larger cases. Sometimes, it takes years of real or CPU time until large counterexamples are found (see e.g. [RR91]), in which case they are published. Notable unimodality disproofs can be found in [Bjö81, Stan90, Ste07], all related to poset inequalities in some way.

Log-concavity is a stronger property than unimodality, but is also more natural. Indeed, in the absence of symmetry there is no natural location of the mode (maximum) of the sequence. While the mode location is critical in establishing unimodality, it is irrelevant for log-concavity. Moreover, as was pointed out in [RT88, p. 38], log-concavity of polynomial coefficients is preserved under multiplication of polynomials, an important property of poset polynomials. Similarly, it was shown in [Lig97] (see also [Gur09]), that ultra-log-concavity is preserved under *convolution*, yet another property of some poset polynomials.

17.2

In his discussion of influence of Rota on matroid theory, Kung writes that Rota was motivated in his unimodality conjecture (see §16.2) in part by the mixed volumes which are "somewhat analogous" to the Whitney numbers, see [Kung95, §3.1]. This seems extremely prescient from the point of view of this paper, as we prove matroid log-concavity with a technology that originates in the "right" proof of the Alexandrov–Fenchel inequality. One could argue that we inadvertently fulfilled Rota's unstated prediction (cf. [AS16]).

17.3

As we mentioned in the introduction, traditionally matroids are viewed as a subclass of lattices, see e.g. [Oxl92, Wel76]. Similarly, greedoids are usually defined by their feasible sets a more general subclass of posets (cf. §16.13). Thus, the title of the paper.

17.4

Our proofs in Section 5 borrows heavily from [SvH19], although they are written in a very different language (see also Remark 5.4). According to the authors, the idea of this proof can be traced back to the work of Lichnerowicz [Lic58], see [SvH19, §6.3] for a further discussion.

The proof of Theorem 7.1 is a modification on the argument in [Ale38], which in turn is based on [Weyl]. In the draft of the paper, we were not aware of the connection and used a similar but longer argument. This simplification was kindly proposed to us by Ramon van Handel (personal communication).

17.5

In the proof of Theorem 1.31 given in §8.4, at a critical step (in the base of induction), we employed *Cauchy's interlacing theorem*. In fact, interlacing of eigenvalues is surprisingly powerful, see e.g. [Hua19, MSS15] for notable recent applications.

17.6

As we mentioned earlier, our proof of Theorem 1.35 is inspired by the approach of Shenfeld and van Handel [SvH19]. Indeed, the mixed volumes in Alexandrov–Fenchel inequality can be converted into inner products in (Hyp), where the vectors are given by the support functions of the polytopes. We present this proof in [CP22a]. Technically, one can object that we assumed the diagonal entries of $\mathbf{M}$ are assumed to be nonnegative. In fact, this assumption is made for convenience as nonnegativity holds in our examples, but allowing M_{ii} to be negative does not change the proof.

Now, it is shown in [SvH19, §5], that the corresponding matrices and vectors for simple, strongly-isomorphic polytopes satisfy all conditions of Theorem 5.2. Note that in that setting (Pull) is always an equality, see [SvH19, Eq. (1.2)] and [SvH20, Eq. (5.23)] for the proof. On the other hand, the inequality (Pull) can be strict in our setting. The comparison between our proof Theorem 15.1 and the proof of Theorem 1.39 in [SvH20] is also curious and we don't fully understand it. We should mention the crucial use of the opposite poset $\mathcal{P}^{\text{op}}$, which does not seem to show up in this context. It would be interesting to find further applications in this "duality" approach (cf. §17.15).

17.7

Although Theorem 1.8 says that there are no interesting examples of equality of log-concavity for matroids, the examples in §1.7 suggest that the family of matroids with equality in Theorem 1.6 is rather rich. While our Theorem 1.9 gives some natural necessary and sufficient conditions, it would be interesting to see if this description can be used to obtain a full classification of such matroids in terms of known classes of set systems.

17.8

Our work is completely independent of the algebraic approach in [AHK18], yet some glimpses of similarity to more recent developments are noticeable if one squints hard enough. For instance, we need the element null in the proof of Theorem 1.31, for roughly the same technical reason that papers [B+20a, B+20b] need to use the *augmented Bergman fan* in place of the (usual) *Bergman fan* employed in [AHK18].

17.9

The connection between our proof and Lorentzian polynomial approach is somewhat indirect to make any formal conclusions. On the one hand, we can use combinatorial atlases to emulate everything Lorentzian polynomial do [CP22a]. On the other hand, the atlas we construct for matroids and polymatroids is sufficiently flexible to allow our refined inequalities. On a technical level, in notation of Section 6, the matrix $\mathbf{K} = (\mathrm{K}_{ij})$ which arises when we emulate Lorentzian polynomials, is always zero (cf. Remark 6.3). Thus, it would be interesting to see if the tools in [ALOV18] and [BH20] can be modified to yield our Theorems 1.6 and 1.21.

17.10

Most recently, Brändén and Leake showed in [BL21] how to obtain the log-concavity of the characteristic polynomial of a matroid using a purely Lorentzian polynomial approach, avoiding the use of algebra altogether.

While it is too early to say, we intend to see if the combinatorial atlas technology can be combined with that approach.

17.11

It would be interesting to see if one can derive Theorems 1.35 from the Alexandrov–Fenchel inequality. If this is possible, do the tools in [SvH20] extend to prove Theorem 1.40?

17.12

In the Example 1.5, the asymptotic constant $3/2$ is probably far from tight for *dense graphs*, say with $\Omega(\mathrm{N}^2)$ edges. What's the right constant then?

17.13

When it comes to interval greedoids, there are more questions than answers. For example, since there is a Tutte polynomial for greedoids defined in [GM97], does it make sense to define an NBC complex? Are there any log-concavity results for characteristic polynomials in some special cases? Can one define morphism of antimatroids or interval greedoids? Are there any other interesting classes of interval greedoids whose log-concavity is worth studying?

17.14

Weak local greedoids introduced in §3.2 by the weak local property (WeakLoc), is a new class of greedoids. It contains poset antimatroids, matroids, discrete polymatroids, and *local poset greedoids*, see Figure 17.1. We do not consider the latter in this paper, but they play an important role in greedoid theory, see [KLS91, Ch. VII]. To understand the relationship between weak local greedoids and local poset greedoids, note the excluded minor characterization of local poset greedoids in [KLS91, Cor. VII.3.2]. By contrast, weak local greedoids exclude the same minor under contraction, but not necessarily deletion.

17.15

Let $\mathcal{G} = (X, \mathcal{L})$ be an interval greedoid, and let $\mathcal{B} \subseteq \mathcal{L}$ be the set of feasible words $\alpha = x_1 \cdots x_\ell$ of maximum length $\ell = \mathrm{rk}(\mathcal{G})$. Denote by $\mathcal{B}^{\mathrm{op}}$ the set of words $\alpha^{\mathrm{op}} := x_\ell \cdots x_1$. An interval greedoid is called *reversible* if $\mathcal{B}^{\mathrm{op}}$ is the set of basis feasible words of an interval greedoid. Note that matroids, polymatroids and poset antimatroids are examples of reversible greedoids.

Let us note that our proof of Stanley's inequality (1.30) can be generalized to reversible interval greedoids. Unfortunately, in the examples above the corresponding generalization of Stanley's inequality is trivial. It would be interesting to characterize reversible greedoids or at least find new interesting examples.

17.16

From the *computational complexity* point of view, one can distinguish "easy inequalities" from "hard inequalities", depending whether the components (or their differences) are computationally easy or hard. For example, *Hoffman's bound* (see e.g. [Big74, Thm 8.8]), relates the *independence number of a graph* which is NP-hard, to the ratio of graph eigenvalues which can be computed in polynomial time. Assuming $\mathrm{P} \neq \mathrm{NP}$, one would expect such bound not to be sharp in many natural cases.

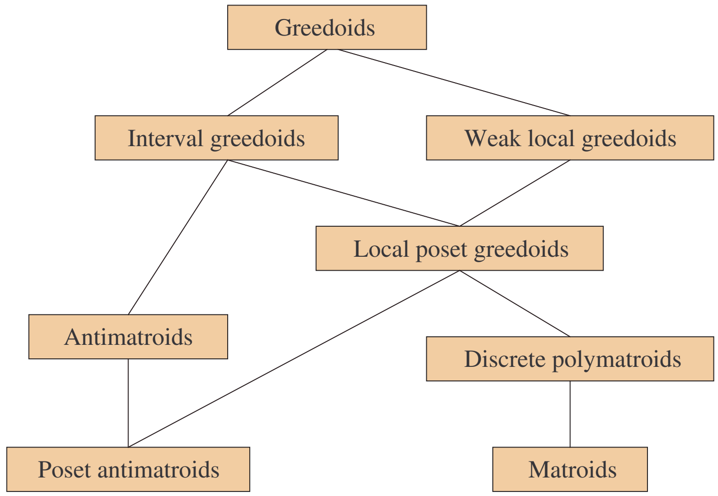


Figure 17.1: Diagram of inclusions of greedoid classes.

By contrast, Alon's lower bound on the number of spanning trees in regular graphs (see [Alon90]) has both sides computable in polynomial time. This suggests that complexity approach may not necessarily capture the mathematical difficulty of the result.

In this context, the inequalities in this paper are the "hardest" of all. For the Mason's conjectures, even in the simplest case of graphical matroids (Example 1.5), the number of k-forests is known to be #P-complete, see e.g. [Wel93]. Similarly, in Stanley's inequality (Theorem 1.34), the number of linear extensions of a poset is #P-complete even for posets of height two or dimension two, see [BrW91, DP18].

17.17

Another computational complexity approach to combinatorial inequalities is to understand whether their difference of two sides is nonnegative for combinatorial reason, i.e. whether it has a combinatorial interpretation. This is a natural question we previously discussed in [Pak19].

For example, observe that both sides in Stanley's inequality (1.30) are #P-functions, i.e., they have a natural combinatorial interpretation. The difference of LHS and RHS is then a function in GAPP = #P − #P. Now the problem whether it lies in #P. Although our proof is elementary, this question remains unresolved.

Similarly, in the case of graphical matroid, the equation (1.1) also corresponds to a nonnegative function in GAPP. Again, no combinatorial interpretation is known in this case. This is in sharp contrast, e.g., with the Heilmann–Lieb theorem (see §16.2) on log-concavity of the matching polynomial, where a combinatorial interpretation of the difference follows from Krattenthaler's combinatorial proof [Kra96], see also [Pak19]. We intend to return to this problem in the future.[19]

17.18

Going back to the discussion in Foreword §1.1 and Final Remark §17.1 above, it seems, the importance of poset log-concavity conjectures is yet to be settled. Back in 1989, Francesco Brenti wrote in this context:

[19] After this paper appeared we continued our investigation in [IP22, Pak22].

> *"In this author's opinion, conjectures and open problems in mathematics are not so much interesting and important 'per se' but because they are symptoms that our knowledge is not complete in some area. Their greatest value is not whether they are true or false but that they stimulate and lead us into deeper knowledge."* [Bre89, p. 6]

One can disagree with these sentiments, but speaking for ourselves we certainly owe these conjectures a debt of gratitude, as we find ourselves in the midst of unexplored territory we neither sought nor expected to discover.

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