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Isometric Immersions with Controlled Curvatures

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Abstract: We δ-approximate strictly short (e.g. constant) maps between Riemannian manifolds $f_0 : X^m \to Y^N$ for $N \gg m^2/2$ by C^∞-smooth isometric immersions $f_\delta : X^m \to Y^N$ with curvatures $curv(f_\delta) < \frac{\sqrt{3}}{\delta}$, for $\delta \to 0$.

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1 Compression and Approximation

A C^∞-immersion of a smooth manifold X to a smooth Riemannian $Y = (Y, h)$,

$$f_0 : X \to Y$$

is called $\mathfrak{II}_h$ or just $\mathfrak{II}$, if the Riemannian metric inducing operator from the space of smooth maps $f : X \to Y$ to the space $\mathcal{G}_+(X)$ of smooth semi definite positive quadratic forms on X,

$$\mathscr{I} = \mathscr{I}_h : \mathcal{F} = C^\infty(X,Y) \to \mathcal{G}_+(X) \text{ for } f \overset{\mathscr{I}}{\mapsto} g = f^*(h)$$

is *infinitesimally invertible in a C^∞-neighbourhood* $\mathcal{F}_0 \subset \mathcal{F}$ of f_0.
   This means that the differential/linearization of $\mathscr{I}$,

$$\mathscr{L}_f : T_f(\mathcal{F}) \to T_{\mathscr{I}(f)}(\mathcal{G}),$$

of $\mathscr{I}$, $f \in \mathcal{F}_0$, is *right invertible* by a *differential operator*

$$\mathscr{M}_f : T_{\mathscr{I}(f)}(\mathcal{G}) \to \mathscr{T}_f(\mathcal{F}), \ \mathscr{L}_f \circ \mathscr{M}_f = Id : T_{\mathscr{I}(f)}(\mathcal{G}) \to T_{\mathscr{I}(f)}(\mathcal{G}),$$

where $f \mapsto \mathscr{M}_f$ is a (possibly non-linear) differential operator defined on $\mathscr{F}_0$.

If $Y = \mathbb{R}^N$, (and in local coordinates for all Y, in general) the operator $\mathscr{L}$ can be written as an operator on maps $\overrightarrow{f} : X \to Y$,

$$\mathscr{L}_f(\overrightarrow{f}) = \mathscr{I}(f + \varepsilon \overrightarrow{f}) - \mathscr{I}(f) + o(\varepsilon), \ \varepsilon \to 0,$$

and where $\mathscr{M}_f(\overrightarrow{g})$ is a differential operator in $(f, \overrightarrow{g})$, which is linear in $\overrightarrow{g}$ and which satisfies

$$\mathscr{L}_f(\mathscr{M}_f(\overrightarrow{g})) = \overrightarrow{g}.$$

"Free" Example. Free immersions f, i.e. where the second osculating spaces $osc_2(f(x)) \in T_{f(x)}(Y)$ have *maximal possible* dimensions,

$$dim(osc_2(f(x)) = \frac{dim(X)(dim(X)+1)}{2} + dim(X)$$

at all points $x \in X$, are $\mathfrak{II}$ by the *Janet-Burstin-Nash Lemma.*

Consequently,

generic f are $\mathfrak{II}$ for $dim(Y) \geq \frac{dim(X)(dim(X)+1)}{2} + 2dim(X)$.[1]

1.A. Definitions of m-Free and of Flat $\mathfrak{II}^{[m]}$ Maps. A smooth immersion $f : X = X^n \to Y$ is m-free, $m \leq n = dim(X)$, if the restrictions of f to all m-dimensional submanifolds in X are free.

For instance,

$$if \ \ N = dim(Y) \geq \frac{m(m+1)}{2} + m(n-m) + 2n,$$

then generic f are m-free.

An immersion $f : X = X^n \to Y$ is *flat* $\mathfrak{II}^{[m]}$, $m \leq n$, if the induced metric in X is Riemannian flat, i.e. locally isometric to $\mathbb{R}^n$, and if the restrictions of f to all *flat*, (i.e. locally isometric to $\mathbb{R}^m \subset \mathbb{R}^n$) m-dimensional submanifolds $X^m \subset Y$ are $\mathfrak{II}$.

For instance, isometric m-free immersions of flat tori are flat $\mathfrak{II}^{[m]}$.

Such immersions, especially of the split tori $\mathbb{T}^n = \underbrace{\mathbb{T}^1 \times ... \times \mathbb{T}^1}_{n}$ to the Euclidean spaces, play a special role in our arguments,[2] where we use below the following.

1.B. Example. Let $\mathbb{T}^n$ be the torus with a flat Riemannian metric, i.e. the universal covering of this $\mathbb{T}^n$ is isometric to $\mathbb{R}^n$. Then:

$\mathbb{T}^n$ *admits a free isometric C^∞-immersion to* $\mathbb{R}^{\frac{n(n+1)}{2}+n+2}$

and

an isometric $\mathfrak{II}$-immersion to $\mathbb{R}^{\frac{n(n+1)}{2}+n+1}$.

1.C. *Remarks on the Proof.* (a) The existence of free isometric immersions of flat split n-tori to N-dimensional Riemannian manifolds is proven for $N \geq \frac{n(n+1)}{2} + n + 2$ in section 3.1.8 in [4] and since non-split flat tori can be approximated by finite coverings of split ones, the general case follows from the Nash implicit function theorem.

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[1]See [4], [5] and references therein.

[2]This is similar to how it is with non-isometric immersions with controlled curvature in [6].

(b) The existence of $\mathfrak{II}$-immersion of split tori to $\mathbb{R}^{\frac{n(n+1)}{2}+n+1}$ is (implicitly) indicated in an exercise on p. 251 in [4][3], while the recent result by DeLeo [1] points toward a similar possibility for

$$dim(Y) \geq \frac{n(n+1)}{2} + n - \sqrt{\frac{n}{2}} + \frac{1}{2}.$$

In fact, it is not impossible that such immersions exist for

$$dim(Y) \geq \frac{n(n+1)}{2} + 1,$$

and it seems not hard to show these don't exist for $dim(Y) \leq \frac{n(n+1)}{2}$.

Definition of curv(f) = curv(f(X)). This is (as in [6]) the curvature of a manifold X immersed by f to a Riemannian Y, that is *the supremum of the "Y-curvatures"*, i.e. curvatures measured in the Riemannian geometry of $Y \supset X$, of geodesics $\gamma \subset X$, for the induced Riemannian metric in X,

$$curv(f) = curv(f(X)) = \sup_{\gamma \in X} curv_Y(\gamma).$$

1.D. $\sqrt{3}$-Remark. One has only a limited control over the curvatures of the above immersions $\mathbb{T}^n \to Y$, even for $Y = \mathbb{R}^N$, but we shall prove in the next section the existence of

a free isometric immersion from the n-torus to the unit N-ball for $N \geq \frac{n(n+1)}{2} + 2n$ with the curvature bounded by a constant D independent of n, where, conceivably, $D < \sqrt{3}$.

In fact, $\sqrt{3}$ is *asymptotically optimal*, since, according to *Petrunin's inequality,*[4]

smooth isometric immersions $f : \mathbb{T}^n \to B^N(1)$ satisfy for all n and N:

$$curv(f) \geq \sqrt{3\frac{n}{n+2}}.$$

1.E. Compression Lemma. Let

$$F : \mathbb{T}^n \hookrightarrow B^N(1) \subset \mathbb{R}^N$$

be a flat $\mathfrak{II}^{[m]}$-immersion.

Let $X^m = (X^m, g)$ be a compact Riemannian *m*-manifold, possibly with a boundary, which admits a smooth $(1+\varepsilon)$-bi-Lipschitz immersion $\phi_\varepsilon : X^m \to \mathbb{T}^n$. If $\varepsilon \leq \varepsilon_0(m) > 0$. [5]

Then there exist C^∞-smooth isometric immersions

$$f_i^\circ : X \hookrightarrow B^N\left(\frac{1}{i}\right), i = 1, 2, ...,$$

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[3]We explain this in section 3. Also notice that there are no known obstructions to the existence of free isometric immersions of flat *n*-tori to $\mathbb{R}^{\frac{n(n+1)}{2}+n}$, but no example of a free (isometric or not) immersion from $\mathbb{T}^n$ to $\mathbb{R}^{\frac{n(n+1)}{2}+n}$ for $n \geq 2$ had been found either.

[4]See https://anton-petrunin.github.io/twist/twisting.pdf and [8].

[5]Possibly, this ε_0 doesn't depend on *m*.

such that
$$curv_{f_i^\circ}(X) \leq i \cdot curv(F(\mathbb{T}^n)) + O(1).$$

Proof. Compose the maps F and ϕ_ε with the homothetic endomorphism $t \mapsto i \cdot t$ of the torus,

$$X \xrightarrow{\phi_\varepsilon} \mathbb{T}^n \xrightarrow{i \cdot} \mathbb{T}^n \xrightarrow{F} B^N(1)$$

and observe that the resulting composed maps, say

$$f_{i,\varepsilon} : X \to B^N(1)$$

satisfy the following conditions.

$\bullet_\varepsilon$ The map $f_{i,\varepsilon}$ is a smooth $(1+\varepsilon)$-bi-Lipschitz immersion with respect to the metrics $i^2 g$.

$\bullet_{1/i}$ The covariant derivatives of the metric $i^2 g$ and the covariant derivatives of the induced metric $f^*(h_{Eucl})$ with respect to $i^2 g$ converge to zero for $i \to \infty$,

$$\max(||\nabla^j_{ig}(i^2 g)||, ||\nabla^j_{ig}(f^*_{i,\varepsilon}(h_{Eucl}))||) \leq const_j i^{-j}.$$

$\bullet_{\mathfrak{JJ}}$ The immersions $f_{i,\varepsilon}$ are *uniformly* $\mathfrak{JJ}_{h_{Eucl}}$:

the $j^2 g$-covariant derivatives of the (the coefficients of the) differential operators $\mathcal{M} = \mathcal{M}_i = \mathcal{M}_{f_{i,\varepsilon}}$, which invert the linearized operator $\mathcal{I} : f_{i,\varepsilon} \mapsto f^*_{i,\varepsilon}(h_{Eucl})$ on X are bounded, for all i independently of i,

$$||\nabla^j_{g_{i,\varepsilon}}(\mathcal{M})|| \leq const = const_{m,j}.$$

It follows from the (generalized) Nash implicit function theorem (section 2.7.2 in [4]) that, for a sufficiently small $\varepsilon > 0$, depending only on m, the maps $f_{i,\varepsilon}$ for sufficiently large i admit C^∞-small, convergent to 0 with $\varepsilon \to 0$, perturbations to smooth isometric immersions $f_i^{\bigcirc} : (X, i^2 g) \to B^N(1)$.

Since the curvatures of these immersions are bounded by the curvature of F plus $O(1/i)$, the immersions

$$f_i^\circ = i^{-1} f_i^{\bigcirc} : (X, i^2 g) \to B^N(1/i)$$

are the required ones. QED.

1.F. Local Compression Corollary. Let X^m be a smooth Riemannian manifold with the sectional curvature bounded by

$$|sect.curv(X^m)| \leq 1$$

and where the injectivity radius at a given point is bounded from below

$$inj.rad_{x_0}(X^m) \geq 1.$$

Then there exists a constant $\rho = \rho_m > 0$,[6] such that the ball $B(\rho) = B_{x_0}(\rho) \subset X$ admits a *smooth isometric immersion* to the Euclidean space

$$f : B(\rho) \to \mathbb{R}^{\frac{m(m+1)}{2}+m+1}.$$

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[6]Probably, what we say here holds for $\rho \geq 10^{-10}$ and all m.

Moreover there exist immersions $f_\delta : B(\rho) \to \mathbb{R}^{\frac{m(m+1)}{2}+m+1}$ for all $0 < \delta \le 1$ with the *diameters of the images bounded by*

$$diam(f_\delta(B(\rho))) \le \delta$$

and the *curvatures of these images bounded by*

$$curv(f_\delta(B_{x_0}(\rho))) \le \frac{C_m}{\delta}.^7$$

   Proof. The assumptions on the curvature and the injectivity radius of X imply that the $B_{x_0}(\rho)$ is $(1+3\rho)$-biLipschitz to the ρ-ball in the flat torus.
   Remark. We shall prove in the next section the existence of smooth isometric $f_\delta : B_{x_0}(\rho) \to \mathbb{R}^{\frac{m(m+1)}{2}+2m+1}$ with $diam(f\delta_{x_0}) \le \delta$ and the curvatures of these images bounded by

$$curv(f_\delta(B_{x_0}(\rho))) \le \frac{C}{\delta}$$

for a universal constant C. [8]
   1.G. Approximation Lemma. Let

$$X^m = (X^m, g_\varepsilon) \xrightarrow{\phi_\varepsilon} \mathbb{T}^n \xrightarrow{F} \mathbb{R}^N$$

be as in 1.E, let $Y = (Y, h)$ be a smooth Riemannian manifold and let $f_0 : X^m \to Y$ be a C^∞-smooth map.
   Let the pullback to X of tangent bundle of Y,

$$f_0^*(T(Y)) \to X,$$

admit N *independent vector fields normal to the image of the differential of f_0.*
   For instance, $dim(Y) \ge N$ and f_0 is a constant map, or f_0 is an immersion homotopic to a constant map and $dim(Y) \ge N + 2m - 1$.
   If $0 < \varepsilon \le \varepsilon_0(m)$, then, for all $i = 1, 2, ...$, there exist a δ_i-approximation of f_0 for $\delta_i \le \frac{1}{i}$ by C^∞-immersions $f_i : X^m \to Y$ with

$$curv_{f_i}(X) \le i \cdot curv(F(\mathbb{T}^n)) + o(i)$$

and where f_i increase the induced Riemannian metric $g_0 = f_0^*(h)$ in X by the above g_ε:

$$f_i^*(h) = g_0 + g_\varepsilon.^9.$$

   Proof. Let $E = E_{N, \delta_0} : X \times B^N(\delta) \to Y$ be the exponential map defined by the N vector fields, where this map is defined for all δ_0 if Y is complete Y and if Y is non-compete, then E_{N, δ_0} is defined if the δ_0-neighbourhood of $f_0(X) \subset Y$ is compact.[10]

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[7]The proof of 1.B in [4] for $Y = \mathbb{R}^{\frac{m(m+1)}{2}+m+2}$ shows that $C_m < (100m)^{100m}$.
[8]Probably $C < 100$.
[9]δ_i-*Approximation* signifies that $dist_Y(f_0(x), f_i(x)) \le \delta_i$, $x \in X$.
[10]We assume here that Y has no boundary.

Let $\delta_i \leq \delta_0$ and let us restrict the map E to the graph of the above map $f_i^\circ : X \to B^N(\delta_i) \subset \mathbb{R}^N$,

$$\Gamma_i = \Gamma_{f_i^\circ} : X \hookrightarrow X \times B^N(\delta_i),$$

where our f_i° is now isometric for the metric g_ε on X.

Since the N fields are *normal* to $f_0(X) \subset Y$, the Riemannian metric in X induced by $\Gamma_i \circ E : X \to Y$ is $(1 + \varepsilon + const \cdot \delta_i)$-bi-Lipschitz to $g_0 + g_\varepsilon$; hence, the (generalized) Nash implicit function theorem applied as in the proof of 1.E, now to the maps $\Gamma_i \circ E$ for small $\varepsilon > 0$ and large i, delivers C^∞-perturbations to these maps to the required immersions $f_i : X \to Y$ with $f_i^*(h) = g_0 + g_\varepsilon$ and with $curv_{f_i}(X) \leq i \cdot curv_f(\mathbb{T}^n) + o(i)$.

1.H. Global Approximation Corollary. Let $X^m = (X^m, g)$ and $Y^N = (Y^N, h)$ be smooth Riemannian manifolds and $f_0 : X \to Y = (Y, h)$ be a smooth *strictly short map*, i.e. the quadratic differential form $g - f^*(h)$ is positive definite.

If X^m is compact, if X admits a smooth immersion to $\mathbb{R}^n$ and if

$$N \geq \frac{n(n+1)}{2} + n + 1,$$

then there exists δ_i-*approximation* of f_0 for $\delta_i \leq \frac{1}{i}$, $i = 1, 2, ...$, by isometric C^∞-immersions $f_i : X^m \to Y$ with

$$curv(f_i(X)) \leq i \cdot C_m + o(i).$$

Proof. Apply the lemma to smooth $(1 + \varepsilon)$-bi-Lipschitz immersions $\phi_\varepsilon : X^m \to \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, delivered by the Nash-Kuiper C^1-immersion theorem and to an isometric $\mathfrak{II}$-immersion $F : \mathbb{T}^n \to \mathbb{R}^{N = \frac{n(n+1)}{2} + n + 1}$ from 1.B.

Remarks. (a) Since all X^m, $m \geq 2$, admit smooth immersions to $\mathbb{R}^{2m-1}$ by the Whitney theorem, the inequality

$$N \geq \frac{2m(2m-1)}{2} + 2m \approx 2m^2$$

suffices for all X^m. This is the worst case dimension-wise.

Our bound on N is much better for $n = m + 1$, e.g. for compact hypersurfaces $X^m \subset \mathbb{R}^{m+1}$, where one needs

$$N \geq \frac{(m+1)(m+2)}{2} + m + 2 = \frac{m(m+1)}{2} + 2m + 3;$$

but this still seems far from optimal.

The best we can get for smaller N, namely for

$$N \geq \frac{m(m+1)}{2} + m + 1,$$

is the following special result.

1.I. Flat Torus Approximation Theorem. Let $\mathbb{T}_g^m$ the torus with an *invariant* (hence *flat*) Riemannian metric g and let

$$f_0 : \mathbb{T}_g^m \to Y$$

be a smooth map, where $Y = (Y, h)$ is a compact N-dimensional C^∞-smooth Riemannian manifold, possibly with a boundary, such that

$$dist(f_0(\mathbb{T}^m_g), \partial Y) > 0.$$

If $N = dim(Y) \geq \frac{m(m+1)}{2} + m + 1$, and if the bundle induced by f_0 from the tangent bundle of Y, that is $f_0^* : T(Y) \to \mathbb{T}^m_g$ is *trivial* (e.g. f_0 is contractible or Y is stably parallelizable), then, for all $\delta > 0$,[11] the map f_0 admits a δ-approximation by *metrically homothetic* maps

$$f_\delta : \mathbb{T}^m_g \to Y,$$

i.e. such that

$$f^*(h) = \lambda \cdot g \text{ for some } \lambda > 0$$

and where

$$curv(f_\delta(\mathbb{T}^m)) \leq \frac{const_m}{\delta} + o\left(\frac{1}{\delta}\right),$$

Proof. Let $E = E_{N,\delta_0} : X \times B^N(\delta) \to Y$ be the exponential map, similar to that in the proof of 1.F but now *not required to be normal* to $f_0(X)$.

Let $F : \mathbb{T}^m \to B^N(1)$, $N = \frac{m(m+1)}{2} + m + 1$ be as in 1.B and let

$$F_{ij} : \mathbb{T}^m \to B^N(1/i) \text{ for } t \mapsto \frac{1}{i}F(jt).$$

Let $\delta_i \leq \delta_0$ and let us restrict the map E to the graph of the map F_{ij}

$$\Gamma_{F_{ij}} = \Gamma_{i,j} : X \hookrightarrow X \times B^N(\delta_i).$$

Let the ratio $\lambda = j/i$ be very large. Then, in terms of the metric λg, the metric induced by $E \circ \Gamma_{ij} : \mathbb{T}^m \to Y$ becomes C^∞-close to λg and the proof follows as in 1.E and 1.G by the (generalized) Nash implicit. function theorem.

Remarks. The proof of Theorem (C) on p. 294 in [4] delivers (a stronger version of) the "local compression" for surfaces, $X^2 \to B^4(\delta)$, and this, seems to imply the torus approximation theorem for $\mathbb{T}^2 \to Y^N$ and all $N \geq 4$.

We don't know if one could comparably improve bounds on N in general, but in the next section we prove an approximation theorem for all X with the constant C *independent of* m and with a slightly improved dimension bounds in some cases.

2 Free Isometric Imbeddings of Tori to the Unit Balls with Small Curvatures

Let

$$\mathbb{T}^N_{Cl} \subset S^{2N-1} \subset B^{2N} \subset \mathbb{R}^{2N}$$

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[11]Since we are concerned with $\delta \to 0$, we assume here and below that $\delta \leq 1$.

be the Clifford torus, which observe, has the Euclidean curvature

$$curv(\mathbb{T}^N_{Cl} \subset \mathbb{R}^{2N}) = \sqrt{N}.$$

2.A. $\Delta(n,N)$-inequality. If $1 \leq \frac{n^2}{2} \leq N$, then there exists a (flat invariant) subtorus $\mathbb{T}^n_0 \subset \mathbb{T}^N_{Cl}$, the Euclidean curvature of which satisfies

$$curv(\mathbb{T}^n_0 \subset \mathbb{R}^{2N}) \leq \sqrt{3}\sqrt{\frac{n}{n+2}} + \Delta(n,N),\text{[12]}$$

where $\Delta(n,N)$ is bounded by a universal constant, which, in fact, *vanishes* for $N \geq 8(n^2 + n)$.[13]

This follows from the $D(m,N)$-inequalities 2.1.E and 3.B in [6].

2.B. m-Freedom Corollary. If $m \leq n$ and

$$\frac{m(m+1)}{2} + m(n-m) + 2n \leq 2N,$$

then, for all $\varepsilon > 0$, the n-torus $\mathbb{T}^n$ admits an immersion $\mathbb{T}^n \overset{F_\varepsilon}{\hookrightarrow} B^{2N} \subset \mathbb{R}^{2N}$ such that

- $\bullet_{flat}$ the induced Riemannian metric $F_\varepsilon^*(h_{Eucl})$ in $\mathbb{T}^n$ is *Riemannian flat*;
- $\bullet_{curv}$ the Euclidean *curvature* of this torus is *bounded* by

$$curv(F_\varepsilon(\mathbb{T}^n)) \leq \sqrt{\frac{3n}{n+2}} + \Delta(n,N) + \varepsilon;$$

- $\bullet_{free}$ The restrictions of F_ε to all m-dimensional submanifolds in $\mathbb{T}^n$ are *free*.

Proof. The required F_ε is obtained by generically ε-perturbing the lift of the above $T^n_0 \subset \mathbb{T}^N_{Cl}$ to a finite covering of $\mathbb{T}^N_{Cl}$ as follows.

Firstly, replace $\mathbb{T}^n_0$ by a generic (flat invariant) subtorus, $\mathbb{T}^n_\varepsilon \subset \mathbb{T}^N_{Cl}$ *tangentially ε-close* to $\mathbb{T}^n_0$, i.e. where the tangent spaces to $\mathbb{T}^n_\varepsilon$ are ε-parallel in $\mathbb{T}^N_{Cl}$ to these of T^n_0,[14] where such an $\mathbb{T}^n_\varepsilon$ can be chosen split, i.e. being Riemannian product of 1-tori.

Moreover, we let

$$\tilde{\mathbb{T}}^N_{Cl} = \underbrace{\mathbb{T}^1_1 \times ... \times \mathbb{T}^1_i \times ... \times \mathbb{T}^1_N}_{N},$$

(the circles $\mathbb{T}^1_i$ may have different lengths l_i), be a split finite covering of the Clifford torus such that

$$\tilde{\mathbb{T}}^n = \underbrace{\mathbb{T}^1_1 \times ... \times \mathbb{T}^1_n}_{N}$$

equals a covering of $\mathbb{T}^n_\varepsilon$.

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[12]The summand $\sqrt{3}\sqrt{\frac{n}{n+2}}$ is optimal by Petrunin's inequality.

[13]Probably, $\Delta(n,N) < 10$ for all $N \geq n(n-1)/2$ and, possibly $\Delta(n,N) \leq 1/n$ for $N \geq \frac{n(n+2)}{2}$.

[14]This $\mathbb{T}^n_\varepsilon$ is far from T^n_0 as a subset in $\mathbb{T}^N_{Cl}$.

Secondly, perturb the embedding $\tilde{\mathbb{T}}^n \subset \tilde{\mathbb{T}}^N_{\text{Cl}}$ keeping the induced metric Riemannian flat in $N - n$ steps, where, at each step, we perturb a split K-torus, which contains $\tilde{\mathbb{T}}^n$, in a $(K+1)$-torus,

$$\tilde{\mathbb{T}}^n \subset \mathbb{T}^{K-1} \times \mathbb{T}^1_i \subset \mathbb{T}^{K-1} \times (\mathbb{T}^1_i \times \mathbb{T}^1_{i+1}) = \mathbb{T}^{K+1} \subset \tilde{\mathbb{T}}^N_{\text{Cl}}, \ K = n, ..., N-1, i = n+1, ...N,$$

by approximating the embedding $\mathbb{T}^1_i \subset \mathbb{T}^1_i \times \mathbb{T}^1_{i+1}$ by a generic isometric embedding $(1+\varepsilon) \cdot \mathbb{T}^1_i \overset{I_\varepsilon}{\to} \mathbb{T}_i \times \mathbb{T}^1_{i+1}$.

Here $(1+\varepsilon) \cdot \mathbb{T}^1_i$ is the same circle as $\mathbb{T}_i$ but with the metric of total length equal to $(1+\varepsilon) \cdot length(\mathbb{T}_i)$ and where, finally, we let

$$\mathbb{T}^{K-1} \times \mathbb{T}^1_i \ni (\theta, t) \to (\theta, I_\varepsilon(t)) \in \mathbb{T}^{K-1} \times \mathbb{T}^1_i \times \mathbb{T}^1_{i+1}.$$

If $\frac{m(m+1)}{2} + m(n-m) + 2n \le 2N$, and granted all was done "sufficiently generically", then the resulting map $\tilde{\mathbb{T}}^n \to B^{2N}(1)$ via $\tilde{\mathbb{T}}^N_{\text{Cl}} \subset B^{2N}(1)$ is free on all $X^m \subset \tilde{\mathbb{T}}^n$..

Checking this, which is similar to "*Making Non-free Maps Free*" on p. 259 in [4], is left to the reader.

2.C. Corollary. *Let X^m and Y^M be smooth Riemannian manifold and $f_0 : X \to Y$ be a smooth strictly short map. If X^m is a compact manifold, which admits a smooth immersion to $\mathbb{R}^n$, and if*

$$M > \frac{m(m+1)}{2} + m(n-m) + 2n,$$

then f_0 can be δ_i-approximated, by isometric C^∞-immersions $f_i : X^m \to Y^M$, for $\delta_i \le \frac{1}{i}, i = 1, 2, ...,$ such that

$$curv(f_i(X)) \le i \left(\sqrt{\frac{3n}{n+2}} + \Delta(n, \lfloor M/2 \rfloor) \right) + o(i).$$

For instance,
if X is a Euclidean hypersurface, then such an approximation is possible for

$$M > \frac{m(m+1)}{2} + 3m + 2$$

with

$$curv(f_i(X)) \le i \left(\sqrt{\frac{3(m+1)}{m+3}} + \Delta(m+1, \lfloor M/2 \rfloor) \right) + o(i).$$

And – this is the worst case – the inequality

$$M > \frac{m(m+1)}{2} + m(m-1) + 4m - 2$$

is sufficient for all compact X^m, where the maps f_i satisfy:

$$curv(f_i(X)) \le i \left(\sqrt{\frac{3(2m-1)}{2m+1}} + \Delta(2m-1, \lfloor M/2 \rfloor) \right) + o(i).$$

2.D. Immersions with Prescribed Curvatures. The *symmetric normal curvature* Ψ_f of an immersion $f : X \to Y$ is the "square" of the second fundamental form, that is the symmetric differential 4-form, such that

$$\Psi_f(\partial, \partial, \partial, \partial) = ||\nabla_{\partial,\partial} f||^2_Y$$

for all tangent vectors $\partial \in T(X)$.

Observe that if f is free, then Ψ_f is *positive definite*. This means that the 4-form Ψ_x on the tangent space $T = T_x(X)(= \mathbb{R}^m)$ is contained, for all $x \in X$, in the interior of the convex hull of the $GL(m)$-orbit of the fourth power of a non-zero 1-form on T.[16]

Also observe that

$$\sup_{||\partial||=1} \Psi_f(\partial, \partial, \partial, \partial) = (curv(f))^2.$$

2.C. C^2-Curvature Theorem.[17] *Let $X = X^m$ and $Y = Y^M$ be smooth Riemannian manifolds, let $f : X \to Y$ a free isometric C^∞-immersion and let Ψ be a symmetric positive definite differential 4-form on X.*

If

$$M = dim(Y) \geq \frac{m(m+1)}{2} + 3m + 5,$$

then f can be arbitrarily finely C^1-approximated by isometric C^2-immersions f' with the increase of their normal curvatures by Ψ.[18]

$$||\nabla_{\partial\partial} f'||^2 = ||\nabla_{\partial\partial} f||^2 + \Psi(\partial, \partial, \partial, \partial)$$

for all tangent vectors $\partial \in T(X)$.

2.E. Euclidean Example. *The standard embedding $f_0 : \mathbb{R}^n \to \mathbb{R}^{(n+2)(n+5)/2}$ can be C^1-approximated by isometric C^2-embeddings f with an arbitrary strictly positive definite normal curvature form Ψ.*

Proof. C^∞-approximate f_0 by *free* isometric embeddings (as in 1.B) and apply 2.C.

2.F. Toric Example. *Let Ψ be a smooth symmetric positive definite differential 4-form on the m torus $\mathbb{T}^m$.*

If $M \geq 16(m^2 + m)$, then there exist a C^2-immersion $f : \mathbb{T}^M \to B^m(1)$, such that the induced metric is flat (split if you wish) and

$$\Psi_f(\partial, \partial, \partial, \partial) = \Psi(\partial, \partial, \partial, \partial) + \frac{3m}{m+2} ||\partial||^4.$$

Proof. Observe that, according to 3.1 from [6] (this also follows from Petrunin's inequality), the isomeric immersion $f_0 : \mathbb{T}^m \to B^M$ with $(curv(f_0))^2 \leq \frac{3m}{m+2}$ has constant curvature,

$$\Psi_{f_0}(\partial, \partial, \partial, \partial) = \frac{3m}{m+2} ||\partial||^4,$$

---

[15]If the right-hand sides of the inequalities $M > ...$ are even, then these may be replaced by $M \geq ....$.

[16]This interior makes the unique open $GL(m)$-invariant (non-empty!) minimal convex cone in the space $T^{\circledS 4}$ (of dimension $m(m+1)(m+2)(m+3)/24$, $m = dim(T)$ of symmetric 4-linear forms (4d-polynomials) on T.
(This cone is strictly smaller than the cone of the forms Φ, which are positive as polynomials, $\Phi(t,t,t,t) > 0, t \neq 0$.)

[17]See 3.1.5.(A) in [4].

[18]In general, such an f' can't be C^∞ for large m , but possibly, these C^2-smooth f' exist for all $m \geq 2$ and all M.

and apply 2.B and 2.D.

3   Perspective and Problems

The $\mathfrak{II}$-property of free immersions was already implicitly present in the proof of the algebraic Janet lemma (1926), where this lemma brings the isometric immersion equations to the Cauchy–Kovalevskaya form and thus implies that real analytic Riemannian n-manifolds locally C^{an}-immerse to $\mathbb{R}^{\frac{n(n+1)}{2}}$. [19] [20]

The simplest non-free $\mathfrak{II}$-immersions, $X \to Y$ are those, which are free on a totally geodesic hypersurface $X_0 \subset X$ and also on the complement $X \setminus X_0$ and where the Hessian of the second derivatives $\partial_{ij}f$ vanishes on X_0 with *finite order* (see 2.3.8, 3.1.8. 3.1.9 in [4]). Then one defines by induction *k-subfree* maps of n-manifolds X to Y, where

$$0\text{-}subfree = free,$$

 and where

$$1\text{-}subfree \ maps \ f$$

generalize the above ones, namely, where there exist tangent hyperplanes $T_x^{n-1} \subset T_x(X)$ at all $x \in X$, such the *restrictions* of f to the *exponential images* $X_x = \exp(T_x) \subset X$ are free at x and also on the complements X_x near x, i.e. in the small balls:

$$B_x(\varepsilon) \cap (X \setminus X_x)$$

with "finite order of non-freedom" on (infinitesimal) neighbourhoods of X_x.

Finally, a map f is

$$k\text{-}subfree, \ k = 0, 1, ..., n,$$

 if it is free on the above hypersurfaces $X_x \subset X$ near x for all $x \in X$ and it is $(k-1)$-subfree on $B_x(\varepsilon) \cap (X \setminus X_x)$ with "finite order of non-$(k-1)$-subfreedom" on (infinitesimal) neighbourhood of X_x.

Clarification. "Finite order of non-subfreedom" means that the coefficients of the relevant linear differential operator $\mathcal{M} = \mathcal{M}_f(g)$ on $X \setminus X_x$ are rational functions in partial derivatives of f, where these functions have their poles on X_x, (see 238 in [4] and [1]).

This seems to imply –I didn't truly check this – that

$$generic \ immersions \ X^n \to Y^N \ for \ N \geq \tfrac{n(n+1)}{2} + n \ are \ \mathfrak{II}.$$

It is also plausible, that

generic bendings of m-subtori in the Clifford torus, as in the proof of 2,B would make them $\mathfrak{II}$ for
$\frac{m(m+1)}{2} + m(n-m) + 2n - m \leq 2N$.

But this, even if true only slightly improve the lower bound on N in the above "worst dimension case", where, in fact, we expect the following.

---

[19] The Cauchy–Kovalevskaya theorem also yields a weak form of the Nash real analytic implicit function theorem. This, by Janet's lemma combined with an analytic version of Nash twisting argument, delivers isometric C^{an}-immersions of compact C^{an}-manifolds to Euclidean spaces, see [2] and p 54 in Appendix 11 in [7].

[20] The existence of local isometric C^∞-immersions of C^∞-manifolds $X^n \to \mathbb{R}^N$ is known for $N = \frac{n(n+1)}{2} + n - 1$ [3] and it is easy to see that there is no local C^∞-immersions of generic smooth Riemannian n-manifolds to $\mathbb{R}^{\frac{n(n+1)}{2}-1}$.

But for *no $n \geq 2$* one can prove the existence of such immersions to $\mathbb{R}^{N=\frac{n(n+1)}{2}+n-2}$ or to find a counterexample for $N = \frac{n(n+1)}{2}$.

It is also unclear for which $i = 2, 3, ...$, (if any) all C^∞-smooth Riemannian manifolds $X = X^n$, admit (local or global) isometric C^i-immersion to $\mathbb{R}^{\frac{n(n+1)}{2}-1}$, where, for all we know, this may be possible, say for $i \leq 0.1\sqrt[4]{n}$, see [5] for more about it.

3.A. Conjecture. Let X^m and Y^N be smooth Riemannian manifolds, where X compact and Y is complete, and let $f_0 : X \to Y$ be a smooth strictly short map. If

$$N \geq \frac{m(m+1)}{2} + 1,$$

then f_0 admits a δ-*approximation* for all $\delta > 0$ by isometric C^∞-immersions $f_\delta : X^m \to Y$ with

$$curv(f(\delta(X))) \leq \frac{1}{\delta} \left(\sqrt{\frac{3m}{m+2}} + \Delta(m,N) \right) + o\left(\frac{1}{\delta}\right),$$

for the same Δ as in 2.A, where, if

$$N \geq \frac{m(m+1)}{2} + m,$$

these f_δ can be chosen *m-subfree*.

<center>FAMILIES OF MAPS.</center>

Given a locally defined class $\mathscr{C}$ of C^∞-maps $f : X \to Y$, where $Y = (Y,h)$ is a smooth Riemannian manifold, e.g. $\mathscr{C}$ consists of smooth immersions, of $\mathfrak{II}$-isometric immersions etc, the δ-approximation problem is accompanied by a similar problem for families of maps.

More generally, such approximation makes sense for maps f from foliated leaf-wise Riemannian manifolds $\mathcal{X} = (\mathcal{X},g)$ [21] to Y, where the restrictions of f to the leaves $X \subset \mathcal{X}$ are in $\mathscr{C}$.

3.B. $\mathscr{C}$-Homotopy Approximation Conjecture. Let $\mathcal{X} = (\mathcal{X},g)$ be a compact manifold foliated into m-dimensional Riemannian leaves, let $Y = (Y,h)$ be complete Riemannian N-manifold.

Let $\mathscr{C}$ be a class of smooth leaf-wise isometric $\mathfrak{II}$-maps and $\phi_0 : \mathcal{X} \to Y$ be a $\mathscr{C}$-map.

Let $f_0 : \mathcal{X} \to Y$ be a smooth leaf-wise strictly short $\mathscr{C}$-map, which is homotopic to ϕ.

Then, at least in in the following three cases, the map f_0 can be δ-approximated for all $\delta > 0$ by smooth leaf-wise $\mathscr{C}$-maps

$$f_\delta : \mathcal{X} \to Y,$$

where the maps f_δ can be joined with ϕ_0 by homotopies of leaf-wise $\mathscr{C}$-maps and where the leaf-wise curvatures of f_δ are bounded by

$$curv(f(\delta(X))) \leq \frac{\Xi}{\delta} + o\left(\frac{1}{\delta}\right),$$

where $\Xi = \Xi(m,N,dim(\mathcal{X})) \leq 100$.

Case 1. $\mathscr{C}$ is the class of leaf-wise *free* isometric maps.

Case 2. $\mathscr{C}$ is the class of leaf-wise *m-subfree* isometric maps.

Case 3. $\mathscr{C}$ is the class of *all* leaf-wise isometric $\mathfrak{II}$-maps.

Remarks/Questions. (i) *How big is* Ξ? Possibly, Ξ is significantly smaller than 100, but it is unlikely to approach $\sqrt{3}$ for $N \gg m^2$.

Yet, it follows by the arguments in section 2 that

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[21]This g – a smooth leaf-wise Riemannian metric on $\mathcal{X}$ – is a positive definite differential quadratic form on the tangent bundle $\mathcal{T} \subset T(\mathcal{X})$ to the leaves $X \subset \mathcal{X}$.

if $N \gg dim(\mathcal{X})^2$, then

$$\Xi \leq \sqrt{\frac{3(2dim(\mathcal{X})-1)}{2dim(\mathcal{X})+1}} < \sqrt{3}.$$

Conceivably, the correct bound on curvature needs $\Xi \sim 3(dim(\mathcal{X})) - m + 1)$.

(ii) *Why Integrable?* The above make sense for possibly non-integrable subbundles $\mathcal{T} \subset T(\mathcal{X})$, where the counterpart of the metric inducing operator sends maps f to the restrictions of the forms $f^*(h)$ to the subbundle $\mathcal{T} \subset \mathcal{X}$, where one can define the corresponding classes $\mathcal{C}_\mathcal{T}$ of maps $f : \mathcal{X} \to Y$ and where the $\mathcal{T}$-curvature of f is defined as follows.

Given a non-zero tangent vector $\tau \in T_x(\mathcal{X})$, $x \in \mathcal{X}$, let $curv_\tau(f)$ be the infimum of Y-curvatures at x of the f-images of the curves $C \subset \mathcal{X}$ which contain x and are tangent to τ,

$$curv_\tau(f) = \inf_C curv_x f(C)$$

and

$$curv_\mathcal{T}(f) = \sup_{\tau \in \mathcal{T}} curv_\tau(f).$$

Here is what one (may be unrealistically) expects in this regard.

3.C. Orthonormal Frame Conjecture. Let $\mathcal{X}$ be a compact smooth Riemannian manifold and Θ_i, $i = 1.,,, m \leq dim(\mathcal{X})$, be smooth orthonormal vector fields on $\mathcal{X}$.

Let $Y = (Y, h)$ be a complete Riemannian N-manifold and $f_0 : \mathcal{X} \to Y$, be a smooth strictly short map. If

$$N \geq \frac{m(m+1)}{2} + 1$$

and the induced (pullback) vector bundle $f^*(T(Y)) \to \mathcal{X}$ admits m linearly independent sections, e.g. f_0 is contractible, then the map f_0 can be δ-approximated for all $\delta > 0$ by smooth maps

$$f_\delta : \mathcal{X} \to Y,$$

such the differential images $d_f(\Theta_i) \in T(Y)$ are orthonormal with respect to h,

$$\langle d_f(\Theta_i), d_f(\Theta_j) \rangle_h = 0, \ ||d_f(\Theta_i)||_h = 1,$$

and such that

$$curv_\mathcal{T}(f) \leq \frac{\Xi}{\delta} + o\left(\frac{1}{\delta}\right),$$

for $\Xi \leq 100$, where $\mathcal{T} \subset T(\mathcal{X})$ is the subbundle spanned by the fields Θ_i.

Below we state without proof the only confirmation we have of the conjectures 3.A and 3.B.

3.D. Parametric $1D$-Approximation Theorem. Let $\mathcal{X}$ be compact smooth Riemannian manifold and $\mathcal{T}^1$ be a smooth line field on $\mathcal{X}$. Let $Y = (Y, h)$ be a complete Riemannian manifold of dimension $N \geq 2$ and let $f_0 : \mathcal{X} \to Y$ be a strictly short map.

If $\mathcal{T}^1$, regarded as a line bundle over $\mathcal{X}$, admits an injective homomorphism to the induced (pullback) bundle $f^*(T(Y)) \to \mathcal{X}$, e.g. $\mathcal{T}^1$ is orientable (defined by a vector field) and the map f_0 is contractible, then the map f_0 can be δ-approximated for all $\delta > 0$ by smooth maps

$$f_\delta : \mathcal{X} \to Y,$$

which are isometric on $\mathcal{T}^1$ and such that the $\mathcal{T}^1$-curvatures of f_δ, i.e. the Y-curvatures of the f_δ-images of the (1-dimensional) orbits/leaves of $\mathcal{T}^1$, are bounded by

$$curv_{\mathcal{T}^1}(f_\delta) \leq \frac{\Xi_1}{\delta} + o\left(\frac{1}{\delta}\right),$$

where $\Xi_1 \leq 4$.[22]

3.E. Example/Corollary. There exists a smooth map $f : S^{2n+1} \to B^2(1)$, for all $n = 1, 2, ...$, such that the f-images of all Hopf circles $S_p^1 \subset S^{2n+1}$, $p \in \mathbb{C}P^n$, have equal length $l = l_n$ ($\leq 100n$) and curvatures

$$curv(f(S_p^1)) \leq 4 + \varepsilon$$

for a given $\varepsilon > 0$.

Acknowledgments

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[22]Probably, the optimal $\Xi_1 = 3$, where one proves that $\Xi_1 > 2$ by showing that:

if a $C^{1,1}$-immersion $f : S^1 \to B^2(2)$ with $curv(f(S^1)) \leq 1$ is regularly homotopic to the figure 8, then f factors through a map to an (8-shaped) union of two unit circles in $B^2(2)$, which touch each other at the center of the disc $B^2(2)$,

where "extremal members" in families of immersions $S^1 \to B^2(3 - \varepsilon)$ with curvatures ≤ 1, are, probably, associated with certain patterns comprised of unit circles in $B^2(3 - \varepsilon)$ tangent to the unit circle centered at zero. Such a pattern associated with a regular isotopy from the unit circle to an immersion with the image 8 is comprised of three mutually tangent unit circles inside $B^2(1 + 2/\sqrt{3})$ (which seems to imply that $\Xi_1 > 1 + 2/\sqrt{3} \approx 2.1547...$).

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Kähler-type Embeddings of Balls into Symplectic Manifolds

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Abstract:

Consider a symplectic embedding of a disjoint union of domains (lying in the standard symplectic $\mathbb{R}^{2n}$) into a symplectic manifold M. We say that such an embedding is Kähler-type, or respectively tame, if it is holomorphic with respect to some (not a priori fixed, Kähler-type) integrable complex structure on M compatible with the symplectic form, or respectively tamed by it. Assume that M is either of the following: a complex projective space (with the standard symplectic form); an even-dimensional torus, or a K3 surface, equipped with an irrational Kähler-type symplectic form. Then any two Kähler-type embeddings of a disjoint union of balls into M can be mapped into each other by a symplectomorphism acting trivially on homology. If the embeddings are holomorphic with respect to complex structures compatible with the symplectic form and lying in the same connected component of the space of Kähler-type complex structures on M, then the symplectomorphism can be chosen to be smoothly isotopic to the identity. For certain M and certain disjoint unions of balls we describe precisely the obstructions to the existence of Kähler-type embeddings of the balls into M. In particular, symplectic volume is the only obstruction for the existence of Kähler-type embeddings of l^n equal balls (for any l) into $\mathbb{C}P^n$ with the standard symplectic form and of any number of possibly different balls into a torus or a K3 surface, equipped with an irrational symplectic form. We also show that symplectic volume is the only obstruction for the existence of tame embeddings of disjoint unions of equal balls, polydisks, or parallelepipeds, into a torus equipped with a generic Kähler-type symplectic form. For balls and parallelepipeds the same is true for K3 surfaces.

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1 Introduction

Symplectic embeddings of balls and other domains into symplectic manifolds have been studied since the early days of symplectic topology. So far, the existence of the embeddings (the so-called symplectic packing problem) got most of attention and much less is known about the connectedness of the spaces of such embeddings. The notable exception is the classical result of McDuff [McD3] about the connectedness of the space of symplectic embeddings of disjoint unions of balls into certain 4-dimensional symplectic manifolds – in particular, into $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ and their blow-ups (see also [McD1, McD2, Lal, Bir1] for earlier partial results on the subject and [McD4], [BLW], [Cr] for more recent extensions of McDuff's result).

In this paper, in an attempt to get more information about the connectedness question for symplectic manifolds of dimension higher than 4, we consider a more restrictive class of symplectic embeddings of disjoint unions of domains.

Namely, let M be a manifold equipped with a **Kähler-type symplectic form** ω – i.e., a symplectic form compatible with some (*not a priori fixed*) integrable complex structure on M; such complex structures will be called **Kähler-type complex structures**. We will denote by $CK(M)$ the space of Kähler-type complex structures on M.

We consider symplectic embeddings (of disjoint unions of domains) into (M, ω) that are holomorphic with respect to some *(not a priori fixed)* integrable (Kähler-type) complex structure I on M which is compatible with ω, or tamed by it. We will say that such a symplectic embedding is of **Kähler type**, or, respectively, **tame**. The existence of Kähler-type embeddings, *for a fixed isotopy class of the complex structure I on M compatible with ω*, has been previously studied – in a different guise – in [Ec], [Fl], [LuWa], [Tr], [WN1], [WN2], see Remark 2.6, Remark 3.10 and part 3 of Remark 3.25.

With this terminology, we have the following hierarchy of properties of the embeddings:

$$\text{Kähler-type} \implies \text{tame} \implies \text{symplectic}.$$

A Kähler-type embedding of a domain in $\mathbb{R}^{2n}$ (equipped with the standard symplectic and complex structures) into (M, ω) which is holomorphic with respect to a complex structure I compatible with ω can be equivalently described as a Kähler-isometric, or just Kähler, embedding of the domain into M equipped with the Kähler metric $\omega(\cdot, I\cdot) + \sqrt{-1}\omega(\cdot, \cdot)$. In other words, an embedding of a domain into (M, ω) is of Kähler type if and only if it is Kähler with respect to some Kähler metric whose imaginary part is ω. The restriction of the Kähler metric to the image of the embedding is the standard flat metric.

There do exist symplectic embeddings of balls that are not of Kähler type (see below). We do not know whether these symplectic embeddings are tame or whether there exist tame embeddings (of balls or any other domains) that are not of Kähler-type – see Remark 3.32 and Question 3.33. Let us note that for a disjoint union of starshaped domains (with a piecewise-smooth boundary) its Kähler-type and tame embeddings form $\mathrm{Symp}(M)$-invariant open subsets of the space of its symplectic embeddings into (M, ω) (with respect to the C^∞-topology) – see Section 5.1.

As far as the connectedness of the spaces of Kähler-type embeddings is concerned, let us note first that a Kähler embedding into (M, ω) may be holomorphic with respect to different complex structures compatible with ω, and, second, that two Kähler-type embeddings lying in the same $\mathrm{Symp}_0(M, \omega)$-orbit have to be holomorphic with respect to complex structures lying in the same orbit of the $\mathrm{Symp}_0(M, \omega)$-

action on the space $CK(M)$ of Kähler-type complex structures on M, hence in the same connected component of the set of complex structures on M compatible with ω.

Unfortunately, we know almost nothing about the latter connected components or about the space $CK(M)/\mathrm{Symp}_0(M, \omega)$. Because of this, we cannot say much about the orbits of the $\mathrm{Symp}_0(M, \omega)$-action on the space of Kähler-type embeddings of given domains into (M, ω). However, we can say more about the actions of $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$ and of the symplectic Torelli group $\mathrm{Symp}_H(M, \omega)$ (the group of symplectomorphisms of (M, ω) acting trivially on the homology) on the spaces of Kähler-type embeddings of balls into (M, ω).

Namely, assume that (M, ω) is either of the following: $\mathbb{C}P^n$ with the standard symplectic form; $\mathbb{T}^{2n}$ or a K3 surface equipped with a Kähler-type symplectic form ω which is irrational (i.e., the real cohomology class $[\omega] \in H^2(M; \mathbb{R})$ is not a real multiple of a rational one). Then any two Kähler-type embeddings of an arbitrary disjoint union of balls into (M, ω) can be mapped into each other by the action of $\mathrm{Symp}_H(M, \omega)$. If these embeddings are holomorphic with respect to complex structures compatible with ω and lying in the same connected component of $CK(M)$, then these embeddings can be mapped into each other by the action of $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$. In particular, these two Kähler-type embeddings are holomorphic with respect to isotopic complex structures compatible with ω.

Note that if (M, ω) is a rational or ruled closed symplectic 4-manifold, or if (M, ω) is obtained from a rational or ruled closed symplectic 4-manifold by removing either a closed 2-dimensional symplectic submanifold or a Lagrangian submanifold diffeomorphic to $\mathbb{S}^2$ or $\mathbb{R}P^2$ (for instance, if (M, ω) is the standard symplectic ball), then any two symplectic embeddings of an arbitrary disjoint union of closed balls into (M, ω) can be mapped into each other by the action of $\mathrm{Symp}_0^c(M, \omega)$ – see [McD3], [BLW] (cf. [McD1, McD2, Lal, Bir1] for earlier partial results on the subject). In particular, this is true for any two Kähler-type, or tame, embeddings and thus the space of such embeddings is connected (see Theorem 3.4 below).

Let us now discuss our results on the existence of Kähler-type and tame embeddings.

Following the terminology in [EV1], we say that Kähler-type, or tame, embeddings of a disjoint union $\mathbb{W} = \bigsqcup_{i=1}^{k} W_i$ of compact domains with piecewise boundary into (M, ω) are **unobstructed** if there are no obstructions for the existence of such embeddings of $\lambda \mathbb{W} = \bigsqcup_{i=1}^{k} \lambda W_i$, $\lambda > 0$, into (M, ω) other than the symplectic volume. (The domains we consider always lie in the standard symplectic $\mathbb{R}^{2n}$.) Note that if Kähler-type embeddings of a disjoint union of domains into (M, ω) are unobstructed, this means, in particular, that one can choose a complex structure I on M compatible with ω so that the Kähler metric $\omega(\cdot, I\cdot) + \sqrt{-1}\omega(\cdot, \cdot)$ on M is flat outside a set of arbitrarily small volume.

Since the classical work of McDuff-Polterovich [McDP], it has been known that proving the existence of symplectic embeddings of closed balls $\bigsqcup_{i=1}^{k} B(r_i)$ of radii $r_1, \ldots, r_k$ into a closed symplectic manifold (M, ω) is equivalent to showing that an appropriate cohomology class of the complex blow-up $\widetilde{M}$ of M at k points can be represented by a symplectic form with suitable properties.

In this paper we extend the results of McDuff and Polterovich [McDP] to Kähler-type embeddings of $\bigsqcup_{i=1}^{k} B(r_i)$. Namely, assume M is closed. For each $x = (x_1, \ldots, x_k)$, $x_1, \ldots, x_k \in M$, and each complex structure I on M, denote by $\widetilde{M}_{I,x}$ the complex blow-up of (M, I) at $x_1, \ldots, x_k$. Let $\widetilde{I}$ be the lift of I to $\widetilde{M}$. We show that $\bigsqcup_{i=1}^{k} B(r_i)$ admits a Kähler-type embedding into (M, ω) if and only if for some complex structure I compatible with ω and some x the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to $\widetilde{I}$ (i.e., can be represented by a Kähler form on $(\widetilde{M}_{I,x}, \widetilde{I})$). In fact, if this cohomology

class is Kähler, then $\bigsqcup_{i=1}^{k} B(r_i)$ admits a Kähler-type embedding into (M, ω) which is holomorphic with respect to a complex structure on M compatible with ω and smoothly isotopic to I. Together with information on the Kähler cones of $(\widetilde{M}_{I,x}, \widetilde{I})$ for various $\widetilde{I}$ (compatible with ω) and x this allows us to show the existence of Kähler-type embeddings of $\bigsqcup_{i=1}^{k} B(r_i)$ into certain (M, ω) (see Section 3.1).

Here are some of the applications for particular symplectic manifolds (see Section 3):

If (M, ω) is either the standard symplectic $\mathbb{C}P^n$ or a standard symplectic ball of real dimension $2n$, then for any $l \in \mathbb{Z}_{>0}$, Kähler-type embeddings of a disjoint union of l^n equal balls are unobstructed. In particular, this implies that for the standard complex structure on $\mathbb{C}P^n$ and l^n equal balls of total symplectic volume smaller than the volume of $\mathbb{C}P^n$, there exists a Kähler form on $\mathbb{C}P^n$, isotopic to the standard Fubini-Study form, so that the resulting Kähler manifold admits a *Kähler* (i.e., both holomorphic and symplectic) embedding of the l^n equal balls – meaning, in particular, that the Kähler metric is a standard flat metric on the image of the balls.

For $M = \mathbb{C}P^2$ we also describe exactly which disjoint unions of $1 \le k \le 8$ (possibly different) balls admit Kähler-type embeddings into M. It then follows from [McD3] that any symplectic embedding of such a disjoint union of balls into $\mathbb{C}P^2$ is, in fact, of Kähler type.

In the cases where M is $\mathbb{C}P^1 \times \mathbb{C}P^1$ or a complex blow-up of $\mathbb{C}P^2$ at one point equipped with any symplectic form compatible with the orientation, we describe exactly which disjoint unions of $1 \le k \le 7$ (possibly different) balls admit Kähler-type embeddings into M that are holomorphic with respect to a complex structure isotopic to the "standard" one and compatible with the symplectic form. It again follows from [McD3] that any symplectic embedding of such a disjoint union of balls into M is, in fact, of Kähler type.

If M is either $\mathbb{T}^{2n}$ or a K3 surface and the Kähler-type symplectic form ω on M is irrational (i.e., the real cohomology class $[\omega] \in H^2(M;\mathbb{R})$ is not a real multiple of a rational one), then Kähler-type embeddings of any disjoint union of balls into (M, ω) are unobstructed. If ω is rational (i.e., $[\omega] \in H^2(M;\mathbb{R})$ is a real multiple of a rational class), then there may be obstructions for the existence of Kähler-type embeddings of disjoint unions of balls into (M, ω) that are independent of the symplectic volume. For instance, in the case $M = \mathbb{T}^4$ there are obstructions coming from the Seshadri constants (see Remark 3.26). In particular, this shows that there exist symplectic embeddings of disjoint unions of balls into $(\mathbb{T}^4, \omega)$ that are not Kähler-type.

Finally, we show that tame embeddings of disjoint unions of equal balls, or equal polydisks, or equal parallelepipeds, into $(\mathbb{T}^{2n}, \omega)$ are unobstructed provided ω lies in a $\mathrm{Diff}_0(\mathbb{T}^{2n})$-invariant open dense set of Kähler-type symplectic forms on $\mathbb{T}^{2n}$ containing all the irrational forms (the set depends on the embedded domains). By choosing possibly smaller $\mathrm{Diff}_0(\mathbb{T}^{2n})$-invariant open dense sets of such ω we make sure that the unobstructedness holds, in fact, for the tame embeddings into $(\mathbb{T}^{2n}, \omega)$ which are holomorphic with respect to complex structures that are not only tamed by ω but also "arbitrarily close" (in the sense of Hodge theory) to complex structures compatible with ω. Kähler-type (not just tame) embeddings of disjoint unions of equal balls, or equal polydisks, or equal parallelepipeds, into $(\mathbb{T}^{2n}, \omega)$ are unobstructed if ω is an irrational Kähler-type symplectic form lying in an appropriate dense $\mathrm{Diff}_0(\mathbb{T}^{2n})$-orbit in the space of forms (the orbit depends on the embedded domains). Similar results concerning balls and parallelepipeds hold also for symplectic manifolds underlying Kähler K3 surfaces.

We also discuss extensions of the results for K3 surfaces to a larger class of hyperkähler manifolds – see Section 3.5.

2 Preliminaries

Assume that M is a smooth manifold of real dimension $2n$.

Denote by $\mathrm{Diff}(M)$ the group of diffeomorphisms of M and by $\mathrm{Diff}_0(M)$ its identity component, and by $\mathrm{Diff}_H(M)$ the subgroup of $\mathrm{Diff}(M)$ formed by the diffeomorphisms acting trivially on the homology of M.

For a symplectic form ω on M denote by $\mathrm{Symp}(M, \omega)$ the group of symplectomorphisms of (M, ω), by $\mathrm{Symp}_0(M, \omega)$ its identity component and by $\mathrm{Symp}_H(M, \omega)$ the **symplectic Torelli group**, i.e. the group of symplectomorphisms of (M, ω) acting trivially on homology. For a non-compact symplectic manifold (M, ω) we denote by $\mathrm{Symp}^c(M, \omega)$ the group of compactly supported symplectomorphisms of (M, ω) and by $\mathrm{Symp}_0^c(M, \omega)$ its identity component.

2.1 Kähler-type complex and symplectic structures

Let ω and I be a symplectic form and a complex structure on M. The form ω **tames** I if $\omega(v, Iv) > 0$ for any non-zero $v \in TM$, and is **compatible with** I if it tames I and is also I-invariant, meaning that $\omega(\cdot, I\cdot) + i\omega(\cdot, \cdot)$ is a Hermitian metric on (M, I). In the latter case the pair (ω, I) is called a **Kähler structure** on M.

A cohomology class in $H^2(M; \mathbb{R})$ is called **Kähler (with respect to a complex structure I on M)**, if it can be represented by a symplectic form on M compatible with I. The cohomology classes that are Kähler with respect to I form the **Kähler cone of** (M, I) in $H^2(M; \mathbb{R})$.

A symplectic/complex structure on M is said to be of **Kähler type**, if it appears in *some* Kähler structure on M.

Example 2.1

Any complex structure on $\mathbb{C}P^2$ is of Kähler type [Yau1]. Whether this holds also for $\mathbb{C}P^n$, $n > 2$, is a well-known open question – see e.g. [LiWo, Tos], cf. [HuKP].

For a closed manifold M, $\dim_{\mathbb{C}} M = 2$, with an even $b_2(M)$, any complex structure on M is of Kähler type [Buch, Lam]. In particular, any complex structure on $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ and $\mathbb{T}^4$ is of Kähler type.

The Kähler-type symplectic/complex structures on $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ are exactly the ones that can be mapped by a diffeomorphism of $\mathbb{T}^{2n}$ to a linear symplectic/complex structure – i.e., a symplectic/complex structure whose lift to $\mathbb{R}^{2n}$ has constant coefficients (see e.g. [EV1, Prop. 6.1]). There do exist complex structures on $\mathbb{T}^{2n}$, for each $n \geq 3$, that are not of Kähler-type [Som, p.212], [BSV].

If M is a smooth manifold underlying a complex K3 surface, then any complex structure on M is of Kähler-type [Siu] and its first Chern class is zero (see e.g. [Bea2])[1]. It is well-known that any complex

---

[1]In fact, all smooth manifolds M underlying a complex K3 surface are diffeomorphic. They all are compact and connected. All complex structures on such an M define the same orientation on M so that $b_+(M) = 3$ and $b_-(M) = 19$ – see e.g. [Bea2]. Further on, we will always equip a K3 surface with this standard orientation. Alternatively, to fix the orientation, one can use the

structure on such an M appears in a hyperkähler structure and so does any Kähler-type symplectic form on M – see e.g. [EV3, Prop. 5.1].

It is a well-known open question whether all symplectic forms on $\mathbb{T}^{2n}$ and K3 surfaces (equipped with the standard smooth structures) are of Kähler type – see e.g. [Don]. (There do exist exotic smooth structures on topological K3 surfaces that admit symplectic forms – see [FiS], cf. [Ch]; it follows from the classification of complex surfaces that these exotic smooth K3 surfaces do not admit complex structures and therefore the symplectic forms on them are not of Kähler type). Using the Kodaira-Spencer stability theorem [KoS2] and Torelli theorems for tori (see e.g. [BaHPV, Ch. I, Thm. 14.2]) and K3 surfaces (see [BurR], cf. [Bea2, p.96]) one can show that the Kähler-type forms on these manifolds form an open subset of the space of all symplectic forms (with respect to the C^∞-topology – see e.g. [AmV1], cf. [StT] for a more general result).

Denote by $SK(M)$ the space of all Kähler-type symplectic forms on M and by $CK(M)$ the space of all Kähler-type complex structures on M.

We equip these spaces with the C^∞-topologies (the complex structures are viewed here as integrable almost-complex structures – i.e., $(1,1)$-tensors).

The space $CK(M)$ is an open subspace of the space of all complex structures on M; both of these spaces are locally C^∞-path connected – see Proposition A.2.

Given $\omega \in SK(M)$, denote by $Cmpt(M, \omega) \subset CK(M)$ the space of all (Kähler-type) complex structures on M compatible with ω.

If $\mathcal{C}_0$ is a connected component of $CK(M)$, we say that ω and $\mathcal{C}_0$ are **compatible**, if some complex structure lying in $\mathcal{C}_0$ is compatible with ω – i.e., if the set $\mathcal{C}_0 \cap Cmpt(M, \omega)$ is non-empty.

With this terminology, every Kähler-type ω is compatible with some connected component of $CK(M)$.

Note that the space $CK(M)$ is invariant under the natural action of $\mathrm{Diff}(M)$ (and hence of $\mathrm{Diff}_0(M)$) on the space of complex structures on M, but the space $Cmpt(M, \omega)$ is not invariant even under the $\mathrm{Diff}_0(M)$-action.

Definition 2.2

Define the **Teichmüller space of Kähler-type complex structures** on M as

$$\mathrm{Teich}(M) := CK(M)/\mathrm{Diff}_0(M),$$

and if $\mathcal{C}_0$ is a connected component of $CK(M)$, then

$$\mathrm{Teich}_{\mathcal{C}_0}(M) := \mathcal{C}_0/\mathrm{Diff}_0(M).$$

The set $\mathrm{Teich}_{\mathcal{C}_0}(M)$ is a connected component of $\mathrm{Teich}(M)$.

Given $I \in CK(M)$, denote by $[I] \in \mathrm{Teich}(M)$ the image of I under the projection $CK(M) \to \mathrm{Teich}(M)$.

Let $\omega \in SK(M)$. Denote the image of $Cmpt(M, \omega)$ under the quotient projection $CK(M) \to \mathrm{Teich}(M)$ by

$$\mathrm{Teich}(M, \omega) := \frac{\mathrm{Diff}_0(M) \cdot Cmpt(M, \omega)}{\mathrm{Diff}_0(M)} \subset \mathrm{Teich}(M).$$

---

fact that all complex K3 surfaces are deformation equivalent and, in particular, oriented diffeomorphic – see e.g. [Bea2].

In other words, $\texttt{Teich}(M,\omega)$ is the set of points of $\texttt{Teich}(M)$ that can be represented by complex structures on M compatible with ω.

Given a connected component $\mathcal{C}_0$ of $CK(M)$, the image of $Cmpt(M,\omega) \cap \mathcal{C}_0$ under the quotient projection $\mathcal{C}_0 \to \texttt{Teich}_{\mathcal{C}_0}(M)$ will be denoted by

$$\texttt{Teich}_{\mathcal{C}_0}(M,\omega) := \frac{\mathrm{Diff}_0(M) \cdot (Cmpt(M,\omega) \cap \mathcal{C}_0)}{\mathrm{Diff}_0(M)} =$$
$$= \texttt{Teich}(M,\omega) \cap \texttt{Teich}_{\mathcal{C}_0}(M).$$

Note that the space $\texttt{Teich}(M)$ may not be Hausdorff – this is the case, for instance, when M is a smooth manifold underlying a complex K3 surface, see Remark 10.11.

2.2 Hodge decomposition

Assume M is closed.

A complex structure I on M defines a (p,q)-decomposition of any (complex-valued) differential k-form η on M, $k = 0,1,\ldots,2n$ – we will denote it by $\eta = \sum_{p+q=k} \eta_I^{p,q}$. If I is of Kähler type, then the (p,q)-decomposition on differential forms, defined by I, induces a decomposition, called the Hodge decomposition[2], on the complex cohomology of M: $H^k(M;\mathbb{C}) = \oplus_{p+q=k} H_I^{p,q}(M;\mathbb{C})$. Given a cohomology class $\alpha \in H^*(M;\mathbb{C})$, we will denote its (p,q)-component in the Hodge decomposition by $\alpha_I^{p,q} \in H_I^{p,q}(M;\mathbb{C})$. We denote $H_I^{p,q}(M;\mathbb{R}) := H_I^{p,q}(M;\mathbb{C}) \cap H^*(M;\mathbb{R})$. For a real cohomology class $\alpha \in H^{2p}(M;\mathbb{R})$ its (p,p)-component is also real. A differential form on M is called a (p,q)-**form**, and a cohomology class of M is called a (p,q)-**class** (with respect to I), if their (p,q)-component is their only non-zero component.

If ω is compatible with I, then it is a $(1,1)$-form with respect to I, and, accordingly, its cohomology class is a $(1,1)$-class.

If a cohomology class $\alpha \in H^2(M;\mathbb{R})$ is Kähler with respect to a Kähler-type complex structure I on M, then for any Kähler-type complex structure I' sufficiently C^∞-close to I the class $\alpha_{I'}^{1,1}$ is Kähler with respect to I' – see Proposition A.2.

2.3 Kähler-type and tame embeddings

Let ω be a Kähler-type symplectic form on M.

The symplectic and complex structures on $\mathbb{R}^{2n}$ are always assumed to be the standard ones and will be denoted, respectively, by ω_0 and J_0.

Let $W_1,\ldots,W_k \subset \mathbb{R}^{2n}$ be compact domains with piecewise-smooth boundaries whose interiors contain the origin. Denote

$$\mathbb{W} := \bigsqcup_{i=1}^{k} W_i.$$

---

[2]Although the proof of the existence of the Hodge decomposition involves both the complex structure I and a Kähler form on (M,I), one can show (see e.g. [Voi, Vol. 1, Prop. 6.11]) that, in fact, the Hodge decomposition on the cohomology depends only on I.

Given $\lambda > 0$, define

$$\lambda \mathbb{W} := \bigsqcup_{i=1}^{k} \lambda W_i.$$

We say that $f : \mathbb{W} \to M$ is a smooth/symplectic/holomorphic embedding if it extends to such an embedding of the disjoint union of some open neighborhoods of W_i, $i = 1, \ldots, k$. We will denote by $f_i : W_i \to M$, $i = 1, \ldots, k$, the restriction of such an extension to a neighborhood of W_i.

Clearly, if $f : \mathbb{W} \to (M, \omega)$ is a symplectic embedding, then

$$\mathrm{Vol}\,(\mathbb{W}, \omega_0) < \mathrm{Vol}\,(M, \omega) = \langle [\omega]^n, [M] \rangle.$$

(Recall that Vol stands for the symplectic volume and $\langle \cdot, \cdot \rangle$ stands for the natural pairing between cohomology and homology classes.)

Definition 2.3

Let $f : \mathbb{W} = \bigsqcup_{i=1}^{k} W_i \to (M^{2n}, \omega)$ be an embedding.

Given a Kähler-type complex structure $I \in Cmpt(M, \omega)$, we say that the embedding f is of $[I]$-**Kähler type**, if it is symplectic with respect to ω and holomorphic with respect to some *(not a priori fixed)* complex structure J on M compatible with ω and isotopic to I, i.e. $[J] = [I] \in \texttt{Teich}(M, \omega)$. (Such a J is tautologically of Kähler-type).

We say that the embedding f is of **Kähler type**, if it is symplectic with respect to ω and holomorphic with respect to some *(not a priori fixed)* complex structure compatible with ω – or, in other words, if f is of $[I]$-Kähler type for some *(not a priori fixed, and possibly non-unique)* $I \in Cmpt(M, \omega)$.

Given a *fixed Kähler structure* (ω, I) on M, we say that $f : \mathbb{W} \to (M^{2n}, \omega, I)$ is a **Kähler** embedding if it is Kähler-isometric – i.e. both symplectic with respect to ω and holomorphic with respect to I.

We say that the embedding $f : \mathbb{W} \to (M^{2n}, \omega)$ is **tame** if it is symplectic with respect to ω and holomorphic with respect to some *(not a priori fixed)* Kähler-type complex structure I tamed by ω.

Assuming that M is closed and $\varepsilon > 0$, we say that the embedding f is ε-**tame** if it is symplectic with respect to ω and holomorphic with respect to some *(not a priori fixed)* Kähler-type complex structure I so that

- I is tamed by ω,

- the class $[\omega]_I^{1,1}$ is Kähler,

- $\left| \left\langle \left([\omega]_I^{2,0} + [\omega]_I^{0,2} \right)^n, [M] \right\rangle \right| < \varepsilon$.

With the terminology as in Definition 2.3, for an embedding into a (closed) symplectic manifold (M, ω) we have the following equivalences and hierarchy of properties:

$$\text{Kähler-type} \iff [I]\text{-Kähler type for some (possibly non-unique!) } I \in Cmpt(M, \omega),$$

$$\text{tame} \iff \varepsilon\text{-tame for some } \varepsilon,$$

$$\text{Kähler-type} \implies \varepsilon\text{-tame for every } \varepsilon \implies \text{tame} \implies \text{symplectic.}$$

In general, there do exist symplectic embeddings of balls that are not of Kähler-type (see Remark 3.26), but we do not know whether there exist symplectic embeddings of balls or any other domains into any M, equipped with a Kähler-type symplectic form, that are not tame, or tame embeddings that are not of Kähler-type – see Remark 3.32 and Question 3.33.

Remark 2.4

For any $I \in Cmpt(M, \omega)$ any finite disjoint union of sufficiently small balls (and hence of arbitrary sufficiently small domains in $(\mathbb{R}^{2n}, \omega_0)$) admits an $[I]$-Kähler-type embedding into (M, ω) – this follows e.g. from [McDP, Prop. 5.5.A]).

Remark 2.5

If a Kähler manifold (M, ω, I) admits a Kähler embedding of a domain in $\mathbb{R}^{2n}$, its restriction to the image of the embedding is flat. This provides a strong obstruction to the existence of Kähler embeddings (if one fixes a Kähler structure on the manifold).

Remark 2.6

The existence of $[I]$-Kähler-type embeddings has been previously studied – in a different guise – in [Ec], [Fl], [LuWa], [Tr], [WN1], [WN2] in the following setup.

Let (M, ω, I) be a closed Kähler manifold (in [Ec], [Fl], [Tr], [WN1], [WN2] it is assumed that the complex manifold (M, I) is projective and the cohomology class $[\omega] \in H^2(M; \mathbb{R})$ is the first Chern class of an ample line bundle on (M, I). Consider the following question: Given a disjoint union $\mathbb{W} = \bigsqcup_{i=1}^k W_i$ of domains in $\mathbb{R}^{2n}$, does there exist a Kähler form η on (M, I), isotopic to ω, such that (M, η, I) admits a Kähler embedding of $\mathbb{W}$?

One easily sees that the answer to the latter question is positive if and only if (M, ω) admits an $[I]$-Kähler-type embedding of $\mathbb{W}$.

Let us introduce additional terminology related to Kähler-type and tame embeddings.

Let $\mathcal{J} \subset CK(M)$ be a non-empty set. If a Kähler-type, or tame, embedding f is holomorphic with respect to some complex structure belonging to $\mathcal{J}$, we say that f **favors** $\mathcal{J}$.

In particular, if $\mathcal{J} = \mathcal{C}_0$ is a connected component $\mathcal{C}_0$ of $CK(M)$, we say that such an f **favors** $\mathcal{C}_0$, or that $\mathcal{C}_0$ is a **favorite connected component of** f.

With this terminology, tautologically,

- For $I \in Cmpt(M, \omega)$, every $[I]$-Kähler-type embedding into (M, ω) favors the $\text{Diff}_0(M)$-orbit of I in $CK(M)$.

- Every Kähler-type embedding into (M, ω) favors a connected component of $Cmpt(M, \omega)$, and consequently a connected component of $CK(M)$.

Note that, in principle, a Kähler-type embedding may be of $[I]$-Kähler-type for several $I \in Cmpt(M, \omega)$ and may favor more than one connected component of $Cmpt(M, \omega)$ or of $CK(M)$.

The group $\mathrm{Symp}(M,\omega)$ acts naturally on the space of Kähler-type, or tame, or ε-tame, embeddings $\mathbb{W} \to (M,\omega)$. Consequently, so do the groups $\mathrm{Symp}_H(M,\omega)$, $\mathrm{Symp}_0(M,\omega)$ and $\mathrm{Symp}(M,\omega) \cap \mathrm{Diff}_0(M)$.

By a standard result concerning extensions of symplectic isotopies (proved as in [McDS, Thms. 3.3.1, 3.3.2]), if the domains $W_1, \ldots, W_k$ are starshaped, then any two symplectic embeddings $f : \mathbb{W} \to (M,\omega)$, $f' : \mathbb{W} \to (M,\omega)$ can be connected by a (smooth) path of symplectic embeddings $\mathbb{W} \to (M,\omega)$ if and only if f and f' lie in the same $\mathrm{Symp}_0(M,\omega)$-orbit – i.e., there exists $\phi \in \mathrm{Symp}_0(M,\omega)$ such that $\phi \circ f = f'$.

Also note, that if $\phi \circ f = f'$ for some $\phi \in \mathrm{Diff}(M)$ and f is holomorphic with respect to a complex structure I on M, then f' is holomorphic with respect to the complex structure $\phi_* I$.

These simple observations easily yield the following corollary.

Proposition 2.7

Assume that the domains $W_1, \ldots, W_k$ are starshaped.

Then any two embeddings $f, f' : \mathbb{W} = \bigsqcup_{i=1}^{k} W_i \to (M,\omega)$ lie in the same connected component of the space of Kähler-type embeddings $\mathbb{W} \to (M,\omega)$ if and only if they lie in the same orbit of the $\mathrm{Symp}_0(M,\omega)$-action. In this case f, f' are holomorphic with respect to complex structures compatible with ω that lie in the same orbit of the $\mathrm{Symp}_0(M,\omega)$-action on $CK(M)$. In particular, such f, f' favor a common connected component of $Cmpt(M,\omega)$ and, consequently, a common connected component of $CK(M)$.

If Kähler-type embeddings $f, f' : \mathbb{W} \to (M,\omega)$ lie in the same $\mathrm{Symp}(M,\omega) \cap \mathrm{Diff}_0(M)$-orbit, then they are holomorphic with respect to complex structures compatible with ω that lie in the same orbit of the $\mathrm{Diff}_0(M)$-action on $CK(M)$. In particular, there exists $[I] \in \mathtt{Teich}(M,\omega)$ such that both f and f' are of $[I]$-Kähler-type, and f, f' favor a common connected component of $CK(M)$. ∎

Let us define a quantity that will be important for the study of the portion of the volume of a target closed symplectic manifold that can be filled out by Kähler-type and tame embeddings.

Definition 2.8

Assume M is a closed manifold equipped with a Kähler-type symplectic form ω.

Define the number $0 < v_K(M,\omega,\mathbb{W}) \le 1$ by

$$v_K(M,\omega,\mathbb{W}) := \sup_{\lambda} \frac{\mathrm{Vol}(\lambda\mathbb{W},\omega_0)}{\mathrm{Vol}(M,\omega)},$$

where the supremum is taken over all $\lambda > 0$ such that $\lambda\mathbb{W}$ admits a Kähler-type embedding into (M,ω).

Given $\varepsilon > 0$, define the number $0 < v_{T,\varepsilon}(M,\omega,\mathbb{W}) \le 1$ by

$$v_{T,\varepsilon}(M,\omega,\mathbb{W}) := \sup_{\lambda} \frac{\mathrm{Vol}(\lambda\mathbb{W},\omega_0)}{\mathrm{Vol}(M,\omega)},$$

where the supremum is taken over all $\lambda > 0$ such that $\lambda\mathbb{W}$ admits an ε-tame embedding into (M,ω). (In view of Remark 2.4, the set of such λ is always non-empty.)

Remark 2.9

1. Clearly,

$$\nu_K(M, \omega, \mathbb{W}) \leq \nu_{T,\varepsilon_1}(M, \omega, \mathbb{W}) \leq \nu_{T,\varepsilon_2}(M, \omega, \mathbb{W})$$

for any $0 < \varepsilon_1 \leq \varepsilon_2$.

2. The equality $\nu_K(M, \omega, \mathbb{W}) = 1$, respectively $\nu_{T,\varepsilon}(M, \omega, \mathbb{W}) = 1$, is equivalent to the claim that the Kähler-type, respectively ε-tame, embeddings of $\lambda\mathbb{W}$ into (M, ω), for all $\lambda > 0$, are unobstructed.

3 Main results

In this section we state the main results of the paper.

For $r > 0$ we denote by $B^{2n}(r)$ the closed Euclidean $2n$-dimensional ball of radius r in $\mathbb{R}^{2n}$ centered at the origin.

3.1 Kähler-type embeddings of balls in general symplectic manifolds and Kähler classes on complex blow-ups

In this section M, $\dim_\mathbb{R} = 2n$, is a closed connected manifold equipped with a Kähler-type symplectic form ω. Let $k \in \mathbb{Z}_{>0}$, $\bar{r} = (r_1, \ldots, r_k) \in (\mathbb{R}_{>0})^k$.

Denote by $\hat{M}^k \subset M^k$ the set of k-tuples of pairwise distinct points of M.

For each $x := (x_1, \ldots, x_k) \in \hat{M}^k$ and each complex structure I on M, let $\widetilde{M}_{I,x}$ denote the complex blow-up of (M, I) at $x_1, \ldots, x_k$.

For any I and x we will use the same notation for the following objects (suppressing the dependence on I and x):

- The lift of I to $\widetilde{M}_{I,x}$ will be denoted by $\widetilde{I}$.

- $\Pi : \widetilde{M}_{I,x} \to M$ will denote the natural projection.

- The cohomology classes Poincaré-dual to the fundamental homology classes of the exceptional divisors will be denoted by $e_1, \ldots, e_k \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$.

For all I and x the groups $H^2(\widetilde{M}_{I,x}; \mathbb{R})$ can be canonically identified (preserving the numbering of the classes $e_1, \ldots, e_n$) and we will use these identifications without further mention.

Definition 3.1

Define the set $\mathcal{K}(\bar{r}) \subset CK(M) \times \hat{M}^k$ as

$$\mathcal{K}(\bar{r}) := \Big\{ (I, x) \mid I \in \mathrm{Diff}_0(M) \cdot Cmpt(M, \omega), x \in \hat{M}^k \text{ and}$$

$$\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R}) \text{ is Kähler with respect to } \widetilde{I} \Big\},$$

Let $pr : \mathcal{K}(\bar{r}) \to CK(M)$ be the natural projection.

Define the set $\mathtt{K}(\bar{r}) \subset \mathtt{Teich}(M, \omega)$ as

$$\mathtt{K}(\bar{r}) := pr(\mathcal{K}(\bar{r}))/\mathrm{Diff}_0(M) =$$

$$= \Big\{ \iota \in \mathtt{Teich}(M, \omega) \ \big| \ \exists I \in Cmpt(M, \omega), [I] = \iota, \ and \ x \in \hat{M}^k \ s.t.$$

$$\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R}) \ is \ K\ddot{a}hler \ with \ respect \ to \ \widetilde{I} \Big\}.$$

Let $\mathcal{C}_0$ be a connected component of $CK(M)$.
Define the set $\mathcal{K}_{\mathcal{C}_0}(\bar{r}) \subset \mathcal{C}_0 \times \hat{M}^k$ as

$$\mathcal{K}_{\mathcal{C}_0}(\bar{r}) := (\mathcal{C}_0 \times \hat{M}^k) \cap \mathcal{K}(\bar{r}) =$$

$$= \Big\{ (I, x) \ \big| \ I \in \mathrm{Diff}_0(M) \cdot \big(\mathcal{C}_0 \cap Cmpt(M, \omega)\big), x \in \hat{M}^k \ and$$

$$\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R}) \ is \ K\ddot{a}hler \ with \ respect \ to \ \widetilde{I} \Big\}.$$

Define the set $\mathtt{K}_{\mathcal{C}_0}(\bar{r}) \subset \mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$ as

$$\mathtt{K}_{\mathcal{C}_0}(\bar{r}) := \mathtt{K}(\bar{r}) \cap \mathtt{Teich}_{\mathcal{C}_0}(M) = pr\left(\mathcal{K}_{\mathcal{C}_0}(\bar{r})\right)/\mathrm{Diff}_0(M) =$$

$$= \Big\{ \iota \in \mathtt{Teich}_{\mathcal{C}_0}(M, \omega) \ \big| \ \exists I \in \mathcal{C}_0 \cap Cmpt(M, \omega), [I] = \iota, \ and \ x \in \hat{M}^k \ s.t.$$

$$\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R}) \ is \ K\ddot{a}hler \ with \ respect \ to \ \widetilde{I} \Big\}.$$

As we will see in Corollary 5.19, the fibers of $pr : \mathcal{K}(\bar{r}) \to CK(M)$ are connected and therefore $\mathtt{K}(\bar{r})$ (respectively, $\mathtt{K}_{\mathcal{C}_0}(\bar{r})$) is connected if and only if so is $\mathcal{K}(\bar{r})$ (respectively, $\mathcal{K}_{\mathcal{C}_0}(\bar{r})$).

Theorem 3.2

Let M be a closed manifold equipped with a Kähler-type symplectic form ω.

(I) The set $\bigsqcup_{i=1}^k B^{2n}(r_i)$ admits a Kähler-type embedding into (M, ω) if and only if for some $I \in Cmpt(M, \omega)$ and some $x \in \hat{M}^k$ the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to $\widetilde{I}$.

More precisely, assume I is a complex structure on M compatible with ω and $\Sigma \subset (M, I)$ is a proper (possibly empty) complex submanifold. If $M = \mathbb{C}P^n \times \mathbb{C}P^1$ (with the product complex structure), we also allow $\Sigma = (\mathbb{C}P^{n-1} \times \mathbb{C}P^1) \cup (\mathbb{C}P^n \times \mathrm{pt})$ (which is not smooth).

Then the following conditions are equivalent:

- There exists $x = (x_1, \ldots, x_k) \in \hat{M}^k$, $x_1, \ldots, x_k \in M \setminus \Sigma$, such that the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to $\widetilde{I}$.

- There exists an $[I]$-Kähler-type embedding $f : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M \setminus \Sigma, \omega)$ which is holomorphic with respect to the complex structure $\phi_1^* I$ compatible with ω, where $\{\phi_t\}_{0 \le t \le 1} \subset \mathrm{Diff}_0(M)$, $\phi_0 = Id$, is an isotopy such that $\phi_t(\Sigma) = \Sigma$ for all $t \in [0,1]$.

(II) Let $\mathcal{C}_0$ be a connected component of $CK(M)$. Assume that the set $\mathtt{K}_{\mathcal{C}_0}(\bar{r})$ (or, equivalently, the set $\mathcal{K}_{\mathcal{C}_0}(\bar{r})$) is connected.

Then any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ favoring $\mathcal{C}_0$ lie in the same $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$-orbit. In particular, there exists $[I] \in \mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$ such that both embeddings are of $[I]$-Kähler type.

If, in addition, $\mathrm{Symp}_H(M)$ acts transitively on the set of connected components of $CK(M)$ compatible with ω, then any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ lie in the same $\mathrm{Symp}_H(M, \omega)$-orbit.

For the proof of Theorem 3.2 see Section 5 (Theorem 5.9).

Remark 3.3

1. For an analogue of part (I) of Theorem 3.2 for tame embeddings see Proposition 5.17 below.

2. [Fl, Remark 5.2] states that the methods of [Fl] can be used to prove a result that, in view of Remark 2.6, is equivalent to Part (I) Theorem 3.2 in the case where ω is compatible with a complex structure I on M, so that the complex manifold (M, I) is projective and the cohomology class $[\omega] \in H^2(M; \mathbb{R})$ is the first Chern class of an ample line bundle on (M, I).

Recall that the symplectic blow-up construction (see e.g. [McDS, Sec. 7], cf. Section 5.5 below) associates to a complex blow-up of a symplectic manifold at a point (with respect to, say, an almost complex structure compatible with the symplectic form and integrable near the point) a class of symplectic forms on the blow-up. We will call such a symplectic form on the blow-up a **blow-up symplectic form**. A **rational (4-dimensional) symplectic manifold** is either $\mathbb{C}P^2$ with a standard Fubini-Study symplectic form, or $\mathbb{C}P^1 \times \mathbb{C}P^1$ with a product symplectic form, or their blow-up at several points equipped with a blow-up symplectic form.

A **ruled (4-dimensional) symplectic manifold** is a closed symplectic 4-manifold which is the total space of an $\mathbb{S}^2$-fiber bundle over a closed connected orientable 2-dimensional surface, so that all the fibers are symplectic submanifolds [LalM].

For rational or ruled closed symplectic manifolds of real dimension 4 we have a stronger connectedness result, which follows directly from similar connectedness results for *symplectic* (not necessarily Kähler-type) embeddings of balls that were proved in [McD3] and [BLW] (for previous partial results see [McD1, McD2, Lal, Bir1]).

Theorem 3.4

Let (M, ω) be a rational or ruled closed symplectic manifold, $\dim_{\mathbb{R}} M = 4$. Let $\Sigma \subset (M, \omega)$ be either of the following:

(1) the empty set;

(2) a finite union of closed compact symplectic submanifolds (without boundary) of real dimension 2 whose pairwise intersections (if they exist) are transverse and ω–orthogonal;

(3) a Lagrangian submanifold which is diffeomorphic to either $\mathbb{S}^2$ or $\mathbb{R}P^2$.

Then for any $k \in \mathbb{Z}_{>0}$ and any $r_1, \ldots, r_k > 0$, the following claims hold:

(I) Any two Kähler-type, respectively tame, embeddings $\bigsqcup_{i=1}^{k} B^4(r_i) \to (M \setminus \Sigma, \omega)$ lie in the same orbit of the $\mathrm{Symp}_0^c(M \setminus \Sigma, \omega)$-action. In particular, (for fixed $r_1, \ldots, r_k$) the space of all such Kähler-type, respectively tame, embeddings is connected.

(II) If there exists an $[I]$-Kähler-type, respectively tame, embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (M \setminus \Sigma, \omega)$, then any symplectic embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (M \setminus \Sigma, \omega)$ is of $[I]$-Kähler type, respectively tame.

Proof of Theorem 3.4:

Any two symplectic embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M \setminus \Sigma, \omega)$ lie in the same orbit of the $\mathrm{Symp}_0^c(M \setminus \Sigma, \omega)$-action:

- In case (1) this was proved in [McD3, Cor. 1.5].

- In case (2), where $\Sigma \subset (M, \omega)$ is just a symplectic submanifold, this was essentially proved in [McD3, Cor. 1.5] – see [BLW, Prop. 2.1] where more details of the proof are spelled out in the case where the symplectic submanifold $\Sigma \subset (M, \omega)$ is a sphere; in fact [McDO, Cor. 1.2.13], used in the proof of [BLW, Prop. 2.1], yields the same claim for any Σ as in (2).

- In case (3), this is proved in [BLW, Thm. 1.1].

Since $[I]$-Kähler-type, respectively tame, embeddings into $(M \setminus \Sigma, \omega)$ are mapped by the $\mathrm{Symp}_0^c(M \setminus \Sigma, \omega)$-action into $[I]$-Kähler-type, respectively tame, embeddings, we get the results of both parts of the theorem. ∎

Now let us discuss applications of Theorem 3.2 and Theorem 3.4 to particular symplectic manifolds.

3.2 The case of projective spaces and their products

Denote by ω_{FS}, or by $\omega_{FS,n}$, the Fubini-Study form on $\mathbb{C}P^n$ normalized so that $\int_{\mathbb{C}P^1} \omega_{FS} = \pi$. Let I_{st} be the standard complex structure on $\mathbb{C}P^n$.

We also set $\mathbb{C}P^0 := \mathrm{pt}$.

Theorem 3.5

Consider the manifold $M := \mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_m}$, $n_1, \ldots, n_m > 0$, $n_1 + \ldots + n_m =: n$, endowed with the symplectic form $\omega_c := c_1 \omega_{FS,n_1} \oplus \ldots \oplus c_m \omega_{FS,n_m}$, $c := (c_1, \ldots, c_m) \in (\mathbb{R}_{>0})^m$. Let I be the complex structure on M which is the product of the standard complex structures on the factors.

Let $l_1, \ldots, l_m \in \mathbb{Z}_{>0}$ so that $[l_1 : \ldots : l_m] = [c_1 : \ldots : c_m] \in \mathbb{R}P^m$. Let

$$k := \frac{(n_1 + \ldots + n_m)!}{n_1! \cdot \ldots \cdot n_m!} l_1^{n_1} \cdot \ldots \cdot l_m^{n_m}.$$

Let $\Sigma \subset (M,I)$ be a proper (possibly empty) complex submanifold. If $M = \mathbb{C}P^{n_1} \times \mathbb{C}P^1$, we also allow $\Sigma = (\mathbb{C}P^{n_1-1} \times \mathbb{C}P^1) \cup (\mathbb{C}P^{n_1} \times \mathrm{pt})$.

Then Kähler-type embeddings of disjoint unions of k equal balls into $(M \setminus \Sigma, \omega_c)$ are unobstructed.

More precisely, if $k\mathrm{Vol}(B^{2n}(r), \omega_0) < \mathrm{Vol}(M, \omega)$, then for each proper (possibly empty) complex submanifold $\Sigma \subset (M,I)$, there exists an $[I]$-Kähler-type embedding of $\bigsqcup_{i=1}^{k} B^{2n}(r)$ into (M, ω) which is holomorphic with respect to a complex structure on M that is compatible with ω and isotopic to I by an isotopy preserving Σ (as a set).

For the proof of Theorem 3.5 see Section 6 (Theorem 6.1).

For $n_1 = \ldots = n_m = 1$, $l_1 = \ldots = l_m = l$, $c_1 = \ldots = c_m = 1$ and $\Sigma = \emptyset$, Theorem 3.5 yields the following corollary.

Corollary 3.6

Let $M := (\mathbb{C}P^1)^m$. Consider a symplectic form ω on M which is the product of equal Fubini-Study forms on the $\mathbb{C}P^1$-factors of M. Let I be the complex structure on M which is the product of the standard complex structures on the $\mathbb{C}P^1$-factors of M.

Then for any $l \in \mathbb{Z}_{>0}$, Kähler-type embeddings of disjoint unions of $m!l^m$ equal balls into (M, ω) are unobstructed.

More precisely, if $m!l^m \mathrm{Vol}(B^{2n}(r), \omega_0) < \mathrm{Vol}(M, \omega)$, then there exists a Kähler-type embedding of $\bigsqcup_{i=1}^{m!l^m} B^{2n}(r)$ into (M, ω) which is holomorphic with respect to a complex structure on M that is compatible with ω and isotopic to I. ∎

Now let us consider the case of a single complex projective space.

Theorem 3.7

A. For each $l \in \mathbb{Z}_{>0}$, Kähler-type embeddings of disjoint unions of l^n equal balls into $(\mathbb{C}P^n, \omega_{FS})$ are unobstructed.

More precisely, if $l^n \mathrm{Vol}(B^{2n}(r), \omega_0) < \mathrm{Vol}(\mathbb{C}P^n, \omega_{FS})$, then for any complex structure J on $\mathbb{C}P^n$ compatible with ω_{FS} (and, in particular, for the standard complex structure I_{st}) and for each proper (possibly empty) complex submanifold $\Sigma \subset (\mathbb{C}P^n, J)$, there exists a $[J]$-Kähler-type embedding of $\bigsqcup_{i=1}^{l^n} B^{2n}(r)$ into $(\mathbb{C}P^n, \omega_{FS})$ which is holomorphic with respect to a complex structure on $\mathbb{C}P^n$ that is compatible with ω_{FS} and isotopic to J by an isotopy preserving Σ (as a set).

B. The group $\mathrm{Symp}_H(\mathbb{C}P^n, \omega_{FS})$ acts transitively on the set of connected components of $CK(\mathbb{C}P^n)$ compatible with ω_{FS}.

C. For any $k \in \mathbb{Z}_{>0}$ and $r_1, \ldots, r_k > 0$, any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (\mathbb{C}P^n, \omega_{FS})$ (if they exist!) lie in the same orbit of the $\mathrm{Symp}_H(\mathbb{C}P^n, \omega_{FS})$-action.

They lie in the same orbit of the $\mathrm{Symp}(\mathbb{C}P^n, \omega_{FS}) \cap \mathrm{Diff}_0(\mathbb{C}P^n)$-action if and only if they favor a common connected component of $CK(\mathbb{C}P^n)$. In the latter case there exists $[I] \in \mathtt{Teich}(\mathbb{C}P^n, \omega_{FS})$ such that both embeddings are of $[I]$-Kähler type.

Part A of Theorem 3.7 follows from Theorem 3.5. For the proof of parts B and C of Theorem 3.7 see Section 6 (Theorem 6.6).

Remark 3.8

Let us say that a Kähler-type complex structure I on a closed manifold M is **rigid**, if the connected component of $CK(M)$ containing I coincides with the $\mathrm{Diff}_0(M)$-orbit of I (or, in other words, if any Kähler-type complex structure on M obtained from I by a smooth deformation is, in fact, isotopic to I).

As we will see in Theorem 6.2, any Kähler-type complex structure on $\mathbb{C}P^n$ is rigid. More examples of rigid complex structures include so-called $\mathcal{Z}$-manifolds (i.e., Calabi-Yau manifolds with $h^{n-1,1} = 0$, [CDP]).

The proof of part C of Theorem 3.7 allows to generalize the last claim of part C as follows: Assume (M, ω) is a closed manifold such that ω is compatible with a rigid complex structure I. Let $\mathcal{C}_0$ be a connected component of $CK(M)$ containing I. Then two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ favoring $\mathcal{C}_0$ lie in the same orbit of the $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$-action.

Applying part A of Theorem 3.7 with $\Sigma = \emptyset$, with $I_{st} = J$ being fixed, and with ω_{FS} being varied by an isotopy, we get the following immediate corollary.

Corollary 3.9

Let $l \in \mathbb{Z}_{>0}$.

Then for any $r > 0$ such that $l^n \mathrm{Vol}\left(B^{2n}(r), \omega_0\right) < \mathrm{Vol}\left(\mathbb{C}P^n, \omega_{FS}\right)$ there exists a Kähler form ω on $(\mathbb{C}P^n, I_{st})$ isotopic to ω_{FS} and such that the Kähler manifold $(\mathbb{C}P^n, I_{st}, \omega)$ admits a Kähler (i.e., both holomorphic and symplectic) embedding of $\bigsqcup_{i=1}^{l^n} B^{2n}(r)$ with the standard flat Kähler metric on it. ∎

In particular, part A of Theorem 3.7 and Corollary 3.9 mean that by changing either I_{st} or ω_{FS} by an isotopy one can get a Kähler metric on $\mathbb{C}P^n$ which is the standard flat Kähler metric on l^n disjoint balls in $\mathbb{C}P^n$ that fill out an arbitrarily large portion of the volume of the manifold.

Remark 3.10

For $l = 1$ and $n = 2$ Corollary 3.9 was proved in [Ec].

Applying Theorem 3.7 with $J = I_{st}$ and with the hyperplane $\Sigma = \mathbb{C}P^{n-1} \subset (\mathbb{C}P^n, I_{st})$, we get the following corollary.

Corollary 3.11

For any $l \in \mathbb{Z}_{>0}$ and any $r > 0$ such that $l^n \mathrm{Vol}\left(B^{2n}(r), \omega_0\right) < \mathrm{Vol}\left(B^{2n}(1), \omega_0\right)$ there exists a Kähler-type embedding of $\bigsqcup_{i=1}^{l^n} B^{2n}(r)$ into $(B^{2n}(1), \omega_0)$ which is holomorphic with respect to a complex structure on $B^{2n}(1)$ isotopic to J_0 and compatible with ω_0. ∎

Now let us focus on the case of $(\mathbb{C}P^2, \omega_{FS})$.

Proposition 3.12

There is only one connected component of $CK(\mathbb{C}P^2)$ compatible with ω_{FS} – it is the connected component of $CK(\mathbb{C}P^2)$ containing I_{st}. Any two complex structures in that connected component are isotopic.

For the proof of Proposition 3.12 see Section 6 (Proposition 6.11). We do not know whether $CK(\mathbb{C}P^2)$ has more than one connected component – see Remark 6.3.

Proposition 3.12 yields the following immediate corollary.

Corollary 3.13

Any Kähler-type embedding (of any disjoint union of domains) into $(\mathbb{C}P^2, \omega_{FS})$ is of $[I_{st}]$-Kähler type: it is symplectic with respect to ω_{FS} and holomorphic with respect to a complex structure on $\mathbb{C}P^2$ isotopic to I_{st} and compatible with ω_{FS}. ∎

Recall the following theorem of McDuff-Polterovich [McDP].

Theorem 3.14

Let $\Sigma \subset (\mathbb{C}P^2, I_{st})$ be a proper (possibly empty) complex submanifold.

A. Assume that $1 \le k \le 8$ and $r_1 \ge r_2 \ge \ldots \ge r_k > 0$.

Then the following conditions are equivalent:

- There exists a symplectic embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$.

- The radii $r_1, \ldots, r_k$ satisfy the following inequalities (listed in [McDP, Cor. 1.3G]):

  (v) $\sum_{i=1}^{k} r_i^4 < 1$ (volume inequality),

  (c1) $r_1^2 + r_2^2 < 1$, if $2 \le k \le 8$,

  (c2) $r_1^2 + \ldots + r_5^2 < 2$, if $5 \le k \le 8$,

  (c3) $2r_1^2 + \sum_{i=2}^{7} r_i^2 < 3$, if $7 \le k \le 8$,

  (c4) $2r_1^2 + 2r_2^2 + 2r_3^2 + r_4^2 + \ldots + r_8^2 < 4$, if $k = 8$,

  (c5) $2\sum_{i=1}^{6} r_i^2 + r_7^2 + r_8^2 < 5$, if $k = 8$,

  (c6) $3r_1^2 + 2\sum_{i=2}^{8} r_i^2 < 6$, if $k = 8$.

B. Symplectic embeddings of a disjoint union of $k = l^2$, $l \in \mathbb{Z}_{>0}$, equal balls into $(\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ are unobstructed: if $k\mathrm{Vol}(B^4(r), \omega_0) < \mathrm{Vol}(\mathbb{C}P^2, \omega_{FS})$, then there exists a symplectic embedding $\bigsqcup_{i=1}^{k} B^4(r) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$.

Here is an extension of Theorem 3.14 to Kähler-type embeddings of balls into $\mathbb{C}P^2$.

Theorem 3.15

A. Assume that $1 \leq k \leq 8$ and $r_1 \geq r_2 \geq \ldots \geq r_k > 0$.

   Then for any proper (possibly empty) complex submanifold $\Sigma \subset (\mathbb{C}P^2, I_{st})$, any symplectic embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ is, in fact, of $[I_{st}]$-Kähler type: it is holomorphic with respect to a complex structure on $\mathbb{C}P^2$ that is compatible with ω_{FS} and isotopic to I_{st} by an isotopy fixing Σ (as a set).

   Accordingly, by part A of Theorem 3.14, such a Kähler-type embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ exists if and only if the radii $r_1, \ldots, r_k$ satisfy the inequalities (v), (c1)-(c6) listed in part A of Theorem 3.14.

B. For any proper (possibly empty) complex submanifold $\Sigma \subset (\mathbb{C}P^2, I_{st})$ and any $k = l^2$, $l \in \mathbb{Z}_{>0}$, any symplectic embedding $\bigsqcup_{i=1}^{k} B^4(r) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ is, in fact, of $[I_{st}]$-Kähler type: it is holomorphic with respect to a complex structure on $\mathbb{C}P^2$ that is compatible with ω_{FS} and isotopic to I_{st} by an isotopy preserving Σ (as a set).

   Accordingly, by part B of Theorem 3.14, such Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(r) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ are unobstructed: they exist if and only if $k\mathrm{Vol}(B^4(r), \omega_0) < \mathrm{Vol}(\mathbb{C}P^2, \omega_{FS})$.

C. Let $\Sigma \subset (\mathbb{C}P^2, \omega_{FS})$ be either of the following:

(1) the empty set;

(2) a finite union of closed compact symplectic submanifolds (without boundary) of real dimension 2 whose pairwise intersections (if they exist) are transverse and ω–orthogonal;

(3) a Lagrangian submanifold which is diffeomorphic to either $\mathbb{S}^2$ or $\mathbb{R}P^2$.

   For any $k \in \mathbb{Z}_{>0}$ and $r_1, \ldots, r_k > 0$, any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ (if they exist!) lie in the same orbit of the $\mathrm{Symp}_0^c(\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$-action, meaning that the space of Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ is connected.

   Part B of Theorem 3.15 follows from part A of Theorem 3.7 and from part (II) of Theorem 3.4. Part C of Theorem 3.15 follows directly from part (I) of Theorem 3.4. For the proof of part A of Theorem 3.15 see Section 6 (Theorem 6.12).

   Applying Theorem 3.15 with the hyperplane $\Sigma = \mathbb{C}P^1 \subset \mathbb{C}P^2$ and recalling that $(\mathbb{C}P^2 \setminus \mathbb{C}P^1, \omega_{FS})$ is symplectomorphic to $(B^4(1), \omega_0)$, we get the following corollary.

Corollary 3.16

A. Assume that $1 \leq k \leq 8$ and $r_1 \geq r_2 \geq \ldots \geq r_k > 0$.

   Then any symplectic embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (B^4(1), \omega_0)$ is, in fact, of Kähler-type: it is holomorphic with respect to a complex structure on $B^4(1)$ that is compatible with ω_0 and isotopic to I_0.

   Accordingly, by part A of Theorem 3.14, such a Kähler-type embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (B^4(1), \omega_0)$ exists if and only if the radii $r_1, \ldots, r_k$ satisfy the inequalities (v), (c1)-(c6) listed in part A of Theorem 3.14.

B. For any $k = l^2$, $l \in \mathbb{Z}_{>0}$, any symplectic embedding $\bigsqcup_{i=1}^{k} B^4(r) \to (B^4(1), \omega_0)$ is, in fact, of Kähler type: it is holomorphic with respect to a complex structure on $B^4(1)$ that is compatible with ω_0 and isotopic to I_0.

Accordingly, by part B of Theorem 3.14, such Kähler-type embeddings $\bigsqcup_{i=1}^k B^4(r) \to (B^4(1), \omega_0)$ are unobstructed: they exist if and only if $k\mathrm{Vol}(B^4(r), \omega_0) < \mathrm{Vol}(B^4(1), \omega_0)$.

C. For any $k \in \mathbb{Z}_{>0}$ and $r_1, \ldots, r_k > 0$, any two Kähler-type embeddings $\bigsqcup_{i=1}^k B^4(r_i) \to (B^4(1), \omega_0)$ (if they exist!) lie in the same orbit of the $\mathrm{Symp}_0^c(B^4(1), \omega_0)$-action, meaning that the space of Kähler-type embeddings $\bigsqcup_{i=1}^k B^4(r_i) \to (B^4(1), \omega_0)$ is connected. ∎

Remark 3.17

1. The description of the Kähler cone of the blow-up of $\mathbb{C}P^2$ at $k \geq 10$ (generic) points is a difficult open question related, in particular, to the Nagata conjecture (which is equivalent to the claim that a certain cohomology class of the blow-up lies in the boundary of the Kähler cone, see e.g. [Bir3]). In particular, if there is a counterexample to the Nagata conjecture for $k \geq 10$, it would yield that the Kähler-type embeddings of k equal balls to $\mathbb{C}P^2$ are not unobstructed – see [McDP, Thm. 1.4.B]. Since by a theorem of Biran [Bir2], *symplectic* embeddings of a disjoint union of $k \geq 10$ equal balls into $(\mathbb{C}P^2, \omega_{FS})$ are unobstructed, this would imply the existence of such symplectic embeddings into $(\mathbb{C}P^2, \omega_{FS})$ that are *not* of Kähler type.

2. The connectedness of the space of *symplectic* embeddings of $\bigsqcup_{i=1}^k B^4(r_i)$ into $(\mathbb{C}P^2, \omega_{FS})$ and $(B^4(1), \omega_0)$ was proved in [McD3] (for previous partial results see [McD1, McD2, Lal, Bir1]).

3. By a theorem of Taubes [Tau1, Tau2, Tau3], any symplectic form on $\mathbb{C}P^2$ can be mapped into a positive multiple of ω_{FS} by a diffeomorphism of $\mathbb{C}P^2$. Therefore, any symplectic form ω on $\mathbb{C}P^2$ is of Kähler type and the results above stated for $(\mathbb{C}P^2, \omega_{FS})$ hold also for $(\mathbb{C}P^2, \omega)$.

3.3 The cases of $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$

Consider $\mathbb{C}P^2$ with the standard complex structure on it. Define $M = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ – as a smooth manifold – as the complex blow-up of $\mathbb{C}P^2$ at one point. Let $E \subset M$ be the exceptional divisor and $\mathbb{C}P^1 \subset M$ a projective line. Given $0 < \lambda < \pi$, let ω_λ be a Kähler form on M such that $\langle [\omega_\lambda], [\mathbb{C}P^1] \rangle = \pi$, $\langle [\omega_\lambda], [E] \rangle = \lambda$.

The manifold $M = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ admits Kähler-type complex structures J_{2l+1}, $l \in \mathbb{Z}_{\geq 0}$, such that (M, J_{2l+1}) is biholomorphic to the odd Hirzebruch surface $\mathbb{F}_{2l+1}$. In particular, we assume $J_1 = \widetilde{I}_{st}$, as we identify the Hirzebruch surface $\mathbb{F}_1$ with $(\widetilde{\mathbb{C}P}^2_{I_{st},x}, \widetilde{I}_{st})$. Each complex structure on M can be mapped by a diffeomorphism into exactly one complex structure J_{2l+1} – this follows from [FriQ]. In fact, this diffeomorphism can be chosen to lie in $\mathrm{Diff}_0(M)$ – see [Mee].

Theorem 3.18

Let $M = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ and let ω_λ be a Kähler-type form on M as above.

Then the following claims hold:

A. Let $2 \leq k \leq 8$, $R_1 \geq R_2 \geq \ldots \geq R_{k-1} > 0$. Assume that $r_1 := R_1 \geq \ldots \geq r_{i-1} := R_{i-1} \geq r_i := \sqrt{\lambda/\pi} \geq r_{i+1} := R_i \geq \ldots \geq r_k := R_{k-1}$.

Then for any proper (possibly empty) complex submanifold $\Sigma \subset (M, J_1)$, the following conditions are equivalent:

- There exists a $[J_1]$-Kähler-type embedding $\bigsqcup_{i=1}^{k-1} B^4(R_i) \to (M \setminus \Sigma, \omega_\lambda)$ holomorphic with respect to a complex structure on M that is compatible with ω_λ and isotopic to J_1 by an isotopy preserving Σ (as a set).

- The numbers $r_1, \ldots, r_k$ satisfy the inequalities (v), (c1)-(c6) in part A of Theorem 3.14.

For $r_1, \ldots, r_k$ satisfying the inequalities (v), (c1)-(c6) in part A of Theorem 3.14, any symplectic embedding $\bigsqcup_{i=1}^{k-1} B^4(R_i) \to (M \setminus \Sigma, \omega_\lambda)$ is, in fact, of $[J_1]$-Kähler type.

B. For any $k \in \mathbb{Z}_{>0}$ and $R_1, \ldots, R_k > 0$, any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(R_i) \to (M, \omega_\lambda)$ (if they exist!) lie in the same orbit of the $\mathrm{Symp}_0(M, \omega_\lambda)$-action, meaning that the space of Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(R_i) \to (M, \omega_\lambda)$ is connected.

Part B of Theorem 3.18 follows directly from part (I) of Theorem 3.4. For the proof of part A of Theorem 3.18 see Section 7 (Theorem 7.3).

Remark 3.19

1. We do not know under which conditions on $r_1, \ldots, r_k$ the complex structure J_1 on M can be replaced in part A of Theorem 3.18 by J_{2l+1}, $l > 0$. In order to obtain these conditions one needs to get an explicit description of the Kähler cones of the blow-ups of the Hirzebruch surfaces $\mathbb{F}_{2l+1}$, $l > 0$, at k points – see Section 7.

2. For any $\bigsqcup_{i=1}^{k} B^4(R_i)$, the connectedness of the space of *symplectic* embeddings of $\bigsqcup_{i=1}^{k} B^4(R_i)$ into (M, ω_λ) was proved in [McD3] (also see [Lal] for a previous partial result for $k = 1$).

3. Theorem 3.18 admits a generalization concerning Kähler-type embeddings of a disjoint union of k balls into a complex blow-up of $\mathbb{C}P^2$ at j points ($1 \le k \le 8 - j$, $j = 1, \ldots, 7$) equipped with an appropriate symplectic form. The proof is a straightforward modification of the proof of Theorem 3.18.

Now let $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ – as a smooth manifold. Let ω_0 be the Fubini-Study form on $\mathbb{C}P^1$ normalized so that $\int_{\mathbb{C}P^1} \omega_0 = \pi$. Given $\mu \ge 1$, let $\omega_\mu = \mu\omega_0 \oplus \omega_0$.

The manifold $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ admits Kähler-type complex structures J_{2l}, $l \in \mathbb{Z}_{\ge 0}$, such that (M, J_{2l}) is biholomorphic to the even Hirzebruch surface $\mathbb{F}_{2l}$. In particular, J_0 can be viewed as the standard complex structure on $M = \mathbb{C}P^1 \times \mathbb{C}P^1$. Each complex structure on $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ can be mapped by a diffeomorphism into exactly one complex structure J_{2l} – this follows from [FriQ]. In fact, this diffeomorphism can be chosen to lie in $\mathrm{Diff}_0(M)$ – see [Mee].

Theorem 3.20

Let $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ and let ω_μ be as above.

Let $\Sigma \subset (M, J_0)$ be either a proper (possibly empty) complex submanifold or $\Sigma = (\mathrm{pt} \times \mathbb{C}P^1) \cup (\mathbb{C}P^1 \times \mathrm{pt})$.

Then the following claims hold:

A. Assume $2 \leq k \leq 8$, $R_1, R_2, \ldots, R_{k-2} > 0$, $0 < R_{k-1} < 1$. Consider the numbers

$$\frac{R_1^2}{\mu + 1 - R_{k-1}^2}, \frac{R_2^2}{\mu + 1 - R_{k-1}^2}, \ldots, \frac{\mu - R_{k-1}^2}{\mu + 1 - R_{k-1}^2}, \frac{1 - R_{k-1}^2}{\mu + 1 - R_{k-1}^2},$$

sort them in the non-increasing order and denote the resulting k numbers by $r_1 \geq \ldots \geq r_k$.

Then the following conditions are equivalent:

- There exists a $[J_0]$-Kähler-type embedding $\bigsqcup_{i=1}^{k-1} B^4(R_i) \to (M \setminus \Sigma, \omega_\mu)$ holomorphic with respect to a complex structure on M that is compatible with ω_μ and isotopic to J_0 by an isotopy preserving Σ (as a set).

- The numbers $r_1, \ldots, r_k$ defined above satisfy the inequalities in part A of Theorem 3.14.

For $r_1, \ldots, r_k$ satisfying the inequalities (v), (c1)-(c6) in part A of Theorem 3.14, any symplectic embedding $\bigsqcup_{i=1}^{k-1} B^4(R_i) \to (M \setminus \Sigma, \omega_\mu)$ is, in fact, of $[J_0]$-Kähler type.

B. Assume that $\mu = 1$.

Then for any $l \in \mathbb{Z}_{>0}$, any symplectic embedding of a disjoint union of $2l^2$ equal balls into $(M \setminus \Sigma, \omega_1)$ is, in fact, of $[J_0]$-Kähler type, and such symplectic (or, equivalently, $[J_0]$-Kähler-type) embeddings are unobstructed.

More precisely, if $2l^2 \text{Vol}(B^4(r), \omega_0) < \text{Vol}(M, \omega_1)$, then there exists a Kähler-type embedding of $\bigsqcup_{i=1}^{2l^2} B^4(r)$ into $(M \setminus \Sigma, \omega_1)$ which is holomorphic with respect to a complex structure on M that is compatible with ω_1 and isotopic to J_0 by an isotopy preserving Σ (as a set).

C. For any $k \in \mathbb{Z}_{>0}$ and $R_1, \ldots, R_k > 0$, any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(R_i) \to (M \setminus \Sigma, \omega_\mu)$ (if they exist!) lie in the same orbit of the $\text{Symp}_0^c(M \setminus \Sigma, \omega_\mu)$-action, meaning that the space of Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(R_i) \to (M \setminus \Sigma, \omega_\mu)$ is connected.

Part B of Theorem 3.20 is a particular case of Corollary 3.6 (for $m = 2$). Part C of Theorem 3.20 follows directly from part (I) of Theorem 3.4. For the proof of part A of Theorem 3.20 see Section 8 (Theorem 8.3).

Remark 3.21

1. For $\Sigma = (\text{pt} \times \mathbb{C}P^1) \cup (\mathbb{C}P^1 \times \text{pt})$ the manifold $(M \setminus \Sigma, \omega_\mu)$ is symplectomorphic to $(B^2(1) \times B^2(\sqrt{\mu}), \omega_0 \oplus \omega_0)$. Therefore Theorem 3.20 yields a straightforward corollary concerning Kähler-type embeddings of balls into $(B^2(1) \times B^2(\sqrt{\mu}), \omega_0 \oplus \omega_0)$.

For *symplectic* embeddings of balls a similar corollary, as well as the result of part B of Theorem 3.20, were proved in [McDP].

2. We do not know under which conditions on $r_1, \ldots, r_k$ the complex structure J_0 on M can be replaced in part A of Theorem 3.20 by J_{2l}, $l > 0$. In order to obtain these conditions one needs to get an explicit description of the Kähler cones of the blow-ups of the Hirzebruch surfaces $\mathbb{F}_{2l}$, $l > 0$, at k points – see Section 8.

3. For any $R_1, \ldots, R_k$ the connectedness of the space of *symplectic* embeddings $\bigsqcup_{i=1}^{k} B^4(R_i) \to (M, \omega_\lambda)$ was proved in [McD3].

3.4 The case of tori and K3 surfaces

Let M, $\dim_{\mathbb{R}} M = 2n$, be either $\mathbb{T}^{2n}$ or a smooth manifold (of real dimension 4) underlying a complex K3 surface.

As before, let ω be a Kähler-type symplectic form on M.

In the case $M = \mathbb{T}^{2n}$ one can assume, without loss of generality, that ω is linear (see Example 2.1).

Recall that ω is called **rational** if the real cohomology class $[\omega] \in H^2(M; \mathbb{R})$ is a real multiple of a rational one, and **irrational** otherwise.

Irrational Kähler-type symplectic forms on tori and K3 surfaces are compatible with so-called Campana-simple complex structures. Namely, given a Kähler-type complex structure I on a closed manifold M, the union of all positive-dimensional proper complex subvarieties of (M, I) is either a countable union of proper analytic subvarieties of M (and hence has a dense connected complement) or the whole M (see e.g. [EV1, Remark 4.2]). In the former case, the complex structure I is called **Campana-simple** [Cam], [CaDV] (for more details see Section 9).

In fact, a Kähler-type complex structure on a torus is Campana-simple if and only if it has no proper positive-dimensional complex subvarieties (see Proposition 10.4). A symplectic form ω on the torus or a K3 surface is compatible with such a complex structure if and only if ω is irrational (see Proposition 10.5 and Proposition 10.20). See Section 3.5 for other examples of manifolds admitting Campana-simple complex structures.

Theorem 3.22

Let M, $\dim_{\mathbb{R}} M = 2n$, be either $\mathbb{T}^{2n}$ or a smooth manifold (of real dimension 4) underlying a complex K3 surface and let ω be a Kähler-type symplectic form on M.

Assume that ω is irrational.

Let $k \in \mathbb{Z}_{>0}$, $r_1, \ldots, r_k > 0$.

Then the following claims hold:

A. Kähler-type embeddings of $\bigsqcup_{i=1}^{k} B^{2n}(r_i)$ into (M, ω) are unobstructed.

More precisely, assume that $\mathrm{Vol}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i), \omega_0\right) < \mathrm{Vol}(M, \omega)$ and ω is compatible with a Campana-simple complex structure I.

Then there exists an $[I]$-Kähler-type embedding $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$.

B. The group $\mathrm{Symp}_H(M, \omega)$ acts transitively on the set of connected components of $CK(M)$ compatible with ω.

C. Any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ (if they exist!) lie in the same orbit of the $\mathrm{Symp}_H(M, \omega)$-action. They lie in the same orbit of the $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$-action if and only if they favor a common connected component of $CK(M)$. In the latter case there exists $[I] \in \mathrm{Teich}(M, \omega)$ such that both embeddings are of $[I]$-Kähler-type.

Remark 3.23

Assume $M = \mathbb{T}^2$ is a torus of real dimension 2. In this case all symplectic forms and complex structures on $\mathbb{T}^2$ are of Kähler-type and any symplectic (i.e., area-preserving) embedding into $\mathbb{T}^2$ is of Kähler-type. In particular, for any symplectic form ω on $\mathbb{T}^2$ (note that any such form is rational!) the result of part A of Theorem 3.22 is true and the space of Kähler-type embeddings of any disjoint union of domains into $(\mathbb{T}^2, \omega)$ is path-connected (since the space of symplectic forms on $\mathbb{T}^2$ is path-connected [Mos]).

For the proof of Theorem 3.22 see Section 10.3 (Theorem 10.21).

For a generalization of Theorem 3.22 to arbitrary closed symplectic manifolds admitting Campana-simple complex structures compatible with the symplectic form see Theorem 9.3. In fact, we will use Theorem 9.3 in the proof of Theorem 3.22.

For tori the existence result in Theorem 3.22 admits the following equivalent formulation.

Corollary 3.24

Assume $\Gamma \subset \mathbb{C}^n$ is a lattice of rank $2n$. Let $M := \mathbb{C}^n/\Gamma$ be the corresponding torus. Let ω, I be the symplectic and the complex structures on M induced by the standard flat symplectic and complex structures on $\mathbb{C}^n = \mathbb{R}^{2n}$.

Assume that (M, I) admits no proper positive-dimensional complex subvarieties. Let $k \in \mathbb{Z}_{>0}$, $r_1, \ldots, r_k > 0$ such that $\mathrm{Vol}\left(\bigsqcup_{i=1}^k B^{2n}(r_i), \omega_0\right) < \mathrm{Vol}(M, \omega)$.

Then there exists a Kähler form ω' on (M, I) isotopic to ω and a Kähler-type embedding $\bigsqcup_{i=1}^k B^{2n}(r_i) \to (M, \omega')$ which is holomorphic with respect to I – i.e., an embedding which is symplectic with respect to ω' and holomorphic with respect to I.

Proof of Corollary 3.24:

It suffices to prove a similar claim for the case where $\Gamma = \mathbb{Z}^{2n}$ and ω and I are compatible symplectic and complex structures on $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, with I being Campana-simple (the proof can be reduced to this case by using a diffeomorphism $M = \mathbb{C}^n/\Gamma \to \mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ induced by an $\mathbb{R}$-linear isomorphism of $\mathbb{R}^{2n} = \mathbb{C}^{2n}$ sending Γ to $\mathbb{Z}^{2n}$). In order to prove the corollary in this case, apply part A of Theorem 3.22 and keep the complex structure fixed while changing the symplectic form by an isotopy (cf. Remark 2.6). ∎

Remark 3.25

1. For an arbitrary Kähler-type symplectic form ω on $M = \mathbb{T}^{2n}$ or on a manifold M underlying a K3 surface, the unobstructedness of symplectic – but not necessarily Kähler-type or tame – embeddings of an arbitrary collection of disjoint balls was proved in [EV1, Thm. 3.1]. In the case $M = \mathbb{T}^4$ a weaker version of this result had been previously proved by Latschev-McDuff-Schlenk [LatMS].

2. It is a well-known open problem (see e.g. [McDS, Ch.15, Prob. 15]) whether $\mathrm{Symp}_H(\mathbb{T}^{2n}, \omega) = \mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$ for a Kähler-type symplectic form ω on $\mathbb{T}^{2n}$, $n > 1$. If the answer to this question is positive, then Part B of Theorem 3.22 would imply that the space of the Kähler-type embeddings

$\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (\mathbb{T}^{2n}, \omega)$ is path-connected for any $k \in \mathbb{Z}_{>0}$ and $r_1, \ldots, r_k > 0$. Conversely, a counter-example to the latter claim would show that $\mathrm{Symp}_H(\mathbb{T}^{2n}, \omega) \neq \mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$.

In the case when M is a manifold underlying a K3 surface it is known by the result of Sheridan-Smith [ShS] that $\mathrm{Symp}_H(M, \omega) \neq \mathrm{Symp}_0(M, \omega)$ for certain Kähler-type symplectic structures ω on M, including some irrational ones. It has been known since the work [Sei] of Seidel that for some (not necessarily irrational) Kähler-type symplectic forms ω on M one has $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M) \neq \mathrm{Symp}_0(M, \omega)$; by a result of Smirnov [Sm], this definitely holds for some irrational Kähler-type symplectic forms ω on M.

3. Corollary 3.24 for one ellipsoid – and, in particular, for one ball – was previously proved by Luef and Wang [LuWa] using a method similar to the one in this paper. Let us note that the paper [LuWa] by Luef and Wang provides a very interesting connection between Kähler-type embeddings of ellipsoids into tori and Gabor frames – an important notion in signal processing.

We expect that the technique used in [LuWa], as well as the techniques developed in [EV2] in relation to symplectic embeddings of ellipsoids into tori and K3 surfaces, can be used to generalize Theorem 3.22 to Kähler-type embeddings of ellipsoids.

Remark 3.26

The assumption that the symplectic form ω on M is irrational is essential in Theorem 3.22: otherwise there may be obstructions for the existence of Kähler-type embeddings of (disjoint unions of) balls into (M, ω) that are independent of the symplectic volume – for instance, the obstructions coming from Seshadri constants.

For simplicity, we will explain this below in the case of one ball and $M = \mathbb{T}^{2n}$, expanding on [LatMS, Sec. 2.2].

First, recall the definition of Seshadri constants. Let (X, I) be a complex projective manifold and let ω be a Kähler form on (X, I) such that the cohomology class $[\omega]$ is integral and equals the first Chern class of an ample line bundle on (X, I). Assume that $x \in X$, $\pi : \widetilde{X} \to X$ is the complex blow-up of X at x, and $E := \pi^{-1}(x)$ is the exceptional divisor. Define the **Seshadri constant**

$$\varepsilon(X, I, [\omega], x) := \sup \left\{ s > 0 \mid \left\langle \pi^*[\omega], [\widetilde{C}] \right\rangle - sE \cdot [\widetilde{C}] \geq 0 \right.$$

$$\left. \text{for all (possibly singular) complex curves } \widetilde{C} \subset \widetilde{X} \right\}.$$

Observe that $E \cdot [\widetilde{C}]$ is the multiplicity of the singularity of C at x.

Then

$$\varepsilon(X, I, [\omega], x) = \sup \left\{ \pi r^2 \mid \text{There exists a Kähler form } \omega', \; [\omega'] = [\omega], \right.$$
$$\text{and a Kähler embedding } f : B^{2n}(r) \to (X, \omega', I), \tag{3.1}$$
$$\left. f^* \omega' = \omega_0, \; f^* I = J_0, \; f(0) = x \right\}.$$

For a proof see [Ec, Thm. 0.6], [Fl, Thm. 1.4], [WN2, Thm. 1.3], cf. [LuWa, Thm. A]. (It had been previously observed in [Laz] that the inequality $\varepsilon(X, I, [\omega], x) \geq$ [the right-hand side in (3.1)] follows from the results of [McDP]).

Assume that ω is a rational symplectic form on $\mathbb{T}^{2n}$. For the study of the symplectic and Kähler-type embeddings in $(\mathbb{T}^{2n}, \omega)$ we may assume then, without loss of generality, that $[\omega] \in H^2(\mathbb{T}^{2n}; \mathbb{Z})$. Let I be a complex structure on $\mathbb{T}^{2n}$ compatible with ω. Consider an ample holomorphic bundle L on $\mathbb{T}^{2n}$ such that $c_1(L) = [\omega]$. The complex manifold $(\mathbb{T}^{2n}, I)$ can be viewed as an abelian variety and L as its polarization.

Since the identity component of the group of biholomorphisms of a complex torus acts transitively on the torus, $\varepsilon(\mathbb{T}^{2n}, I, [\omega], x)$ depends only on I and $[\omega]$ and not on $x \in \mathbb{T}^{2n}$, so we will denote $\varepsilon(\mathbb{T}^{2n}, I, [\omega]) := \varepsilon(\mathbb{T}^{2n}, I, [\omega], x)$.

Since $\mathrm{Vol}(B^{2n}(r), \omega_0) = \pi^n r^{2n}$, (3.1) implies that

$$\sup \left\{ \mathrm{Vol}(B^{2n}(r), \omega_0) \mid \text{ There exists a Kähler-type embedding} \right.$$

$$\left. f : (B^{2n}(r), \omega_0) \to (\mathbb{T}^{2n}, \omega) \right\} = \sup_{I \in Cmpt(M, \omega)} \varepsilon^n(\mathbb{T}^{2n}, I, [\omega]),$$

where the supremum in the right-hand side is taken over all I compatible with ω.

There exist examples where

$$\sup_{I \in Cmpt(M, \omega)} \varepsilon^n(\mathbb{T}^{2n}, I, [\omega]) < \mathrm{Vol}(\mathbb{T}^{2n}, \omega).$$

For instance, consider the torus $\mathbb{T}^4 = \mathbb{R}^4 / \mathbb{Z}^4$ equipped with the symplectic form $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, where p_1, p_2, q_1, q_2 are the coordinates on $\mathbb{R}^4$. For any complex structure I on $\mathbb{T}^4$ compatible with ω one can biholomorphically identify $(\mathbb{T}^4, I)$ with a principally polarized abelian variety. In this case, by a result of Steffens [Ste, Prop. 2], we have $\varepsilon(\mathbb{T}^4, I, [\omega]) \le 4/3$ for all I compatible with ω. At the same time $\mathrm{Vol}(\mathbb{T}^4, \omega) = \int_{\mathbb{T}^4} \omega^2 = 2$. Thus,

$$\sup_{I \in Cmpt(M, \omega)} \varepsilon^2(\mathbb{T}^4, I, [\omega]) \le (4/3)^2 < 2 = \mathrm{Vol}(\mathbb{T}^4, \omega).$$

This yields an obstruction, independent of the symplectic volume, for the existence of Kähler-type embeddings of a ball into $(\mathbb{T}^4, \omega)$: a ball whose volume is between $(4/3)^2$ and $2 = \mathrm{Vol}(\mathbb{T}^4, \omega)$ does not admit Kähler-type embeddings into $(\mathbb{T}^4, \omega)$. At the same time, by [LatMS, Thm. 1.1], such a ball does admit a symplectic embedding into $(\mathbb{T}^4, \omega)$. *Thus, there exists a symplectic embedding of a ball which is not of Kähler type.* We do not know whether such an embedding is tame.

Let us now consider tame embeddings of balls into tori and K3 surfaces.

Theorem 3.27

Let M, $\dim_{\mathbb{R}} M = 2n$, be either $\mathbb{T}^{2n}$ or a smooth manifold (of real dimension 4) underlying a complex K3 surface.

Let ω_1, ω_2 be Kähler-type forms on M that are irrational and satisfy $\int_M \omega_1^n = \int_M \omega_2^n > 0$.

Let $W_i \subset \mathbb{R}^{2n}$, $i = 1, \ldots, k$, be compact domains with piecewise-smooth boundary whose interiors contain the origin. Assume that $H^2(W_i; \mathbb{R}) = 0$ for all i. Set $\mathbb{W} := \bigsqcup_{i=1}^k W_i$.

Then, for any $\varepsilon > 0$,

$$v_{T,\varepsilon}(M, \omega_1, \mathbb{W}) = v_{T,\varepsilon}(M, \omega_2, \mathbb{W}).$$

For the proof of Theorem 3.27 see Section 11 (Theorem 11.6).

Remark 3.28

Theorem 3.27 generalizes a similar result in [EV1] about symplectic (not necessarily tame) embeddings and its proof is similar to the proof in [EV1].

Let us present applications of Theorem 3.27.

Corollary 3.29

Assume:

- M is either $\mathbb{T}^{2n}$ or a smooth manifold underlying a K3 surface.

- $\mathbb{W} := \bigsqcup_{i=1}^{k} B^{2n}(r_i)$ is a disjoint union of k (possibly different) balls.

- $\varepsilon > 0$.

Then the following claims hold:

A. For any irrational Kähler-type symplectic form ω on M we have

$$\nu_K(M, \omega, \mathbb{W}) = 1,$$

meaning that Kähler-type embeddings of $\lambda \mathbb{W}$ into (M, ω) are unobstructed.

B. There exists a $\mathrm{Diff}^+(M)$-invariant open dense set of Kähler-type symplectic forms on M, depending on $\mathbb{W}$ and ε and containing, in particular, all irrational Kähler-type symplectic forms on M, so that for each ω in this set $\nu_{T,\varepsilon}(M, \omega, \mathbb{W}) = 1$ – meaning that ε-tame embeddings of $\lambda \mathbb{W}$ into (M, ω) are unobstructed.

Corollary 3.30

Assume:

- $M = \mathbb{T}^{2n}$.

- $\mathbb{W} := \bigsqcup_{i=1}^{k} W_i$ is either a disjoint union of k identical copies of a $2n$-dimensional polydisk

$$B^{2n_1}(R_1) \times \ldots \times B^{2n_l}(R_l),\ n_1 + \ldots + n_l = n,\ R_1, \ldots, R_l > 0, l > 1,$$

or a disjoint union of k identical copies of a parallelepiped

$$P(e_1, \ldots, e_{2n}) := \left\{ \sum_{j=1}^{2n} s_j e_j, 0 \le s_j \le 1, j = 1, \ldots, 2n \right\},$$

where $e_1, \ldots, e_{2n}$ is a basis of the vector space $\mathbb{R}^{2n}$.

- $\varepsilon > 0$.

Then the following claims hold:

A. For any positive volume there exists a dense $\mathrm{Diff}^+(\mathbb{T}^{2n})$-orbit (of an irrational Kähler-type symplectic form depending on $\mathbb{W}$) in the space of Kähler-type symplectic forms of that volume on $\mathbb{T}^{2n}$ such that for any ω' in this orbit we have $v_K(\mathbb{T}^{2n}, \omega', \mathbb{W}) = 1$ – or, in other words, Kähler-type embeddings $\lambda \mathbb{W} \to (\mathbb{T}^{2n}, \omega')$ are unobstructed.

B. There exists a $\mathrm{Diff}^+(\mathbb{T}^{2n})$-invariant open dense set of Kähler-type symplectic forms on $\mathbb{T}^{2n}$, depending on $\mathbb{W}$ and ε and containing, in particular, all irrational Kähler-type symplectic forms on $\mathbb{T}^{2n}$, so that for each ω' in this set $v_{T,\varepsilon}(\mathbb{T}^{2n}, \omega', \mathbb{W}) = 1$ – or, in other words, ε-tame embeddings $\lambda \mathbb{W} \to (\mathbb{T}^{2n}, \omega')$ are unobstructed.

Corollary 3.31

Assume:

- M is a smooth manifold underlying a K3 surface.

- $\mathbb{W} := \bigsqcup_{i=1}^k W_i$ is a disjoint union of k identical copies of a parallelepiped

$$P(e_1, \ldots, e_{2n}) := \left\{ \sum_{j=1}^{2n} s_j e_j, 0 \le s_j \le 1, j = 1, \ldots, 2n \right\},$$

where $e_1, \ldots, e_{2n}$ is a basis of the vector space $\mathbb{R}^{2n}$.

- $\varepsilon > 0$.

Then there exists a $\mathrm{Diff}^+(M)$-invariant open dense set of Kähler-type symplectic forms on M, depending on $\mathbb{W}$ and ε and containing, in particular, all irrational Kähler-type symplectic forms on M, so that for each ω' in this set $v_{T,\varepsilon}(M, \omega', \mathbb{W}) = 1$, meaning that ε-tame embeddings of $\lambda \mathbb{W}$ into (M, ω') are unobstructed.

For the proof of Corollary 3.29, Corollary 3.30 and Corollary 3.31 see Section 11. These corollaries generalize similar results for symplectic – but not necessarily tame – embeddings that were proved in [EV1].

Remark 3.32

We know only one way to show the possible existence of embeddings (of a disjoint union of domains) into (M, ω) that are symplectic and non-tame, or tame and not Kähler-type – via the volume restrictions: e.g., if Kähler-type embeddings cannot fill more than a certain portion of the volume of (M, ω) while tame embeddings can. This is how we got in Remark 3.26 that there exist symplectic embeddings (of balls) that are not of Kähler-type. It would be interesting to find out whether there are other ways of detecting symplectic and non-tame, or tame and not Kähler-type, embeddings.

Together with Corollary 3.29, this leads to the following questions.

Question 3.33

Does there exist a closed manifold equipped with a Kähler-type symplectic form that admits a symplectic embedding which is not tame, or a tame embedding which is not of Kähler-type? In particular, does there exist a rational Kähler-type symplectic form ω on $M = \mathbb{T}^{2n}$, $n > 1$, or on a smooth manifold M underlying a complex K3 surface, and a disjoint union of balls $\mathbb{W}$, $\mathrm{Vol}(\mathbb{W}, \omega_0) < \mathrm{Vol}(M, \omega)$, so that $\mathbb{W}$ does not admit a tame embedding into (M, ω)?

3.5 The IHS-hyperkähler case

K3 surfaces are particular examples of so called IHS-hyperkähler manifolds, that is, compact hyperkähler manifolds of maximal holonomy. Most of the results of Section 3.4 generalize to such manifolds. In this section we present relevant definitions and statements.

Recall that a **hyperkähler manifold** is a manifold equipped with three complex structures I_1, I_2, I_3 satisfying the quaternionic relations and three symplectic forms $\omega_1, \omega_2, \omega_3$ compatible, respectively, with I_1, I_2, I_3, so that the three Riemannian metrics $\omega_i(\cdot, I_i \cdot)$, $i = 1, 2, 3$, coincide. Such a collection $\mathfrak{H} = \{I_1, I_2, I_3, \omega_1, \omega_2, \omega_3\}$ of complex structures and symplectic forms on a manifold is called a **hyperkähler structure**. All the complex and symplectic structures appearing in a hyperkähler structure induce the same orientation on the manifold.

Assume that M is a closed connected and simply connected manifold.

The Levi-Civita connection of a Kähler manifold preserves its complex structure and the Kähler form. Therefore, the Levi-Civita connection of a hyperkähler manifold preserves the three complex structures I_1, I_2, I_3 and the three symplectic forms $\omega_1, \omega_2, \omega_3$. The subgroup of $GL(\mathbb{R}^{4n})$ preserving both the quaternionic Hermitian metric and the quaternionic action is called **the compact symplectic group** and is denoted by $Sp(n)$. This is the group of all quaternionic unitary matrices; it can also be obtained as a maximal compact form of a complex symplectic group $\mathrm{Sp}(\mathbb{C}^{2n})$. Since the Levi-Civita connection of (M, g, I_1, I_2, I_3) preserves the metric and the quaternionic operators, its holonomy belongs to $Sp(n)$. However, a priori the holonomy group can be smaller; for example, for a hyperkähler flat torus, the holonomy is trivial. The de Rham theorem and Berger's classification of irreducible holonomies imply that any compact hyperkähler manifold has a finite covering which is decomposed into a product of a torus and manifolds with maximal holonomy; this result is called **the Bogomolov decomposition theorem**, [Bo1]. This theorem is the reason why the maximal holonomy hyperkähler manifolds are also called **irreducible holomorphically symplectic (IHS)**.

A maximal holonomy compact hyperkähler manifold can be characterized cohomologically: M has maximal holonomy if and only $h^{2,0}(M) = 1$ and $b_1(M) = 0$. Here $h^{2,0}(M)$ denotes the dimension of the group of closed, holomorphic 2-forms on M, for any of its complex structures.

K3 surfaces, as well as the Hilbert schemes of points for K3 surfaces, are known to admit IHS-hyperkähler structures.

We say that a symplectic/complex structure on M is **of IHS-hyperkähler type**, if it appears in *some* maximal holonomy hyperkähler structure. In particular, all IHS-hyperkähler-type symplectic and complex structures are of Kähler type.

Remark 3.34

1. Note that even if a symplectic and a complex structure are of IHS-hyperkähler type and compatible, it still does not mean that they can be included in *the same* IHS-hyperkähler structure.

2. All complex structures and all Kähler-type symplectic structures on a K3 surface are in fact of IHS-hyperkähler type – see e.g. [EV3, Prop. 3.1]. As we have already mentioned, it is an open question whether any symplectic form on a K3 surface is of Kähler type.

3. Manifolds admitting IHS-hyperkähler structures are also called hyperkähler manifolds of maximal holonomy, because the holonomy group of a hyperkähler manifold is $\mathrm{Sp}(n)$ (the group of invertible quaternionic $n \times n$-matrices) – and not its proper subgroup – if and only if the hyperkähler structure is IHS [Bes].

4. Note that the property of being IHS is a topological invariant: a holomorphically symplectic Kähler manifold M, $\dim_{\mathbb{C}} M = 2n$, is IHS if and only if $\pi_1(M) = 0$ and there exists a primitive integral quadratic form q on $H^2(M;\mathbb{R})$ (called the Bogomolov-Beauville-Fujiki form [Bea1], [Fu]) that satisfies, for some rational $C > 0$, the Fujiki relation $\int_M \eta^{2n} = Cq(\eta,\eta)^n$.

Therefore, the set of IHS-hyperkähler type symplectic/complex structures on a given IHS-hyperkähler manifold coincides with the set of hyperkähler-type symplectic/complex structures – i.e., the symplectic/complex structures appearing in some hyperkähler structure.

Now we can go over Section 2 and define **IHS-hyperkähler-type** embeddings by replacing the words "Kähler-type" in the relevant definitions by the words "IHS-hyperkähler-type". In particular, in this way we can define **IHS-hyperkähler-type symplectic embeddings** of domains in $\mathbb{R}^{4n}$ into M equipped with an IHS-hyperkähler-type symplectic form.

Note that, in view of part 2 of Remark 3.34, for symplectic embeddings into a K3 surface being of IHS-hyperkähler type is the same as being of Kähler type.

The results of Section 3.4 on Kähler-type embeddings generalize to the IHS-hyperkähler case as follows:

- Part A of Theorem 3.22 generalizes directly to the IHS-hyperkähler case as long as one assumes that the Campana-simple complex structure I is of IHS-hyperkähler-type.

- The claim in part C of Theorem 3.22 concerning the action of $\mathrm{Symp}(M,\omega) \cap \mathrm{Diff}_0(M)$ generalizes to the IHS-hyperkähler case, if $CK(M)$ is replaced by the space of the IHS-hyperkähler-type complex structures on M. For a general IHS-hyperkähler manifold (M,ω) we cannot say anything about the transitivity of the $\mathrm{Symp}_H(M,\omega)$-action.

Let us discuss the results on ε-tame embeddings into (M,ω) for an IHS-hyperkähler-type form ω. One can define tame and ε-tame embeddings into (M,ω) exactly as in Definition 2.3 using *Kähler-type* complex structures. Another possibility is to modify the definition of tame and ε-tame embeddings into (M,ω) by using *IHS-hyperkähler-type* complex structures in Definition 2.3 instead of Kähler-type ones. The claim $v_{T,\varepsilon}(M,\omega_1,\mathbb{W}) = v_{T,\varepsilon}(M,\omega_2,\mathbb{W})$ of Theorem 3.27 holds for the original $v_{T,\varepsilon}$ defined in Definition 2.8, as well as for a version of $v_{T,\varepsilon}$ defined using the ε-tame embeddings in the second

sense above, if ω_1 and ω_2 lie in the same connected component of the space of IHS-hyperkähler-type symplectic forms on M.

In Section 12 we discuss how to modify the proofs of the results of Section 3.4 in order to obtain the above-mentioned results in the IHS-hyperkähler case.

4 Plan of the paper and an outline of the proofs

In this section we describe the plan of the paper and outline the proofs.

In order to simplify the exposition, let us fix a connected component $\mathcal{C}_0$ of $CK(M)$ and the radii $r_1, \dots, r_k > 0$ of the balls from the beginning. Let $\bar{r} := (r_1, \dots, r_k)$.

In Section 5 we prove Theorem 3.2 and Theorem 9.3. Namely, we define the following spaces:

- The space $Pairs_{\mathcal{C}_0}$ formed by pairs (η, f), where η is a Kähler-type symplectic form on M isotopic to ω and $f : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \eta)$ is a Kähler-type embedding favoring $\mathcal{C}_0$.

- The Teichmüller space $\mathtt{Pairs}_{\mathcal{C}_0} := Pairs_{\mathcal{C}_0}/\mathrm{Diff}_0(M)$.

- The space $Triples_{\mathcal{C}_0}$ formed by triples (I, x, h), where $I \in \mathrm{Diff}_0(M) \cdot Cmpt(M, \omega)$, $x = (x_1, \dots, x_k) \in \hat{M}^k$, so that the pair (I, x) lies in the set $\mathcal{K}_{\mathcal{C}_0}(\bar{r})$ defined in Definition 3.1, and $h := \bigsqcup_{i=1}^{k} h_i : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, I)$ is a holomorphic embedding such that $h_i(0) = x_i$, $i = 1, \dots, k$.

- The Teichmüller space $\mathtt{Triples}_{\mathcal{C}_0} := Triples_{\mathcal{C}_0}/\mathrm{Diff}_0(M)$.

We then construct a continuous surjective map $\Phi : \mathtt{Triples}_{\mathcal{C}_0} \to \mathtt{Pairs}_{\mathcal{C}_0}$. The construction uses the symplectic blow-up and blow-down operations similar to the ones used in [McDP]. Let us note that for the local constructions on which these operations are based we use the regularized maximum of plurisubharmonic functions while in [McDP] a different analytic technique is used. We also need the regularized maximum tool to construct a Kähler form on $(\widetilde{M}_{I,x}, \widetilde{I})$ in the class $\Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ which is standard *on a neighborhood of each exceptional divisor* – this is needed to obtain a Kähler form on M by the symplectic blow-down construction (unlike in [McDP], where the goal was to produce a symplectic, not necessarily Kähler, form on M, and this was done much easier, using Moser's method for forms on $\widetilde{M}_{I,x}$).

The symplectic blow-down construction, enhanced as we have just described, and a relative version of Moser's method are used to construct Φ, and the symplectic blow-up is used to verify that Φ is surjective. The continuity of Φ is verified using the observation that the set $\mathtt{Pairs}_{\mathcal{C}_0}$ is discrete and therefore it suffices to check that the preimage of a point is open, which is then proved using again a relative version of Moser's method.

We then prove Theorem 3.2 as follows.

The mere existence of Φ implies that if $\mathcal{K}_{\mathcal{C}_0}(\bar{r})$ (and hence $\mathtt{Triples}_{\mathcal{C}_0}$) is non-empty, there exists a Kähler-type embedding $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ favoring $\mathcal{C}_0$.

The connectivity of the set $\mathtt{K}_{\mathcal{C}_0}(\bar{r})$ (see Definition 3.1) yields the transitivity of the $\mathrm{Symp}(M) \cap \mathrm{Diff}_0(M)$-action on the space of Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ favoring $\mathcal{C}_0$ as follows. Using methods of complex geometry (the Kodaira-Spencer stability and the Demailly-Paun description of the Kähler cones of fibers of an analytic deformation family) we show that the connectivity of $\mathtt{K}_{\mathcal{C}_0}(\bar{r})$ is equivalent to the connectivity of $\mathcal{K}_{\mathcal{C}_0}(\bar{r})$. The latter easily implies the connectivity of $\mathtt{Triples}_{\mathcal{C}_0}$. Since

Φ is continuous and surjective, we get that $\texttt{Pairs}_{\mathcal{C}_0}$ is connected, and since it is discrete, it is just a point. This easily implies the transitivity of the $\mathrm{Symp}(M) \cap \mathrm{Diff}_0(M)$-action.

In Section 9 we consider the case of a closed symplectic manifold for which the symplectic form is compatible with a so-called Campana-simple complex structure – i.e., a Kähler-type complex structure admitting "few" complex subvarieties. Examples of such manifolds include tori and smooth manifolds underlying complex K3 surfaces equipped with irrational Kähler-type symplectic forms. The general results on Kähler-type embeddings of balls into such manifolds, proved in Section 9, are used further in the paper to deduce the results on such embeddings into tori and K3 surfaces stated in Section 3.4. The key existence result on such embeddings is based on the observation, made already in [LatMS], [EV1], that for any Campana-simple complex structure I compatible with ω and any Campana-generic $x_1, \ldots, x_k \in (M, I)$, $x := (x_1, \ldots, x_k)$, we have $(I, x) \in \mathcal{K}(\bar{r})$, as long as $\mathrm{Vol}\left(\bigsqcup_{i=1}^k B^{2n}(r_i)\right) < \mathrm{Vol}(M, \omega)$. This follows from the Demailly-Paun description of the Kähler cone of a closed Kähler manifold [DP].

In Sections 6, 7, 8, 10 we discuss the cases of the specific M: complex projective spaces and their products, Hirzebruch surfaces, tori and K3 surfaces. In each of these cases we discuss the structure of the Teichmüller space of Kähler-type complex structures on M and of its subset formed by equivalence classes of complex structures compatible with a given symplectic form on M, as well as the Kähler cones of the complex blow-ups of M. We use this information in order to apply our general results about Kähler-type embeddings of balls and deduce the results in Section 3 about these specific M.

In Section 11 we discuss the proofs of the results in Section 3.4. concerning tame embeddings of arbitrary domains. The strategy of the proof of Theorem 3.27 is the same as for a similar claim in [EV1]. Namely, with M being either $\mathbb{T}^{2n}$ or a smooth manifold underlying a K3 surface, the group $\mathrm{Diff}^+(M)$ of orientation-preserving diffeomorphisms of M acts on the space $SK_1(M)$ of Kähler-type forms on M of total volume 1. The function $\omega \mapsto v_{T,\varepsilon}(M, \omega, \mathbb{W})$ is clearly invariant under the action. We will show that this function is lower semicontinuous (with respect to the C^1-topology on $SK(M)$) – the proof is a modification of the proof of a similar claim in [EV1]. Then we use the result from [EV1] saying that the orbit of a form $\omega \in SK_1(M)$ under the action of $\mathrm{Diff}^+(M)$ is dense in $SK_1(M)$, as long as ω is irrational. Since the orbits of both ω_1 and ω_2 are dense in $SK_1(M)$, we get, by the lower semicontinuity, that $v_{T,\varepsilon}(M, \omega_1, \mathbb{W}) \leq v_{T,\varepsilon}(M, \omega_2, \mathbb{W})$ and $v_{T,\varepsilon}(M, \omega_1, \mathbb{W}) \geq v_{T,\varepsilon}(M, \omega_2, \mathbb{W})$, which means that $v_{T,\varepsilon}(M, \omega_1, \mathbb{W}) = v_{T,\varepsilon}(M, \omega_2, \mathbb{W})$. Then the proofs of Corollary 3.29, Corollary 3.30 and Corollary 3.31 follow the same lines as the proof of [EV1, Cor. 3.3] where the existence of symplectic embeddings of polydisks into tori was proved.

In Section 12 we discuss how the proofs in the K3 case can be modified (or just carried over without any changes) in order to prove the results in the hyperkähler case stated in Section 3.5.

Finally, in Section A (the Appendix) we discuss various well-known facts concerning the dependence of the Hodge decomposition on the complex structure, the deformations of complex structures, Moser's method, and Alexander's trick that we use in the paper and for which we could not find a direct reference in the literature.

5 Kähler-type embeddings of balls and blow-ups

As before, let M, $\dim_{\mathbb{R}} M = 2n$, be a closed connected manifold admitting Kähler structures. Let ω be a Kähler-type symplectic form on M.

Let $k \in \mathbb{Z}_{>0}$, $r_1, \ldots, r_k > 0$, $\bar{r} = (r_1, \ldots, r_k)$.

Let $\mathcal{C}_0$ be a connected component of $CK(M)$.

From now on, all the spaces of maps between manifolds and of differential forms and complex structures on manifolds will be endowed with the C^∞-topologies. All the product/quotient spaces will be endowed with the product/quotient topologies.

5.1 The $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$-action on the space of Kähler-type embeddings

Denote:

$Emb_{M,\omega}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\right)$ – the space of the symplectic embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$;

$EK_{M,\omega}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\right) \subset Emb_{M,\omega}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\right)$ – the space of the Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$;

$EK_{M,\omega,\mathcal{C}_0}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\right)$ – the space of the Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ favoring $\mathcal{C}_0$.

Given a symplectic embedding $f : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$, any symplectic embedding $f' : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ sufficiently C^∞-close to f can be mapped in f by an element of $\mathrm{Symp}_0(M, \omega)$, and moreover, this element can be chosen arbitrarily close to the identity, provided f' is sufficiently C^∞-close to f. This fact is standard and follows from the well-known Alexander trick and the symplectic isotopy extension results – see Corollary A.5 for details.

Since symplectomorphisms of (M, ω) map Kähler-type embeddings into Kähler-type embeddings, this yields that $EK_{M,\omega}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\right)$ is an open $\mathrm{Symp}(M, \omega)$-invariant subset of $Emb_{M,\omega}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\right)$ (in the C^∞-topology). This yields the following corollary.

Proposition 5.1

The space

$$\frac{EK_{M,\omega}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\right)}{\mathrm{Symp}_0(M, \omega)},$$

and consequently the space

$$\frac{EK_{M,\omega}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\right)}{\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)},$$

and its subspace

$$\frac{EK_{M,\omega,\mathcal{C}_0}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\right)}{\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)},$$

are discrete. ∎

Definition 5.2

Define *Pairs*$_{\mathcal{C}_0}$ as the set of pairs (η, f), where $\eta \in \mathrm{Diff}_0(M) \cdot \omega$ and $f : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \eta)$ is Kähler-type embedding into (M, η) favoring $\mathcal{C}_0$. (Note that the embedding here is into (M, η), not (M, ω)!)

Define

$$\texttt{Pairs}_{\mathcal{C}_0} := \textit{Pairs}_{\mathcal{C}_0} / \mathrm{Diff}_0(M).$$

For each $(\eta, f) \in \textit{Pairs}_{\mathcal{C}_0}$, denote by $\{\eta, f\} \in \texttt{Pairs}_{\mathcal{C}_0}$ the element of $\texttt{Pairs}_{\mathcal{C}_0}$ represented by (η, f) – i.e., the $\mathrm{Diff}_0(M)$-orbit of (η, f).

The following proposition follows easily from the definitions.

Proposition 5.3

The natural map

$$\frac{EK_{M,\omega,\mathcal{C}_0}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\right)}{\mathrm{Symp}(M,\omega) \cap \mathrm{Diff}_0(M)} \to \texttt{Pairs}_{\mathcal{C}_0}$$

is a homeomorphism. In particular, in both cases $\texttt{Pairs}_{\mathcal{C}_0}$ is discrete. ∎

A discrete topological space is connected if and only if it is a point. Together with Proposition 5.3, this immediately yields the following corollary.

Corollary 5.4

The $\mathrm{Symp}(M,\omega) \cap \mathrm{Diff}_0(M)$-action on $EK_{M,\omega,\mathcal{C}_0}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\right)$ is transitive if and only if $\texttt{Pairs}_{\mathcal{C}_0}$ is connected. ∎

Proposition 5.5

Assume (η_t, f_t), $0 \le t \le 1$, is a smooth family in *Pairs*$_{\mathcal{C}_0}$.

Then all (η_t, f_t), $0 \le t \le 1$, lie in the same orbit of the $\mathrm{Diff}_0(M)$-action on *Pairs*$_{\mathcal{C}_0}$.

Proof of Proposition 5.5:

The forms η_t, $0 \le t \le 1$, form a smooth family of cohomologous symplectic forms. Hence, by Moser's theorem [Mos], all these forms can be identified by the $\mathrm{Diff}_0(M)$-action with η_0. Under such an identification, $\{f_t\}_{0 \le t \le 1}$ becomes a smooth family of symplectic embeddings $\{f_t : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \eta_0)\}_{0 \le t \le 1}$. By the symplectic isotopy extension result (see e.g. [McDS, Thm. 3.3.1]), all such f_t lie in the same $\mathrm{Symp}_0(M, \eta_0)$-orbit. This proves the proposition. ∎

5.2 From holomorphic to Kähler-type embeddings of balls

As before, for each $x := (x_1, \ldots, x_k) \in \hat{M}^k$ and each complex structure I on M, let $\widetilde{M}_{I,x}$ denote the complex blow-up of (M, I) at $x_1, \ldots, x_k$.

For any I and x we will use the same notation for the following objects (suppressing the dependence on I and x):

- The lift of I to $\widetilde{M}_{I,x}$ will be denoted by $\widetilde{I}$. This complex structure is of Kähler-type (see e.g. [Voi, Vol I, Prop. 3.24]). If I' is another complex structure on M coinciding with I near each x_i, $i = 1, \ldots, k$, then it also lifts to $\widetilde{M}_{I,x}$ and its lift will be denoted by $\widetilde{I'}$.

- $\Pi : \widetilde{M}_{I,x} \to M$ will denote the natural projection.

- $E_i := \Pi^{-1}(x_i)$, $i = 1, \ldots, k$, is an exceptional divisor.

- The cohomology classes Poincaré-dual to the fundamental homology classes of E_i, $i = 1, \ldots, k$, will be denoted by $e_1, \ldots, e_k \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$.

For all I and x the groups $H^2(\widetilde{M}_{I,x}; \mathbb{R})$ can be canonically identified (preserving the numbering of the classes $e_1, \ldots, e_n$) and we will use these identifications without further mention.

Definition 5.6

Define $Triples_{\mathcal{C}_0}$ as the set of triples (I, x, h), where

- $(I, x) \in \mathcal{K}_{\mathcal{C}_0}(\bar{r})$, meaning that

$$I \in \text{Diff}_0(M) \cdot (Cmpt(M, \omega) \cap \mathcal{C}_0),$$

$$x := (x_1, \ldots, x_k) \in \hat{M}^k,$$

and the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to $\widetilde{I}$.

- $h = \bigsqcup_{i=1}^k h_i : \bigsqcup_{i=1}^k B^{2n}(r_i) \to (M, I)$ is a holomorphic embedding into (M, I) so that $h_i(0) = x_i$, $i = 1, \ldots, x_k$.

Set $\text{Triples}_{\mathcal{C}_0} := Triples_{\mathcal{C}_0} / \text{Diff}_0(M)$.

For each $(I, x, h) \in Triples_{\mathcal{C}_0}$, denote by $\{I, x, h\} \in \text{Triples}_{\mathcal{C}_0}$ the element of $\text{Triples}_{\mathcal{C}_0}$ represented by (I, x, h) – i.e., the $\text{Diff}_0(M)$-orbit of (I, x, h).

For each $\kappa > 0$ define

$$R_\kappa : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \ \mathcal{R}_\kappa(x) := \kappa x.$$

The proof of Theorem 3.2 is based on the following key propositions.

Proposition 5.7

There exists a continuous surjective map $\Phi : \text{Triples}_{\mathcal{C}_0} \to \text{Pairs}_{\mathcal{C}_0}$ with the following properties:

1. For each $(I, x, h) \in Triples_{\mathcal{C}_0}$,

$$\Phi(\{I, x, h\}) = \{\eta, h \circ R_\kappa\}$$

for a symplectic form η compatible with I and cohomologous to ω and for a sufficiently small $0 < \kappa \leq 1$, so that the embedding $h \circ R_\kappa : \bigsqcup_{i=1}^k B^{2n}(r_i) \to M$ is symplectic with respect to η and holomorphic with respect to I.

2. If η is a symplectic form on M compatible with a complex structure I and cohomologous to ω and $f = \bigsqcup_{i=1}^k f_i : \bigsqcup_{i=1}^k B^{2n}(r_i) \to (M, \eta)$ is a symplectic embedding into (M, η) holomorphic with respect to I, then, for $x_i := f_i(0)$, $i = 1, \ldots, k$, and $x := (x_1, \ldots, x_k)$, we have $(I, x, f) \in \textit{Triples}_{\mathcal{C}_0}$ and $\Phi(\{I, x, f\}) = \{\eta, f\}$.

Proposition 5.8

The space $\texttt{Triples}_{\mathcal{C}_0}$ is connected if only if the set $K_{\mathcal{C}_0}(\bar{r})$ (see Definition 3.1) is connected.

For the proof of Proposition 5.7 see Section 5.5. The proof of Proposition 5.8 will be given in Section 5.6.

Let us now prove Theorem 3.2. For convenience, we restate it here.

Theorem 5.9 (=Theorem 3.2)

Let M be a closed manifold equipped with a Kähler-type symplectic form ω.

(I) The set $\bigsqcup_{i=1}^k B^{2n}(r_i)$ admits a Kähler-type embedding into (M, ω) if and only if for some $I \in Cmpt(M, \omega)$ and some $x \in \hat{M}^k$ the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$, is Kähler with respect to $\widetilde{I}$.

More precisely, assume I is a complex structure on M compatible with ω and $\Sigma \subset (M, I)$ is a proper (possibly empty) complex submanifold. If $M = \mathbb{C}P^{n_1} \times \mathbb{C}P^1$ (with the product complex structure), we also allow $\Sigma = (\mathbb{C}P^{n_1 - 1} \times \mathbb{C}P^1) \cup (\mathbb{C}P^{n_1} \times \text{pt})$.

Then the following conditions are equivalent:

1. There exists $x = (x_1, \ldots, x_k) \in \hat{M}^k$, $x_1, \ldots, x_k \in M \setminus \Sigma$, so that the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to $\widetilde{I}$.

2. There exists an $[I]$-Kähler-type embedding $f : \bigsqcup_{i=1}^k B^{2n}(r_i) \to (M \setminus \Sigma, \omega)$ which is holomorphic with respect to a complex structure $\phi_1^* I$ compatible with ω, where $\{\phi_t\}_{0 \leq t \leq 1} \subset \text{Diff}_0(M)$, $\phi_0 = Id$, is an isotopy such that $\phi_t(\Sigma) = \Sigma$ for all $t \in [0, 1]$.

(II) Let $\mathcal{C}_0$ be a connected component of $CK(M)$. Assume that the set $K_{\mathcal{C}_0}(\bar{r})$ (or, equivalently, the set $\mathcal{K}_{\mathcal{C}_0}(\bar{r})$) is connected.

Then any two Kähler-type embeddings $\bigsqcup_{i=1}^k B^{2n}(r_i) \to (M, \omega)$ favoring $\mathcal{C}_0$ lie in the same $\text{Symp}(M, \omega) \cap \text{Diff}_0(M)$-orbit. In particular, there exists $[I] \in \texttt{Teich}_{\mathcal{C}_0}(M, \omega)$ such that both embeddings are of $[I]$-Kähler type.

If, in addition, $\text{Symp}_H(M)$ acts transitively on the set of connected components of $CK(M)$ compatible with ω, then any two Kähler-type embeddings $\bigsqcup_{i=1}^k B^{2n}(r_i) \to (M, \omega)$ lie in the same $\text{Symp}_H(M, \omega)$-orbit.

Proof of Theorem 5.9 (=Theorem 3.2):

Let us prove part (I) of Theorem 5.9.

Let $\mathcal{C}_0$ be a connected component of $CK(M)$ containing the complex structure I.

Define the set $Y \supset \Sigma$ as follows: If Σ is a complex submanifold of (M,I), set $Y = \Sigma$. If $M = \mathbb{C}P^{n_1} \times \mathbb{C}P^1$ and $\Sigma = (\mathbb{C}P^{n_1-1} \times \mathbb{C}P^1) \cup (\mathbb{C}P^{n_1} \times \text{pt})$, let Y be the union of Σ and of the closure of an open neighborhood of $(\mathbb{C}P^{n_1-1} \times \mathbb{C}P^1) \cap (\mathbb{C}P^{n_1} \times \text{pt})$.

Let us prove $1 \Longrightarrow 2$. Assume that $x = (x_1,\dots,x_k) \in \hat{M}^k$, $x_1,\dots,x_k \in M \setminus \Sigma$, so that the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to $\widetilde{I}$. Choose a holomorphic embedding $h = \bigsqcup_{i=1}^k h_i : \bigsqcup_{i=1}^k B^{2n}(r_i + \varepsilon) \to (M,I)$, for some $\varepsilon > 0$, so that $h_i(0) = x_i$, $i = 1,\dots,k$, and $\text{Im}\,h \cap Y = \emptyset$. Then $(I,x,h) \in \textit{Triples}_{\mathcal{C}_0}$.

By part 1 of Proposition 5.7, there exist a symplectic form η compatible with I and cohomologous to ω and a sufficiently small $0 < \kappa \leq 1$, so that the embedding $h \circ R_\kappa : \bigsqcup_{i=1}^k B^{2n}(r_i) \to M$ is symplectic with respect to η and holomorphic with respect to I. In particular, $\text{Im}\,(h \circ R_\kappa) \cap Y = \emptyset$. The linear family of 2-forms connecting η and ω is formed by cohomologous symplectic forms compatible with I and, in particular, their restrictions to Σ are non-degenerate (in the case $M = \mathbb{C}P^{n_1} \times \mathbb{C}P^1$ and $\Sigma = (\mathbb{C}P^{n_1-1} \times \mathbb{C}P^1) \cup (\mathbb{C}P^{n_1} \times \text{pt})$, the restrictions of the forms to both $\mathbb{C}P^{n_1-1} \times \mathbb{C}P^1$ and $\mathbb{C}P^{n_1} \times \text{pt}$ are non-degenerate). By a relative version of Moser's theorem (see Proposition A.3, parts II and III), there exists an isotopy $\{\phi_t\}_{0 \leq t \leq 1} \subset \text{Diff}_0(M)$, $\phi_0 = Id$, $\phi_t(\Sigma) = \Sigma$ for all $t \in [0,1]$, so that $(\phi_1^{-1})^*\omega = \eta$ outside Y. Then $\phi^{-1} \circ h \circ R_\kappa$ is an embedding of $\bigsqcup_{i=1}^k B^{2n}(r_i)$ into $M \setminus \Sigma$ which is symplectic with respect to ω and holomorphic with respect to the complex structure ϕ^*I on M compatible with ω.

Now let us prove $2 \Longrightarrow 1$. Assume $f = \bigsqcup_{i=1}^k f_i : \bigsqcup_{i=1}^k B^{2n}(r_i) \to (M \setminus \Sigma, \omega)$ is a Kähler-type embedding holomorphic with respect to a complex structure $J := \phi_1^*I$, which is compatible with ω, where $\{\phi_t\}_{0 \leq t \leq 1} \subset \text{Diff}_0(M)$ is an isotopy such that $\phi_0 = Id$, $\phi_t(\Sigma) = \Sigma$ for all $t \in [0,1]$. Let $x_i := f_i(0)$, $i = 1,\dots,k$. Set $x := (x_1,\dots,x_k)$. Then, by property 2 of Φ listed in Proposition 5.7, $(J,x,f) \in \textit{Triples}_{\mathcal{C}_0}$, meaning, in particular, that the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i$ in $H^2(\widetilde{M}_{J,x}; \mathbb{R})$ is Kähler with respect to $\widetilde{J}$. Since $x_1,\dots,x_k \in M \setminus \Sigma$ and $\phi_1(\Sigma) = \Sigma$, we get that the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i$ in $H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to $\widetilde{I}$.

This finishes the proof of part (I) of Theorem 5.9.

Let us prove part (II) of Theorem 5.9.

Assume that the set $K_{\mathcal{C}_0}(\bar{r})$ is connected. By Proposition 5.8, this means that $\texttt{Triples}_{\mathcal{C}_0}$ is connected. Since, by Proposition 5.7, $\Phi : \texttt{Triples}_{\mathcal{C}_0} \to \texttt{Pairs}_{\mathcal{C}_0}$ is a surjective continuous map, we get that $\texttt{Pairs}_{\mathcal{C}_0}$ is connected. Hence, Corollary 5.4 implies that $\text{Symp}(M,\omega) \cap \text{Diff}_0(M)$ acts transitively on $EK_{M,\omega,\mathcal{C}_0}\left(\bigsqcup_{i=1}^k B^{2n}(r_i)\right)$. In particular, by Proposition 2.7, there exists $[I] \in \texttt{Teich}_{\mathcal{C}_0}(M,\omega)$ such that both embeddings are of $[I]$-Kähler type.

Assume, in addition, that $\text{Symp}_H(M)$ acts transitively on the set of connected components of $CK(M)$ compatible with ω. Then any element of $EK_{M,\omega}\left(\bigsqcup_{i=1}^k B^{2n}(r_i)\right)$ can be mapped by $\text{Symp}_H(M)$ into an element of the space $EK_{M,\omega,\mathcal{C}_0}\left(\bigsqcup_{i=1}^k B^{2n}(r_i)\right)$. The transitivity of the $\text{Symp}(M,\omega) \cap \text{Diff}_0(M)$-action on the latter space yields that any two Kähler-type embeddings of $\bigsqcup_{i=1}^k B^{2n}(r_i)$ into (M,ω) can be mapped into each other by an element of $\text{Symp}_H(M,\omega)$.

This finishes the proof of part (II) of Theorem 5.9. ∎

5.3 Regularized maximum

Given a $(1,1)$-form θ on a complex manifold with a complex structure I, we write $\theta > 0$ if $\theta(v,Iv) > 0$ for any non-zero tangent vector v of the manifold. If θ' is another $(1,1)$-form on the same complex manifold, we write $\theta > \theta'$ if $\theta - \theta' > 0$.

Recall from [Dem, Lemma 5.18] that for any smooth functions Q_1, Q_2 on a complex manifold and any $\delta > 0$, one can define their regularized maximum $\max_\delta\{Q_1, Q_2\}$. It is a smooth function on the same manifold, depending smoothly on δ, and satisfying the following conditions:

1. $\max\{Q_1, Q_2\} \leq \max_\delta\{Q_1, Q_2\} \leq \max\{Q_1, Q_2\} + \delta$;

2. $\max_\delta\{Q_1, Q_2\}(y) = Q_j(y)$, if $Q_i(y) + 2\delta \leq Q_j(y)$, $i, j = 1, 2$, $i \neq j$.

3. If ϑ is a Hermitian form on the manifold and $\sqrt{-1}\partial\bar\partial Q_i > -\vartheta$, $i = 1, 2$, then

$$\sqrt{-1}\partial\bar\partial \max_\delta\{Q_1, Q_2\} > -\vartheta.$$

5.4 Blow-up – local constructions

First, we recall some standard local constructions concerning the blow-up of $\mathbb{C}^n$ at the origin.

Denote by $z_1, \ldots, z_n$ the complex coordinates on $\mathbb{C}^n$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ denote

$$|z| := \sqrt{|z_1|^2 + \ldots + |z_n|^2}.$$

As before, let ω_0 be the standard symplectic form on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. It can be written as

$$\omega_0 = \sqrt{-1}\partial\bar\partial|z|^2/2. \tag{5.2}$$

Let σ be the Fubini-Study symplectic form on $\mathbb{C}P^{n-1}$ normalized by $\int_{\mathbb{C}P^1} \sigma = \pi$.

Let $\mathcal{L} \subset \mathbb{C}^n \times \mathbb{C}P^{n-1}$ be the incidence relation, i.e. $\mathcal{L} := \{(z, \mathfrak{l}) \mid z \in \mathfrak{l}\}$. It is the complex blow-up of $\mathbb{C}^n$ at the origin. Let $pr_1 : \mathcal{L} \to \mathbb{C}^n$, $pr_2 : \mathcal{L} \to \mathbb{C}P^{n-1}$ be the natural projections.

For $r > 0$, denote $\mathcal{L}(r) := pr_1^{-1}(B^{2n}(r))$. We will also write $\mathcal{L}(0) := pr_1^{-1}(0)$.

Given $c, r > 0$, define a differential 2-form $\rho(c, r)$ on $\mathcal{L}$ as

$$\rho(c, r) := c^2 pr_1^* \omega_0 + r^2 pr_2^* \sigma.$$

The restriction of pr_1 to $\mathcal{L} \setminus pr_1^{-1}(0)$ is a diffeomorphism $\mathcal{L} \setminus pr_1^{-1}(0) \to \mathbb{C}^n \setminus 0$. One can easily see that it identifies the form $\rho(c, r)$ on $\mathcal{L} \setminus \mathcal{L}(0)$ with the form $(pr_1)_* \rho(c, r)$ on $\mathbb{C}^n \setminus 0$ that can be written as

$$(pr_1)_* \rho(c, r) = \frac{\sqrt{-1}}{2} \partial\bar\partial \left(c^2|z|^2 + r^2 \log|z|^2 + d \right) \tag{5.3}$$

for any $d \in \mathbb{R}$.

The following two propositions and their proofs describe local constructions needed further on to define symplectic blow-up and blow-down operations. Here we use the regularized maximum for these constructions – cf. e.g. [McDP], [McDS], where different tools were used.

Proposition 5.10

Let $r, \varepsilon > 0$.

Then for any sufficiently small $c > 0$ there exists a Kähler form ρ^c on $\mathbb{C}^n \setminus 0$, depending smoothly on c, so that

- $\rho^c = \omega_0$ on a neighborhood of $\mathbb{C}^n \setminus Int\, B^{2n}(r + \varepsilon)$,

- $\rho^c = (pr_1)_* \rho(c, r)$ on a neighborhood of $B^{2n}(r) \setminus 0$.

Proof of Proposition 5.10:

The functions $s^2 - r^2$ and $r^2 \log\left((s/r)^2\right)$ (as functions of s) coincide with their first derivatives at $s = r$. The first function is convex while the second one is concave and $s^2 - r^2 > r^2 \log\left((s/r)^2\right)$ for all $s > 0$, $s \neq r$. Therefore for any sufficiently small $c > 0$ and a sufficiently small $\sigma > 0$ (depending on c) the functions $F(s) := s^2 - r^2 - \sigma$ and $G(s) := cs^2 + r^2 \log s^2$ have exactly two points s_1, s_2, $0 < s_1 < r < s_2 < r + \varepsilon$, where they coincide, so that $F(s) < G(s)$ on $[s_1, r)$, while $F(s) > G(s)$ on $(s_2, r + \varepsilon]$.

Choose $\delta > 0$ so that $F(r) < G(r) - 2\delta$ and $F(r + \varepsilon) > G(r + \varepsilon) + 2\delta$.

Define $H : \mathbb{C}^n \setminus 0 \to \mathbb{R}$ by $H(z) := \max_\delta \{F(|z^2|), G(|z^2|)\}$, where $\max_\delta$ is the regularized maximum mentioned in Section 5.3. It follows from (5.2) and (5.3) and the properties of the regularized maximum (see Section 5.3) that $(\sqrt{-1}/2) \cdot \partial \bar{\partial} H$ is a Kähler form on $\mathbb{C}^n \setminus 0$ equal to $(\sqrt{-1}/2) \cdot \partial \bar{\partial} F = \omega_0$ on a neighborhood $\mathbb{C}^n \setminus Int\, B^{2n}(r + \varepsilon)$, and to $(\sqrt{-1}/2) \cdot \partial \bar{\partial} G = (pr_1)_* \rho(c, r)$ on a neighborhood of the sphere $\partial B^{2n}(r)$.

Define the form ρ^c on $\mathbb{C}^n \setminus 0$ as $(pr_1)_* \rho(c, r)$ on $B^{2n}(r) \setminus 0$, $(\sqrt{-1}/2) \cdot \partial \bar{\partial} G$ on the neighborhood of $\partial B^{2n}(r)$ and $(\sqrt{-1}/2) \cdot \partial \bar{\partial} H$ outside $\partial B^{2n}(r)$. It is a well-defined form. The parameters σ, s_1, s_2, δ in the construction can be chosen to depend smoothly on c and thus ρ^c can be assumed to depend smoothly on c. One easily sees that ρ^c satisfies the bulleted properties listed in the proposition. This finishes the proof. ∎

Proposition 5.11

Let $r, \varepsilon > 0$.

Then for any $c > 0$ there exists a Kähler form ρ_c on $\mathbb{C}^n$, depending smoothly on c, so that

- $\rho_c = \omega_0$ on a neighborhood of $B^{2n}(r)$,

- $\rho_c = (pr_1)_* \rho(c, r)$ on a neighborhood of $\mathbb{C}^n \setminus Int\, B^{2n}(r + \varepsilon)$.

Proof of Proposition 5.11:

For an appropriate constant $d \in \mathbb{R}$ the functions $F(s) := s^2 + d$ and $G(s) := c^2 s^2 + r^2 \log\left((s/r)^2\right)$ (as functions of s) are equal at some point $s_1 \in (r, r + \varepsilon)$, so that $F(s) > G(s)$ on $[r, s_1)$, $F(s) < G(s)$ on $(s_1, s_2]$ for some $s_2 \in (s_1, r + \varepsilon)$.

Having chosen such s_1, s_2, pick $\delta > 0$ so that $F(r) > G(r) + 2\delta$ and $F(s_2) < G(s_2) - 2\delta$.

Define $H : \mathbb{C}^n \setminus 0 \to \mathbb{R}$ by $H(z) := \max_\delta \{ F(|z^2|), G(|z^2|) \}$, where $\max_\delta$ is the regularized maximum mentioned in Section 5.3. It follows from (5.2) and (5.3) and the properties of the regularized maximum (see Section 5.3) that $(\sqrt{-1}/2) \cdot \partial\bar{\partial}H$ is a Kähler form on $\mathbb{C}^n \setminus 0$ equal to $(\sqrt{-1}/2) \cdot \partial\bar{\partial}F = \omega_0$ on $B^{2n}(r + \sigma) \setminus Int\, B^{2n}(r)$ for some sufficiently small $\sigma > 0$, and to $(\sqrt{-1}/2) \cdot \partial\bar{\partial}G = (pr_1)_* \rho(c, r)$ on a neighborhood of $\partial B^{2n}(s_2)$.

Define a form ρ_c on $\mathbb{C}^n$ as ω_0 on $B^{2n}(r)$, as $(\sqrt{-1}/2) \cdot \partial\bar{\partial}H$ on $B^{2n}(s_2) \setminus B^{2n}(r)$ and as $(\sqrt{-1}/2) \cdot \partial\bar{\partial}G = (pr_1)_* \rho(c, r)$ on $\mathbb{C}^n \setminus B^{2n}(s_2)$. It is a well-defined smooth form. The parameters d, s_1, s_2, δ in the construction can be chosen to depend smoothly on c and thus ρ_c can be assumed to depend smoothly on c. One easily sees that ρ_c satisfies the bulleted properties listed in the proposition. This finishes the proof. ∎

Lemma 5.12 ($\partial\bar{\partial}$-lemma for a complex blow-up of a ball)

Let $B \subset \mathbb{C}^n$ be a ball centered at the origin $0 \in \mathbb{C}^n$. Let $\widetilde{B}$ be the complex blow-up of B at 0.

Then any exact $(1,1)$-form υ on $\widetilde{B}$ can be written as $\upsilon = \sqrt{-1}\partial\bar{\partial}Q$ for a smooth real-valued function Q on $\widetilde{B}$.

Proof of Lemma 5.12:

Let $\pi : \widetilde{B} \to B$ be the natural projection. Let $\mathcal{O}_{\widetilde{B}}$ be the structure sheaf (i.e., the sheaf of germs of holomorphic functions) on $\widetilde{B}$.

Note that $R^i \pi_* \mathcal{O}_{\widetilde{B}} = 0$ for all $i > 1$ (see [Ue, Prop. 2.14], cf. [Hi, (2), p.144]). Consequently, $H^1(\widetilde{B}, \mathcal{O}_{\widetilde{B}}) = H^1(B, \pi_* \mathcal{O}_{\widetilde{B}}) = 0$.

By the Dolbeault theorem, this implies that any $\bar{\partial}$-closed $(0,1)$-form on $\widetilde{B}$ is $\bar{\partial}$-exact. The latter fact yields, by the complex conjugation symmetry, that any ∂-closed $(1,0)$-form on $\widetilde{B}$ is ∂-exact.

Since υ is exact, it can be written as $\upsilon = d\sigma$ for some 1-form σ on $\widetilde{B}$. Write σ as a sum of its $(1,0)$ and $(0,1)$-parts: $\sigma = \sigma^{1,0} + \sigma^{0,1}$. Then the 1-form $\sigma^{0,1}$ is $\bar{\partial}$-closed and hence $\bar{\partial}$-exact, meaning that $\sigma^{0,1} = \bar{\partial}S$ for a smooth function S on $\widetilde{B}$. One readily checks that the form $\sigma - dS$ is of type $(1,0)$ and ∂-closed, hence ∂-exact, meaning that $\sigma - dS = \partial(-\sqrt{-1}Q)$ for a smooth function Q on $\widetilde{B}$. Finally, a direct computation yields $\upsilon = \bar{\partial}\partial(-\sqrt{-1}Q) = \sqrt{-1}\partial\bar{\partial}Q$. Since the form υ is real, the function Q can be also chosen to be real-valued. This finishes the proof of the lemma. ∎

5.5 Symplectic forms on complex blow-ups of M

Let I be a complex structure on M and let $x = (x_1, \ldots, x_k) \in \hat{M}^k$.

Fix an I-holomorphic embedding $h = \bigsqcup_{i=1}^{k} h_i : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to M$, such that $h_i(0) = x_i$, $i = 1, \ldots, k$, and h extends to a holomorphic embedding $h = \bigsqcup_{i=1}^{k} h_i : \bigsqcup_{i=1}^{k} B^{2n}(r_i + \varepsilon) \to M$ for some $\varepsilon > 0$. Each map $h_i : B^{2n}(r_i + \varepsilon) \to M$, $i = 1, \ldots, k$, lifts to the unique map $\widetilde{h}_i : \mathcal{L}(r_i + \varepsilon) \to \widetilde{M}_{I,x}$ such that $h_i \circ pr_1 = \Pi \circ \widetilde{h}_i$, where $pr_1 : \mathcal{L} \to \mathbb{C}^n$ is the projection defined in Section 5.4. For any $c, r > 0$ the map $\widetilde{h}_i$, $i = 1, \ldots, k$, identifies the form $\rho(c, r)$ on $\mathcal{L}(r_i + \varepsilon)$ with a form on a neighborhood of the exceptional divisor $E_i \subset \widetilde{M}_{I,x}$ that will be denoted by $\rho_{x_i}(c, r)$. Each $\rho_{x_i}(c, r)$ is a symplectic form compatible with $\widetilde{I}$.

Given a cohomology class $\alpha \in H^2(M;\mathbb{R})$, define the cohomology class $\widetilde{\alpha}_{\tilde{r}} \in H^2(\widetilde{M}_{I,x};\mathbb{R})$ by

$$\widetilde{\alpha}_{\tilde{r}} := \Pi^*\alpha - \pi \sum_{i=1}^{k} r_i^2 e_i. \tag{5.4}$$

Proposition 5.13

Let $\alpha \in H^2(M;\mathbb{R})$. Assume that the cohomology class $\widetilde{\alpha}_{\tilde{r}} \in H^2(\widetilde{M}_{I,x};\mathbb{R})$ is Kähler with respect to $\widetilde{I}$.
Then there exists a symplectic form $\widetilde{\theta}$ on $\widetilde{M}_{I,x}$ which is compatible with $\widetilde{I}$, represents $\widetilde{\alpha}_{\tilde{r}}$, and equals to $\rho_{x_i}(1,r_i)$ on a neighborhood of E_i for each $i = 1,\ldots,k$.

Remark 5.14

If one drops the requirement that $\widetilde{\theta}$ is compatible with $\widetilde{I}$, the existence of a symplectic form $\widetilde{\theta}$, $[\widetilde{\theta}] = \widetilde{\alpha}_{\tilde{r}}$, coinciding with $\rho_{x_i}(1,r_i)$ on a neighborhood of each E_i can be proved much easier using Moser's method, as in [McDP, Pf. of Prop. 2.1.C]. The need to make $\widetilde{\theta}$ compatible with $\widetilde{I}$ comes from the fact that otherwise the blow-down construction (see [McDP, Sec. 5]) applied to $\widetilde{\theta}$ will produce only a symplectic, but not necessarily a holomorphic, embedding of balls. Unlike Moser's method – which can be applied, as in [McDP, Pf. of Prop. 2.1.C], also when I is not necessarily integrable outside neighborhoods of the base points $x_1,\ldots,x_k$ – here we do rely on the integrability of I on the whole M, as we use methods of complex geometry.

Proof of Proposition 5.13:

The argument below uses a result based on an idea of Demailly and published in [OV, Thm. 4.1] and, moreover, mimics the idea of its proof.

Since the cohomology class $\widetilde{\alpha}_{\tilde{r}} \in H^2(\widetilde{M}_{I,x};\mathbb{R})$ is Kähler, we can pick a symplectic form $\widetilde{\eta}$ that represents it and that is compatible with $\widetilde{I}$.

For each $i = 1,\ldots,k$ denote:
$$U_{x_i} := h\left(B^{2n}(r_i+\varepsilon)\right),$$
$$U'_{x_i} := h\left(B^{2n}(r_i+\varepsilon/2)\right),$$
$$\widetilde{U}_{x_i} := \widetilde{h}_i\left(\mathcal{L}(r_i+\varepsilon)\right) = \Pi^{-1}(U_{x_i}),$$
$$\widetilde{U}'_{x_i} := \widetilde{h}_i\left(\mathcal{L}(r_i)\right) = \Pi^{-1}(U'_{x_i}).$$

Thus, $U'_{x_i} \subset U_{x_i}$, $\widetilde{U}'_{x_i} \subset \widetilde{U}_{x_i}$.

Let $\delta > 0$.

Let $E := \bigcup_{i=1}^{k} E_i$ be the union of the exceptional divisors of $\widetilde{M}_{I,x}$. Since E is a complex submanifold of $(\widetilde{M}_{I,x}, \widetilde{I})$, by a result of Demailly-Paun [DP, Lem. 2.1], there exists a smooth real-valued function G on $\widetilde{M}_{I,x} \setminus E$ so that

$$\sqrt{-1}\partial\bar{\partial}G > -\widetilde{\eta} \tag{5.5}$$

and G has logarithmic poles along E_i, $i = 1,\ldots,k$.

The set $\widetilde{U}_{x_i}$ is biholomorphic to the blow-up of the ball at the origin and $\rho_{x_i}(1,r_i) - \widetilde{\eta}$ is an exact $(1,1)$-form on $\widetilde{U}_{x_i}$. Therefore, by Lemma 5.12,

$$\rho_{x_i}(1,r_i) - \widetilde{\eta}|_{\widetilde{U}_{x_i}} = \sqrt{-1}\partial\bar{\partial}F_i \tag{5.6}$$

for a smooth function $F_i : \widetilde{U}_{x_i} \to \mathbb{R}$. Adding, if necessary, a constant to F_i, we may assume without loss of generality that

$$F_i < G - 2\delta \text{ on an open neigborhood of } \widetilde{U}_{x_i} \setminus \widetilde{U}'_{x_i}, i = 1, \ldots, k. \tag{5.7}$$

Since the function G tends to $-\infty$ as its argument converges to E, for any $i = 1, \ldots, k$ there exists a neighborhood $V_{x_i} \subset U'_{x_i}$ of x_i, whose preimage $\Pi^{-1}(V_{x_i}) \subset \widetilde{U}'_{x_i}$ will be denoted by $\widetilde{V}_{x_i} := \Pi^{-1}(V_{x_i})$, so that

$$G + 2\delta < F_i \text{ on } \widetilde{V}_{x_i}, i = 1, \ldots, k. \tag{5.8}$$

Define the function $K : \widetilde{M}_{I,x} \to \mathbb{R}$ as follows

$$K := \max_{\delta}\{F_i, G\}, \text{ on } \widetilde{U}_{x_i}, i = 1, \ldots, k,$$

$$K := G, \text{ on } \widetilde{M}_{I,x} \setminus \bigcup_{i=1}^{k} \widetilde{U}'_{x_i}.$$

By (5.7), (5.8) and the properties of the regularized maximum (see Section 5.3), K is a well-defined smooth function on $\widetilde{M}_{I,x}$, equal to F_i on $\widetilde{V}_{x_i}$, to G on $\widetilde{M}_{I,x} \setminus \bigcup_{i=1}^{k} \widetilde{U}'_i$, and to $\max_{\delta}\{F_i, G\}$ on $\widetilde{U}'_{x_i} \setminus \widetilde{V}_{x_i}$, $i = 1, \ldots, k$.

Define the smooth 2-form $\widetilde{\theta}$ on $\widetilde{M}_{I,x}$ by

$$\widetilde{\theta} := \widetilde{\eta} + \sqrt{-1}\partial\bar{\partial}K.$$

The form $\widetilde{\theta}$ is of type $(1,1)$ with respect to $\widetilde{I}$. Let us verify that $\widetilde{\theta}$ is positive – i.e. symplectic. We need to verify that

$$\sqrt{-1}\partial\bar{\partial}K > -\widetilde{\eta} \text{ everywhere on } \widetilde{M}_{I,x}. \tag{5.9}$$

Let us verify the latter inequality case by case for the points of $\widetilde{V}_{x_i}, \widetilde{U}'_{x_i} \setminus \widetilde{V}_{x_i}, i = 1, \ldots, k$, and $\widetilde{M}_{I,x} \setminus \bigcup_{i=1}^{k} \widetilde{U}'_{x_i}$.

Consider first the case of $\widetilde{V}_{x_i}, i = 1, \ldots, k$. By (5.6), and since $\rho_{x_i}(1, r_i) > 0$, we have

$$\sqrt{-1}\partial\bar{\partial}F_i = -\widetilde{\eta} + \rho_{x_i}(1, r_i) > -\widetilde{\eta}, \text{ on } \widetilde{U}_{x_i}, i = 1, \ldots, k, \tag{5.10}$$

Hence, for the points of $\widetilde{V}_{x_i} \subset \widetilde{U}_{x_i}, i = 1, \ldots, k$,

$$\sqrt{-1}\partial\bar{\partial}K = \sqrt{-1}\partial\bar{\partial}F_i > -\widetilde{\eta},$$

meaning that the inequality (5.9) is satisfied in this case.

For the points of $\widetilde{U}'_{x_i} \setminus \widetilde{V}_{x_i}, i = 1, \ldots, k$, we have

$$\sqrt{-1}\partial\bar{\partial}F_i > -\widetilde{\eta}$$

by (5.10), and

$$\sqrt{-1}\partial\bar{\partial}G > -\widetilde{\eta}$$

by (5.5). Hence, for such points

$$\sqrt{-1}\partial\bar{\partial}K = \sqrt{-1}\partial\bar{\partial}\max_{\delta}\{F_i, G\} > -\widetilde{\eta},$$

by the third property of the regularized maximum (see Section 5.3). Thus, the inequality (5.9) is satisfied in this case too.

Finally, for the points of $\widetilde{M}_{I,x} \setminus \bigcup_{i=1}^{k} \widetilde{U}'_{x_i}$,

$$\sqrt{-1}\partial\bar{\partial}G > -\widetilde{\eta}$$

by (5.5). Hence,

$$\sqrt{-1}\partial\bar{\partial}K = \sqrt{-1}\partial\bar{\partial}G > -\widetilde{\eta},$$

meaning that the inequality (5.9) is satisfied also in this case.

This finishes the verification of (5.9).

We have shown that $\widetilde{\theta}$ is a positive $(1,1)$-form with respect to $\widetilde{I}$, which means that it is compatible with $\widetilde{I}$. It is cohomologous to $\widetilde{\eta}$, meaning that it represents the cohomology class $\widetilde{\alpha}_{\bar{r}}$. By (5.6),

$$\widetilde{\theta} = \widetilde{\eta} + \sqrt{-1}\partial\bar{\partial}K = \widetilde{\eta} + \sqrt{-1}\partial\bar{\partial}F_i = \rho_{x_i}(1,r_i) \text{ on } \widetilde{V}_{x_i}, i = 1,\ldots,k.$$

This finishes the proof of the proposition. ∎

Recall that for each $\kappa > 0$ we define

$$R_{\kappa} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \; R_{\kappa}(x) := \kappa x.$$

The map R_{κ} induces a diffeomorphism of $\mathcal{L} \subset \mathbb{C}^n \times \mathbb{C}P^{n-1}$ that will be denoted by $\widetilde{R}_{\kappa}$:

$$\widetilde{R}_{\kappa} : \mathcal{L} \to \mathcal{L}.$$

For $\kappa \in (0,1]$, denote by $h \circ R_{\kappa}$ the holomorphic embedding

$$h \circ R_{\kappa} := \bigsqcup_{i=1}^{k} h_i \circ R_{\kappa} : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, I).$$

For each $i = 1,\ldots,k$ and each $0 < \kappa \leq 1$ denote

$$U_{x_i,\kappa} := h_i \circ R_{\kappa}\left(B^{2n}(r_i + \varepsilon)\right) = h_i\left(B^{2n}\left(\kappa(r_i + \varepsilon)\right)\right).$$

The following proposition describes the exact version of the symplectic blow-down operation that we need for our purposes.

Proposition 5.15

Let $\alpha \in H^2(M; \mathbb{R})$. Let $0 < \kappa \leq 1$, $c_1,\ldots,c_k > 0$.

Assume that the cohomology class $\widetilde{\alpha}_{\bar{r}} \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is represented by a symplectic form $\widetilde{\theta}$ coincides with $\rho_{x_i}(c_i, r_i)$ on the neighborhood $\Pi^{-1}(U_{x_i,\kappa})$ of E_i, $i = 1,\ldots,k$.

Then there exists a symplectic form $\eta_{\kappa,\widetilde{\theta}}$ on M with the following properties:

- $[\eta_{\kappa,\widetilde{\theta}}] = \alpha$,

- If $\widetilde{\theta}$ is compatible with $\widetilde{I}$, then $\eta_{\kappa,\widetilde{\theta}}$ is compatible with I; if I' is a complex structure on M coinciding with I on $\bigcup_{i=1}^{k} U_{x_i,\kappa}$ and such that $\widetilde{\theta}$ tames $\widetilde{I}'$, then $\eta_{\kappa,\widetilde{\theta}}$ tames I'.

- The embedding $h \circ R_\kappa : \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to M$ is symplectic with respect to $\eta_{\kappa,\widetilde{\theta}}$ (and holomorphic with respect to I).

- $\eta_{\kappa,\widetilde{\theta}}$ depends smoothly on κ.

Proof of Proposition 5.15:

Assume $\kappa > 0$ is sufficiently small so that $\widetilde{\theta} = \rho_{x_i}(c_i, r_i)$ on $\Pi^{-1}(U_{x_i,\kappa})$ for all $i = 1, \ldots, k$. Then $\widetilde{R}_\kappa^* \circ \widetilde{h}_i^* \widetilde{\theta} = \rho(\kappa c_i, r_i)$ on $\mathcal{L}(r_i + \varepsilon)$ for all $i = 1, \ldots, k$.

For each $i = 1, \ldots, k$ we can use Proposition 5.11 in order to construct a form $\rho_{\kappa c_i}$ on $B^{2n}(r_i + \varepsilon)$, depending smoothly on κ, that equals ω_0 on a neighborhood of $B^{2n}(r_i)$, and $(pr_1)_* \rho(\kappa c_i, r_i)$ on a neighborhood of $\partial B^{2n}(r_i + \varepsilon)$. Then the form $(h_i)_* \rho_{\kappa c_i}$ is defined on $U_{x_i,\kappa}$ and coincides with $\Pi_* \widetilde{\theta} = \Pi_* (\rho_{x_i}(\kappa c_i, r_i))$ near $\partial U_{x_i,\kappa}$.

Consequently, we can define a 2-form $\eta_{\kappa,\widetilde{\theta}}$ on M by setting it equal to $\Pi_* \widetilde{\theta}$ outside $\bigcup_{i=1}^{k} U_{x_i,\kappa}$ and to $(h_i)_* \rho_{\kappa c_i}$ on each $U_{x_i,\kappa}$, $i = 1, \ldots, k$. One easily checks that $\eta_{\kappa,\widetilde{\theta}}$ is symplectic and that the bulleted properties in the proposition are satisfied. ∎

The following proposition is a version of the symplectic blow-up construction – cf. e.g. [McDP], [McDS].

Proposition 5.16

Assume that η is a symplectic form on M such that the embedding h is symplectic with respect to η and η is compatible with I on the image of h.

Then for any sufficiently small $c > 0$ there exists a symplectic form $\widetilde{\theta}_c$ on $\widetilde{M}_{I,x}$ with the following properties:

1. $\widetilde{\theta}_c = \rho_{x_i}(c, r_i)$ on $\Pi^{-1}(h_i(B(r_i)))$, $i = 1, \ldots, k$,

2. $\Pi_* \widetilde{\theta}_c = \eta$ outside $\bigsqcup_{i=1}^{k} h\left(B^{2n}(r_i + \varepsilon)\right)$,

3. $\widetilde{\theta}_c$ depends smoothly on c,

4. If η is compatible with, respectively tames, I on the whole M, then $\widetilde{\theta}_c$ is compatible with, respectively tames, $\widetilde{I}$ on the whole $\widetilde{M}_{I,x}$.

5. $[\widetilde{\theta}_c] = \Pi^*[\eta] - \pi \sum_{i=1}^{k} r_i^2 e_i$.

Proof of Proposition 5.16:

Using Proposition 5.10 for each $B^{2n}(r_i)$, $i = 1, \ldots, k$, for any sufficiently small $c > 0$ we can construct a form $\widetilde{\theta}_c$ on $\widetilde{M}_{I,x}$ with properties 1-4. Properties 1 and 2 imply that $\widetilde{\theta}_c$ is a symplectic form satisfying also property 5. This finishes the proof. ∎

Proof of Proposition 5.7:

Let $\alpha := [\omega] \in H^2(M; \mathbb{R})$.

Consider a triple $(I, x, h) \in Triples_{\mathcal{C}_0}$, where $x = (x_1, \ldots, x_k)$. By Definition 5.6, the cohomology class $\widetilde{\alpha}_{\widetilde{r}} \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to I. Hence, by Proposition 5.13, there exists a symplectic form

$\widetilde{\theta}$ on $\widetilde{M}_{I,x}$ which is compatible with $\widetilde{I}$, represents $\widetilde{\alpha}_{\widetilde{r}}$, and equals to $\rho_{x_i}(1, r_i)$ on a neighborhood of E_i for each $i = 1, \ldots, k$. Consequently, Proposition 5.15 implies that for any sufficiently small $\kappa > 0$ there exists a symplectic form $\eta_{\kappa, \widetilde{\theta}}$ on M with the following properties:

- $[\eta_{\kappa, \widetilde{\theta}}] = \alpha$,

- $\eta_{\kappa, \widetilde{\theta}}$ is compatible with I,

- $h \circ R_\kappa : \bigsqcup_{i=1}^k B^{2n}(r_i) \to M$ is symplectic with respect to $\eta_{\kappa, \widetilde{\theta}}$ (and holomorphic with respect to I),

- $\eta_{\kappa, \widetilde{\theta}}$ depends smoothly on κ.

By Definition 5.6, there exists $\phi \in \mathrm{Diff}_0(M)$ such that $\phi^* I$ is compatible with ω. Then $\phi^* \eta_{\kappa, \widetilde{\theta}}$ and ω are cohomologous symplectic forms compatible with $\phi^* I$. The linear family of 2-forms connecting $\phi^* \eta_{\kappa, \widetilde{\theta}}$ and ω is then a family of cohomologous symplectic forms. Hence, by Moser's theorem [Mos], $\phi^* \eta_{\kappa, \widetilde{\theta}}$ lies in the $\mathrm{Diff}_0(M)$-orbit of ω, and consequently so does $\eta_{\kappa, \widetilde{\theta}}$. Thus, for any sufficiently small $\kappa > 0$ we have

$$(\eta_{\kappa, \widetilde{\theta}}, h \circ R_\kappa) \in Pairs_{\mathcal{C}_0}.$$

Define the map

$$\Phi : \mathtt{Triples}_{\mathcal{C}_0} \to \mathtt{Pairs}_{\mathcal{C}_0}$$

by

$$\Phi(\{I, x, h\}) := \left\{ \eta_{\kappa, \widetilde{\theta}}, h \circ R_\kappa \right\}.$$

We need to show that the map Φ is well-defined and has the required properties.

Φ is well-defined:
    Let us first show that $\left\{ \eta_{\kappa, \widetilde{\theta}}, h \circ R_\kappa \right\}$ does not depend on the choice of a sufficiently small $\kappa > 0$ (for fixed $\widetilde{\theta}$). Given such $0 < \kappa_1 < \kappa_2$, we have a smooth family of pairs $(\eta_{\kappa, \widetilde{\theta}}, h \circ R_\kappa)$, $\kappa \in [\kappa_1, \kappa_2]$, lying in $Pairs_{\mathcal{C}_0}$. By Proposition 5.5, this family lies in the same $\mathrm{Diff}_0(M)$-orbit on $Pairs_{\mathcal{C}_0}$. Thus, $\left\{ \eta_{\kappa, \widetilde{\theta}}, h \circ R_\kappa \right\}$ does not depend on the choice of κ.

    Let us verify that $\left\{ \eta_{\kappa, \widetilde{\theta}}, h \circ R_\kappa \right\}$ does not depend on the choice of $\widetilde{\theta}$. Assume that $\widetilde{\theta}'$ is another symplectic form which is compatible with $\widetilde{I}$, represents $\widetilde{\alpha}_{\widetilde{r}}$, and equals $\rho_{x_i}(1, r_i)$ on a neighborhood of E_i for each $i = 1, \ldots, k$. Pick a sufficiently small $\kappa > 0$ so that $\widetilde{\theta} = \widetilde{\theta}' = \rho_{x_i}(c_i, r_i)$ on $\Pi^{-1}\left(h_i\left(B^{2n}(\kappa(r_i + \varepsilon)) \right) \right)$. Then $\eta_{\kappa, \widetilde{\theta}}$ and $\eta_{\kappa, \widetilde{\theta}'}$ are cohomologous symplectic forms compatible with ω and coinciding on $h_i\left(B^{2n}(\kappa(r_i + \varepsilon)) \right)$. By a relative version of Moser's theorem (see Proposition A.3, part I), there exists $\phi \in \mathrm{Diff}_0(M)$ that fixes $h_i\left(B^{2n}(\kappa(r_i + \varepsilon)) \right)$ pointwise (thus mapping $h \circ R_\kappa$ to itself) and maps $\eta_{\kappa, \widetilde{\theta}}$ to $\eta_{\kappa, \widetilde{\theta}'}$. Thus, $\left\{ \eta_{\kappa, \widetilde{\theta}}, h \circ R_\kappa \right\}$ does not depend on the choice of $\widetilde{\theta}$.

    Finally, a direct check shows that $\Phi(\{I, x, h\})$ does not depend on a representative (I, x, h) of $\{I, x, h\} \in \mathtt{Triples}_{\mathcal{C}_0}$: acting by $\phi \in \mathrm{Diff}_0(M)$ on (I, x, h) and applying the previous construction to the resulting element of $Triples_{\mathcal{C}_0}$ we get a pair in $Pairs_{\mathcal{C}_0}$ obtained from $\left(\eta_{\kappa, \widetilde{\theta}}, h \circ R_\kappa \right)$ by the action of the same ϕ.

(Here we identify the complex blow-ups of (M,I) at $x_1, \ldots, x_k$ and of $(M, \phi_* I)$ at $\phi(x_1), \ldots, \phi(x_k)$ by the diffeomorphism induced by ϕ.)

Thus, we have shown that Φ is well-defined.

Φ is continuous:

By Proposition 5.3, the space $\mathtt{Pairs}_{\mathcal{C}_0}$ is discrete. Thus, showing that $\Phi \colon \mathtt{Triples}_{\mathcal{C}_0} \to \mathtt{Pairs}_{\mathcal{C}_0}$ is continuous is equivalent to showing that the preimage of a point in $\mathtt{Pairs}_{\mathcal{C}_0}$ under Φ is open in $\mathtt{Triples}_{\mathcal{C}_0}$, or, in other words, that for any $(I', x', h') \in \mathit{Triples}_{\mathcal{C}_0}$ sufficiently close to (I, x, h) its $\mathrm{Diff}_0(M)$-orbit $\{(I', x', h')\} \in \mathtt{Triples}_{\mathcal{C}_0}$ is mapped by Φ to the same point in $\mathtt{Pairs}_{\mathcal{C}_0}$ as $\{(I, x, h)\}$.

Given $(I', x', h') \in \mathit{Triples}_{\mathcal{C}_0}$, one easily constructs $\phi \in \mathrm{Diff}_0(M)$ so that

- ϕ sends x' into x,

- $\phi \circ h'$ coincides with h near each x_i, $i = 1, \ldots, k$.

If h and h', and x and x', are sufficiently C^∞-close, then ϕ can be chosen to be close to the identity and therefore it will send (I', x', h') into a triple close to (I, x, h). Thus, without loss of generality we may replace (I', x', h') by a triple (I', x, h') that is close to (I, x, h) and satisfies $h \circ R_\kappa = h' \circ R_\kappa$ for any sufficiently small $\kappa > 0$. In particular, for any such κ the complex structures I and I' coincide on each $h_i \left(B^{2n} \left(\kappa (r_i + \varepsilon) \right) \right)$, $i = 1, \ldots, k$.

Arguing as above, we can construct a symplectic form $\widetilde{\theta}$ on $\widetilde{M}_{I,x} = \widetilde{M}_{I',x}$ which is compatible with $\widetilde{I'}$, represents $\widetilde{\alpha}_{\bar{r}}$, and for any sufficiently small $\kappa > 0$ equals to $\rho_{x_i}(1, r_i)$ on the neighborhood $\Pi^{-1} \left(h_i \left(B^{2n} \left(\kappa (r_i + \varepsilon) \right) \right) \right)$ of E_i for each $i = 1, \ldots, k$. Consequently, for any sufficiently small $\kappa > 0$ we have a well-defined symplectic form $\eta_{\kappa, \widetilde{\theta}'}$ on M. This form is compatible with I', represents the class α and coincides with $\eta_{\kappa, \widetilde{\theta}}$ on each $h_i \left(B^{2n} \left(\kappa (r_i + \varepsilon) \right) \right)$, $i = 1, \ldots, k$.

Assume that I' is sufficiently close to I so that it is tamed by the form $\eta_{\kappa, \widetilde{\theta}}$ (which is compatible with I). Then $\eta_{\kappa, \widetilde{\theta}}$ and $\eta_{\kappa, \widetilde{\theta}'}$ are cohomologous symplectic forms on M that both tame I' and coincide on each $h_i \left(B^{2n} \left(\kappa (r_i + \varepsilon) \right) \right)$, $i = 1, \ldots, k$. A relative version of Moser's theorem (see Proposition A.3, part I) implies then that there exists $\varphi \in \mathrm{Diff}_0(M)$ that maps $\eta_{\kappa, \widetilde{\theta}'}$ into $\eta_{\kappa, \widetilde{\theta}}$ and fixes pointwise each $h_i \left(B^{2n} \left(\kappa (r_i + \varepsilon) \right) \right)$, $i = 1, \ldots, k$. Hence, φ maps $\left(\eta_{\kappa, \widetilde{\theta}'}, h' \circ R_\kappa \right) \in \mathit{Pairs}_{\mathcal{C}_0}$ into $\left(\eta_{\kappa, \widetilde{\theta}}, h \circ R_\kappa \right)$.

Thus, for any $(I', x', h') \in \mathit{Triples}_{\mathcal{C}_0}$ sufficiently close to (I, x, h), its $\mathrm{Diff}_0(M)$-orbit $\{I', x', h'\} \in \mathtt{Triples}_{\mathcal{C}_0}$ is mapped by Φ to the same point in $\mathtt{Pairs}_{\mathcal{C}_0}$ as $\{I, x, h\} \in \mathtt{Triples}_{\mathcal{C}_0}$. This finishes the proof of the claim that Φ is continuous.

Φ is surjective:

The proof uses the well-known fact (see e.g. [McDP]) that the symplectic blow-down is an "inverse" of the symplectic blow-up, up to an isotopy.

Namely, let $(\eta, f) \in \mathit{Pairs}_{\mathcal{C}_0}$, where $f = \bigsqcup_{i=1}^{k} f_i \colon \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \eta)$ is a symplectic embedding holomorphic with respect to a complex structure I on M compatible with η. Let us show that $\{\eta, f\} \in \mathtt{Pairs}_{\mathcal{C}_0}$ lies in the image of Φ.

Set

$$x_i := f_i(0), i = 1, \ldots, k,$$

$$x := (x_1, \ldots, x_k).$$

By Proposition 5.16, the cohomology class $\widetilde{\alpha}_{\bar{r}} \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to $\widetilde{I}$. Thus, by Definition 5.6, $(I, x, f) \in \textit{Triples}_{\mathfrak{C}_0}$. Moreover, by Proposition 5.16, the class $\widetilde{\alpha}_{\bar{r}}$ can be represented, for a sufficiently small $c > 0$, by a symplectic form $\widetilde{\theta}_c$ on $\widetilde{M}_{I,x}$ with the following properties:

- $\widetilde{\theta}_c = \rho_{x_i}(c, r_i)$ on $\Pi^{-1}(f_i(B(r_i)))$, $i = 1, \ldots, k$,

- $[\widetilde{\theta}_c] = \Pi^*[\eta] - \pi \sum_{i=1}^k r_i^2 e_i$,

- $\widetilde{\theta}_c$ is compatible with $\widetilde{I}$.

Consequently, Proposition 5.15 implies that for $\kappa = 1$ there exists a symplectic form $\eta_1(\widetilde{\theta}_c)$ on M with the following properties:

- $[\eta_1(\widetilde{\theta}_c)] = \alpha$,

- $\eta_1(\widetilde{\theta}_c)$ is compatible with I,

- The $f \circ R_1 = f : \bigsqcup_{i=1}^k B^{2n}(r_i) \to M$ is symplectic with respect to $\eta_1(\widetilde{\theta}_c)$ (and holomorphic with respect to I).

This yields that

1. $\widetilde{\alpha}_{\bar{r}} \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to $\widetilde{I}$. Thus, by Definition 5.6, $(I, x, f) \in \textit{Triples}_{\mathfrak{C}_0}$.

2. $\eta_1(\widetilde{\theta}_c)$ coincides with η on each $f_i(B^{2n}(\kappa r_i))$, $i = 1, \ldots, k$.

A relative version of Moser's theorem (see Proposition A.3, part I) implies then that there exists $\psi \in \text{Diff}_0(M)$ that maps $\eta_1(\widetilde{\theta}_c)$ into η and fixes pointwise each each $f_i(B^{2n}(\kappa r_i))$, $i = 1, \ldots, k$. Hence,

$$\psi\left(\eta_1\left(\widetilde{\theta}_c\right), f\right) = (\eta, f).$$

Thus,

$$\Phi(\{I, x, f\}) = \left\{\eta_1\left(\widetilde{\theta}_c\right), f\right\} = \{\eta, f\} \in \texttt{Pairs}_{\mathfrak{C}_0}.$$

This shows that $\{\eta, f\} \in \texttt{Pairs}_{\mathfrak{C}_0}$ lies in the image of Φ. Hence, Φ is surjective.

We have shown that Φ is well-defined, continuous and surjective. By construction, it clearly satisfies the properties 1 and 2 listed in the statement of Proposition 5.7. This finishes the proof of Proposition 5.7. ∎

The tools developed above allow us also to prove the following existence result for tame embeddings of balls, which is an analogue of part (I) of Theorem 3.2.

Proposition 5.17

The set $\bigsqcup_{i=1}^k B^{2n}(r_i)$ admits a tame embedding into (M, ω) if and only if for some $I \in CK(M, \omega)$, tamed by ω, and some $x \in \hat{M}^k$, the class $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$, can be represented by a form taming $\widetilde{I}$ and equal to $\rho_{x_i}(c_i, r_i)$, for some $c_i > 0$, near each E_i, $i = 1, \ldots, k$.

More precisely, given $I \in CK(M, \omega)$, tamed by ω, the following conditions are equivalent:

- There exists $x = (x_1, \ldots, x_k) \in \hat{M}^k$ so that the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ can be represented by a form taming $\widetilde{I}$ and equal to $\rho_{x_i}(c_i, r_i)$, for some $c_i > 0$, near each E_i, $i = 1, \ldots, k$.

- There exist a tame embedding $f : \bigsqcup_{i=1}^k B^{2n}(r_i) \to (M, \omega)$ and $\phi \in \mathrm{Diff}_0(M)$, so that f is holomorphic with respect to the complex structure $\phi^* I$ on M tamed by ω.

Proof of Proposition 5.17:

Assume that for some $I \in CK(M, \omega)$, tamed by ω, and some $x \in \hat{M}^k$ the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ can be represented by a form taming $\widetilde{I}$ and equal to $\rho_{x_i}(c_i, r_i)$, for some $c_i > 0$, near each $E_i = \Pi^{-1}(x_i)$, $i = 1, \ldots, k$. Then one can deduce from Proposition 5.15 that there exist a symplectic form η on M, cohomologous to ω and taming I, and an embedding $g : \bigsqcup_{i=1}^k B^{2n}(r_i) \to M$ symplectic with respect to η and holomorphic with respect to I. The linear family of closed 2-forms connecting ω and η is formed by cohomologous forms each of which tames I and is therefore symplectic. Therefore, by Moser's theorem [Mos], $\omega = \phi^* \eta$ for some $\phi \in \mathrm{Diff}_0(M)$. The embedding $f := \phi^{-1} \circ g : \bigsqcup_{i=1}^k B^{2n}(r_i) \to (M, \omega)$ is symplectic with respect to ω and holomorphic with respect to the complex structure $\phi^* I$ on M tamed by ω, meaning that f is tame. This proves one of the implications in the required equivalence.

Let us now prove the opposite implication.

Assume that $f = \bigsqcup_{i=1}^k f_i : \bigsqcup_{i=1}^k B^{2n}(r_i) \to (M, \omega)$ is a symplectic embedding holomorphic with respect to the complex structure $J := \phi^* I$, $\phi \in \mathrm{Diff}_0(M)$, tamed by ω. Set $y_i := f_i(0)$, $i = 1, \ldots, k$, and $y := (y_1, \ldots, y_k)$.

By Proposition 5.16, there exists a symplectic form $\widetilde{\theta}$ on $\widetilde{M}_{J,y}$ with the following properties:

1. $\widetilde{\theta} = \rho_{y_i}(c, r_i)$, for a sufficiently small $c > 0$, on a neighborhood of the exceptional divisor $\Pi^{-1}(y_i) \subset \widetilde{M}_{J,y}$, $i = 1, \ldots, k$,

2. $\widetilde{\theta}$ tames $\widetilde{J}$,

3. $[\widetilde{\theta}] = \Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{J,y}; \mathbb{R})$.

Set $x_i := \phi(y_i)$, $i = 1, \ldots, k$, and $x := (x_1, \ldots, x_k)$. The diffeomorphism $\phi : M \to M$ induces a diffeomorphism $\widetilde{M}_{J,y} \to \widetilde{M}_{I,x}$ identifying the form $\widetilde{\theta}$ on $\widetilde{M}_{J,y}$ with a form on $\widetilde{M}_{I,x}$ taming $\widetilde{I}$, equal to $\rho_{x_i}(c, r_i)$ near each $E_i = \Pi^{-1}(x_i) \subset \widetilde{M}_{I,x}$, $i = 1, \ldots, k$, and lying in the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$.

This finishes the proof of the opposite implication and of the proposition. $\blacksquare$

5.6 Connectedness of $\mathtt{Triples}_{\mathcal{C}_0}$

Let I be a Kähler-type complex structure on M. Equip $\hat{M}^k$ and $\hat{M}^k \times M$ with the product complex structures induced by I.

Denote by $\Xi \subset \hat{M}^k \times M$ the following incidence variety:

$$\Xi := \left\{ ((x_1, \ldots, x_k), x) \in \hat{M}^k \times M \mid x = x_i \text{ for some } i = 1, \ldots, k \right\}.$$

It is a smooth complex submanifold of $\hat{M}^k \times M$. Denote by $\mathcal{M}$ the complex blow-up of $\hat{M}^k \times M$ along Ξ. It is a complex manifold that projects holomorphically to $\hat{M}^k \times M$ and consequently to $\hat{M}^k$. Denote the latter projection by $\pi : \mathcal{M} \to \hat{M}^k$. This projection defines a complex analytic family of complex manifolds equipped with Kähler-type complex structures.

The fiber of π over each $x = (x_1, \ldots, x_k) \in \hat{M}^k$ can be canonically identified with the complex blow-up $\widetilde{M}_{I,x}$ of (M, I) at $x_1, \ldots, x_k$. The holonomy of the Gauss-Manin connection on π is trivial and the homology/cohomology groups of all the fibers of π are then identified with the homology/cohomology groups of the complex blow-up of M at k distinct ordered points (the identification does not depend on the choice of the points, since the homology/cohomology groups of the complex blow-ups for different choices can be canonically identified). The identifications preserve the Hodge decompositions and, in particular, cohomology classes of type $(1,1)$ of one fiber are identified with the $(1,1)$-classes of another fiber.

Proposition 5.18

Assume that β is a $(1,1)$-cohomology class of the complex blow-up of (M, I) at k points.
Denote by $\mathcal{B}_\beta \subset \hat{M}^k$ the set of all $x \in \hat{M}^k$ such that $\beta \in H_{\tilde{I}}^{1,1}(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler.
Then $\mathcal{B}_\beta$ is connected.

Proof of Proposition 5.18:

If $\mathcal{B}_\beta$ is empty, the claim is trivial. Assume it is non-empty. Since $\beta \in H_{\tilde{I}}^{1,1}(\widetilde{M}_{I,x}; \mathbb{R})$ for all $x \in \hat{M}^k$, the Kodaira-Spencer stability theorem implies (see Proposition A.2) that the set $\mathcal{B}_\beta$ is open.

A theorem of Demailly-Paun [DP, Thm. 0.9] about the Kähler cones of the fibers of a complex analytic deformation family implies in our situation that there exists a countable union $\bigcup_\nu S_\nu$ of proper analytic subsets $S_\nu \subsetneq \hat{M}^k$ such that the Kähler cones of the fibers $\pi^{-1}(x)$ coincide for all $x \in \hat{M}^k \setminus \bigcup_\nu S_\nu$ under our identifications of the cohomologies of the fibers.

Since the set $\mathcal{B}_\beta \subset \hat{M}^k$ is open, it intersects $\hat{M}^k \setminus \bigcup_\nu S_\nu$. Therefore $\beta \in H_{\tilde{I}}^{1,1}(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler for all $x \in \hat{M}^k \setminus \bigcup_\nu S_\nu$, meaning that $\hat{M}^k \setminus \bigcup_\nu S_\nu \subset \mathcal{B}_\beta$. Being the complement of a countable union of proper analytic subsets of a connected complex manifold, the set $\hat{M}^k \setminus \bigcup_\nu S_\nu$ is connected and dense in $\hat{M}^k$. Hence, the set $\mathcal{B}_\beta$ is connected too. ∎

Corollary 5.19

The fibers of the projection $pr : \mathcal{K}(\bar{r}) \to CK(M)$ are connected.

Proof of Corollary 5.19:

For each $I \in \mathrm{Diff}_0(M, \omega) \cdot Cmpt(M, \omega)$ and each $x \in \hat{M}^k$ the cohomology class $\widetilde{\alpha}_{\bar{r}} = \Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is of type $(1,1)$. The image of pr lies in $\mathrm{Diff}_0(M, \omega) \cdot Cmpt(M, \omega)$ and the fiber of pr over $I \in \mathrm{Im}\,pr$ is the set of all $x \in \hat{M}^k$ such that the $(1,1)$-class $\widetilde{\alpha}_{\bar{r}}$ is Kähler. By Proposition 5.18, applied to $\beta = \widetilde{\alpha}_{\bar{r}}$, this fiber is connected. ∎

Proposition 5.20

Assume I is a complex structure on M, $x \in M$, $r > 0$.

Then the space of holomorphic embeddings $h : B^{2n}(r) \to (M, I)$, such that $h(0) = x$, is path-connected.

Proof of Proposition 5.20:

Assume $h : B^{2n}(r) \to (M, I)$, $h(0) = x$, is a holomorphic embedding. Let $g : B^{2n}(r) \to (M, I)$, $g(0) = x$, be another holomorphic embedding. We want to show that g can be connected to h by a smooth path of holomorphic embeddings $B^{2n}(r) \to (M, I)$ mapping 0 to x.

We can assume without loss of generality that $\mathrm{Im}\, g \subset \mathrm{Im}\, h$. Indeed, for a sufficiently small $\kappa > 0$ we have $\mathrm{Im}(g \circ R_\kappa) \subset \mathrm{Im}\, h$ and $g = g \circ R_1$ can be connected to $g \circ R_\kappa$ by the family $\{g \circ R_t\}_{t \in [\kappa, 1]}$.

Assuming $\mathrm{Im}\, g \subset \mathrm{Im}\, h$, in order to connect g to h by a smooth path of holomorphic embeddings $B^{2n}(r) \to (M, I)$ mapping 0 to x it suffices to show that the holomorphic embedding $f := h^{-1} \circ g : B^{2n}(r) \to B^{2n}(r)$, $h^{-1} \circ g(0) = 0$, can be connected to the identity by a smooth path of holomorphic embeddings $B^{2n}(r) \to B^{2n}(r)$ mapping 0 to 0. The latter result is standard and is proved by a version of the Alexander trick – see Proposition A.4, part II.

This finishes the proof of the proposition. ∎

Proof of Proposition 5.8:

The space $\mathtt{Triples}_{\mathcal{C}_0} = Triples_{\mathcal{C}_0}/\mathrm{Diff}_0(M)$ is connected if only if the space $Triples_{\mathcal{C}_0}$ is connected.

The projection $(I, x, h) \mapsto (I, x)$ defines a surjective continuous map $Triples_{\mathcal{C}_0} \to \mathcal{K}_{\mathcal{C}_0}(\bar{r})$. In view of Proposition 5.20, the fibers of this map are connected. Hence, $Triples_{\mathcal{C}_0}$ is connected if and only $\mathcal{K}_{\mathcal{C}_0}(\bar{r})$ is connected. By Corollary 5.19, the fibers of the projection $pr : \mathcal{K}_{\mathcal{C}_0}(\bar{r}) \to \mathcal{C}_0$ are connected. Thus, $\mathcal{K}_{\mathcal{C}_0}(\bar{r})$ is connected if and only if $pr(\mathcal{K}_{\mathcal{C}_0}(\bar{r}))$ is connected. The set $pr(\mathcal{K}_{\mathcal{C}_0}(\bar{r}))$ is connected if and only if the set $\mathtt{K}_{\mathcal{C}_0}(\bar{r}) = pr(\mathcal{K}_{\mathcal{C}_0}(\bar{r}))/\mathrm{Diff}_0(M)$ is connected. Summing up, we see that $\mathtt{Triples}_{\mathcal{C}_0}$ is connected if and only if $\mathtt{K}_{\mathcal{C}_0}(\bar{r})$ is connected. This finishes the proof of the proposition. ∎

6 The case of complex projective spaces and their products

As above, we denote by I_{st} the standard complex structure on $\mathbb{C}P^n$ and by ω_{FS}, or by $\omega_{FS,n}$, the standard Fubini-Study symplectic form on $\mathbb{C}P^n$ normalized so that $\int_{\mathbb{C}P^1} \omega_{FS} = \pi$. Let $h \in H^2(\mathbb{C}P^n; \mathbb{Z})$ be the generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$ so that $[\omega_{FS}] = \pi h$.

Let us prove Theorem 3.5. For convenience, we restate it here.

Theorem 6.1 (= Theorem 3.5)

Consider the manifold $M := \mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_m}$, $n_1, \ldots, n_m > 0$, $n_1 + \ldots + n_m =: n$, endowed with the symplectic form $\omega_c := c_1 \omega_{FS,n_1} \oplus \ldots \oplus c_m \omega_{FS,n_m}$, $c := (c_1, \ldots, c_m) \in (\mathbb{R}_{>0})^m$. Let I be the complex structure on M which is the product of the standard complex structures on the factors.

Let $l_1, \ldots, l_m \in \mathbb{Z}_{>0}$ so that $[l_1 : \ldots : l_m] = [c_1 : \ldots : c_m] \in \mathbb{R}P^m$. Let

$$k := \frac{(n_1 + \ldots n_m)!}{n_1! \cdot \ldots \cdot n_m!} l_1^{n_1} \cdot \ldots \cdot l_m^{n_m}.$$

Let $\Sigma \subset (M, I)$ be a proper (possibly empty) complex submanifold. If $M = \mathbb{C}P^{n_1} \times \mathbb{C}P^1$, we also allow $\Sigma = (\mathbb{C}P^{n_1-1} \times \mathbb{C}P^1) \cup (\mathbb{C}P^{n_1} \times \mathrm{pt})$.

Then Kähler-type embeddings of disjoint unions of k equal balls into $(M \setminus \Sigma, \omega_c)$ are unobstructed.

More precisely, if $k\mathrm{Vol}(B^{2n}(r), \omega_0) < \mathrm{Vol}(M, \omega)$, then for each proper (possibly empty) complex submanifold $\Sigma \subset (M, I)$, there exists an $[I]$-Kähler-type embedding of $\bigsqcup_{i=1}^k B^{2n}(r)$ into (M, ω) which is holomorphic with respect to a complex structure on M that is compatible with ω and isotopic to I by an isotopy preserving Σ (as a set).

Proof of Theorem 6.1 (= Theorem 3.5):

The proof mimics the proof of [McDP, Thm. 1.5.A].

Namely, as it is shown in [McDP, Pf. of Thm. 1.5.A], there exist $x_1, \ldots, x_k \in M$, $x := (x_1, \ldots, x_k) \in \hat{M}^k$, and a holomorphic map $F : \widetilde{M}_{I,x} \to (\mathbb{C}P^{n-1}, I_{st})$ so that

- F induces a biholomorphism on each exceptional divisor,

- $\Pi^*[\omega] - F^*[\omega_{FS,n-1}] \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is a linear combination of the cohomology classes $e_1, \ldots, e_k$ Poincaré-dual to the fundamental homology classes of the exceptional divisors of $\widetilde{M}_{I,x}$. Here $\Pi : \widetilde{M}_{I,x} \to M$ is the blow-up projection, as before.

Let $c > 0$. A direct check using the above-mentioned properties of F shows that the form $\widetilde{\omega}_c := (1+c)^{-1}(\Pi^*\omega + cF^*\omega_{FS,n-1})$ is a symplectic form on $\widetilde{M}_{I,x}$ compatible with $\widetilde{I}$ and representing the cohomology class $\Pi^*[\omega] - \pi \sum_{i=1}^k (1+c)^{-1} ce_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$. Therefore, by part (I) of Theorem 3.2 (=Theorem 5.9), for $r_c := \sqrt{(1+c)^{-1}c}$, there exists a symplectic embedding $\bigcup_{i=1}^k B(r_c) \to (M \setminus \Sigma, \omega)$ and an isotopy $\{\phi_t\}_{0 \le t \le 1} \subset \mathrm{Diff}_0(M)$, $\phi_0 = Id$, $\phi_t(\Sigma) = \Sigma$ for all $t \in [0, 1]$, so that f is holomorphic with respect to the complex structure $\phi_1^* I$ which is compatible with ω. As it is shown in [McDP, Pf. of Thm. 1.5.A], as $c \to +\infty$, the symplectic volume of $\bigcup_{i=1}^k B(r_c)$ tends from below to the symplectic volume of (M, ω). This readily yields the claim of the theorem. ∎

Let us know consider the case $M = \mathbb{C}P^n$.

The following result is classical – see Remark 6.3 for the references.

Theorem 6.2

(I) Any Kähler-type complex structure on $\mathbb{C}P^n$ can be identified with I_{st} by a diffeomorphism lying in $\mathrm{Diff}_H(\mathbb{C}P^n)$.

(II) Any two Kähler-type complex structures on $\mathbb{C}P^n$ lying in the same component of $CK(\mathbb{C}P^n)$ can be mapped into each other by an element of $\mathrm{Diff}_0(\mathbb{C}P^n)$. ∎

Remark 6.3

1. Part (I) of Theorem 6.2 is due to Hirzebruch and Kodaira [HiK] for odd n and to Yau [Yau1] for even n – see [Tos] for an exposition and a proof. (The result in [Tos] does not state explicitly that the diffeomorphism in part (I) of the theorem acts trivially on homology but this can be easily deduced from the proof there.)

2. It is not known whether each complex structure on $\mathbb{C}P^n$, $n \geq 3$, is of Kähler type. For $n = 1$ this is obviously true. For $n = 2$ this is true by a deep result of Yau [Yau1].

3. Part (II) of the theorem implies that $\mathrm{Diff}_H(\mathbb{C}P^n) = \mathrm{Diff}_0(\mathbb{C}P^n)$ if and only if $CK(\mathbb{C}P^n)$ is connected. It is known that $\mathrm{Diff}_H(\mathbb{C}P^n) \neq \mathrm{Diff}_0(\mathbb{C}P^n)$ for $n = 3$ [KrS, Thm. 1.2] and $n = 4$ [Bru, Rem. II.11]. Thus, $CK(\mathbb{C}P^n)$, $n = 3, 4$, has more than one connected component. We do not know whether the same facts hold for $n = 2$ and $n > 4$.

4. Part (II) of Theorem 6.2 follows from a theorem of Fröhlicher and Nijenhuis [FroN], cf. [KoS1, Thm. 6.3].

Corollary 6.4

Each connected component of the space $\mathtt{Teich}(\mathbb{C}P^n)$ is a point. The $\mathrm{Diff}_H(\mathbb{C}P^n)$-action on $\mathtt{Teich}(\mathbb{C}P^n)$ is transitive. ∎

Corollary 6.5

The group $\mathrm{Symp}_H(\mathbb{C}P^n, \omega_{FS}) \subset \mathrm{Diff}_H(\mathbb{C}P^n)$ acts transitively on the set of connected components of $\mathtt{Teich}(\mathbb{C}P^n)$ compatible with ω_{FS}.

Proof of Corollary 6.5:

Assume J is a complex structure on $\mathbb{C}P^n$ compatible with ω_{FS}. Then, by part (I) of Theorem 6.2, there exists $\phi \in \mathrm{Diff}_H(\mathbb{C}P^n)$ such that $\phi^* J = I_{st}$. Then $\phi^* \omega_{FS}$ is a symplectic form compatible with I_{st} and cohomologous to ω_{FS}. Therefore the straight path in the space of 2-forms connecting these forms is formed by cohomologous symplectic forms, and consequently, by Moser's theorem [Mos], there exists $\psi \in \mathrm{Diff}_0(\mathbb{C}P^n)$ such that $\psi^* \phi^* \omega_{FS} = \omega_{FS}$. Then $\phi \psi \in \mathrm{Symp}_H(\mathbb{C}P^n, \omega_{FS})$ and $(\phi \psi)^* J = \psi^* I_{st}$ lies in the $\mathrm{Diff}_0(\mathbb{C}P^n)$-orbit of I_{st}. This means that the connected component of $\mathtt{Teich}(\mathbb{C}P^n)$ containing $[J]$ is mapped under the action of $\phi \psi \in \mathrm{Symp}_H(\mathbb{C}P^n, \omega_{FS})$ into the connected component of $\mathtt{Teich}(\mathbb{C}P^n)$ containing $[I_{st}]$. This yields the proposition. ∎

We are ready to prove Theorem 3.7 – let us first recall it here.

Theorem 6.6 (=Theorem 3.7)

A. For each $l \in \mathbb{Z}_{>0}$, Kähler-type embeddings of disjoint unions of l^n equal balls into $(\mathbb{C}P^n, \omega_{FS})$ are unobstructed.

More precisely, if $l^n \text{Vol}(B^{2n}(r), \omega_0) < \text{Vol}(\mathbb{C}P^n, \omega_{FS})$, then for any complex structure J on $\mathbb{C}P^n$ compatible with ω_{FS} (and, in particular, for the standard complex structure I_{st}) and for each proper (possibly empty) complex submanifold $\Sigma \subset (\mathbb{C}P^n, J)$, there exists a $[J]$-Kähler-type embedding of $\bigsqcup_{i=1}^{l^n} B^{2n}(r)$ into $(\mathbb{C}P^n, \omega_{FS})$ which is holomorphic with respect to a complex structure on $\mathbb{C}P^n$ that is compatible with ω_{FS} and isotopic to J by an isotopy preserving Σ (as a set).

B. The group $\text{Symp}_H(\mathbb{C}P^n, \omega_{FS})$ acts transitively on the set of connected components of $CK(\mathbb{C}P^n)$ compatible with ω_{FS}.

C. For any $k \in \mathbb{Z}_{>0}$ and $r_1, \ldots, r_k > 0$, any two Kähler-type embeddings $\bigsqcup_{i=1}^k B^{2n}(r_i) \to (\mathbb{C}P^n, \omega_{FS})$ (if they exist!) lie in the same orbit of the $\text{Symp}_H(\mathbb{C}P^n, \omega_{FS})$-action.

They lie in the same orbit of the $\text{Symp}(\mathbb{C}P^n, \omega_{FS}) \cap \text{Diff}_0(\mathbb{C}P^n)$-action if and only if they favor a common connected component of $CK(\mathbb{C}P^n)$. In the latter case there exists $[I] \in \texttt{Teich}(\mathbb{C}P^n, \omega_{FS})$ such that both embeddings are of $[I]$-Kähler type.

Proof of Theorem 6.6 (=Theorem 3.7):

As we already explained in Section 3.2, we only need to prove parts B and C.

Part B is exactly Corollary 6.5 that we have just proved.

Let us prove part C. By Corollary 6.4, for each connected component $\mathcal{C}_0$ of $CK(\mathbb{C}P^n)$ the corresponding connected component $\texttt{Teich}_{\mathcal{C}_0}(\mathbb{C}P^n)$ of $\texttt{Teich}(\mathbb{C}P^n)$ is just a point. Consequently, for any $\bar{r}$ the set $K_{\mathcal{C}_0}(\bar{r}) \subset \texttt{Teich}_{\mathcal{C}_0}(\mathbb{C}P^n)$ is connected. Now applying Theorem 3.2 along with part B of this theorem we get part C.

This finishes the proof. ∎

Now consider the case of $\mathbb{C}P^2$. Similarly to our previous notation, we write $(\widehat{\mathbb{C}P^2})^k$ for the set of all ordered k-tuples of pairwise distinct points in $\mathbb{C}P^2$. As before, given $x = (x_1, \ldots, x_k) \in (\widehat{\mathbb{C}P^2})^k$, the manifold $\widetilde{\mathbb{C}P^2}_{I_{st}, x}$ is the complex blow-up of $(\mathbb{C}P^2, I_{st})$ at $x_1, \ldots, x_k$. Given $\bar{r} = (r_1, \ldots, r_k) \in (\mathbb{R}_{>0})^k$ and $J \in CK(\mathbb{C}P^2)$, set $\alpha := [\omega]$ and

$$\widetilde{\alpha}_{\bar{r}} := \Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{\mathbb{C}P^2}_{J,x}; \mathbb{R}).$$

Definition 6.7

Assume that $k \in \mathbb{Z}$, $1 \leq k \leq 8$.

We say that k points in $\mathbb{C}P^2$ are in **general position** if

(i) no three points are on a line;

(ii) no six points are on a conic;

(iii) no cubic passes through the points with one of the points being a singular point.

Proposition 6.8

Let $1 \leq k \leq 8$.

Then the set of tuples $x = (x_1, \ldots, x_k) \in (\widehat{\mathbb{C}P^2})^k$ such that $x_1, \ldots, x_k$ are in general position is connected and dense in $(\widehat{\mathbb{C}P^2})^k$.

Proof of Proposition 6.8:

Consider the product complex structure on $(\widehat{\mathbb{C}P^2})^k \subset (\mathbb{C}P^2)^k$. In view of Definition 6.7, the k-tuples $(x_1, \ldots, x_k) \in (\widehat{\mathbb{C}P^2})^k$ such that $x_1, \ldots, x_k$ are *not* in general position lie in a proper analytic subset of $(\widehat{\mathbb{C}P^2})^k$. This means that the set of k-tuples $(x_1, \ldots, x_k) \in (\widehat{\mathbb{C}P^2})^k$ such that $x_1, \ldots, x_k$ *are* in general position contains the complement of a proper analytic subset of $(\widehat{\mathbb{C}P^2})^k$. Thus, it is connected (because $(\widehat{\mathbb{C}P^2})^k$ itself is connected) and dense (and open) in $(\widehat{\mathbb{C}P^2})^k$. ∎

Proposition 6.9

Assume that $1 \leq k \leq 8$. Let $\bar{r} = (r_1, \ldots, r_k) \in (\mathbb{R}_{>0})^k$, $r_1 \geq r_2 \geq \ldots \geq r_k$. Let $\mathcal{C}_0$ be the connected component of $CK(\mathbb{C}P^2)$ containing I_{st}.

Then for any $x = (x_1, \ldots, x_k) \in (\widehat{\mathbb{C}P^2})^k$ such that $x_1, \ldots, x_k \in \mathbb{C}P^2$ are in general position, the class $\widetilde{\alpha}_{\bar{r}} \in H^2(\widehat{\mathbb{C}P^2}_{I_{st},x}; \mathbb{R})$ is Kähler with respect to $\widetilde{I}_{st}$ if and only if the numbers $r_1, \ldots, r_k$ satisfy the following inequalities (listed in [McDP, Cor. 1.3G]):

(v) $\sum_{i=1}^{k} r_i^4 < 1$,

(c1) $r_1^2 + r_2^2 < 1$, if $2 \leq k \leq 8$,

(c2) $r_1^2 + \ldots + r_5^2 < 2$, if $5 \leq k \leq 8$,

(c3) $2r_1^2 + \sum_{i=2}^{7} r_i^2 < 3$, if $7 \leq k \leq 8$,

(c4) $2r_1^2 + 2r_2^2 + 2r_3^2 + r_4^2 + \ldots + r_8^2 < 4$, if $k = 8$,

(c5) $2\sum_{i=1}^{6} r_i^2 + r_7^2 + r_8^2 < 5$, if $k = 8$,

(c6) $3r_1^2 + 2\sum_{i=2}^{8} r_i^2 < 6$, if $k = 8$.

Proof of Proposition 6.9:

If $1 \leq k \leq 8$ and the points $x_1, \ldots, x_k \in \mathbb{C}P^2$ are in general position (such points exist by Proposition 6.8), then $(\widehat{\mathbb{C}P^2}_{I_{st},x}, \widetilde{I}_{st})$, for $x = (x_1, \ldots, x_k)$, is a del Pezzo surface (see e.g. [Dol, Thm. 8.1.25]). As such, it is a good generic surface in the sense of [FriM, Def. 2.1] (see [FriM, Rem. 2.9]). The Kähler cone of $(\widehat{\mathbb{C}P^2}_{I_{st},x}, \widetilde{I}_{st})$ is the cone of classes in $H^2(\widehat{\mathbb{C}P^2}_{I_{st},x}; \mathbb{R})$ that have a positive square and evaluate positively on the homology classes of holomorphically embedded 2-spheres with self-intersection index equal to -1 and on the class $3[\mathbb{C}P^1] - \sum_{i=1}^{k} [E_i] \in H_2(\widehat{\mathbb{C}P^2}_{I_{st},x}; \mathbb{Z})$ [FriM]. Together with the information on the homology classes of holomorphically embedded 2-spheres with self-intersection index equal to -1 given in [Dmz], this yields that $\widetilde{\alpha}_{\bar{r}}$ is Kähler with respect to $\widetilde{I}_{st}$ if and only if the numbers $r_1, \ldots, r_k$ satisfy the inequalities (v), (c1)-(c6) – see [McDP, Cor. 1.3G].

This finishes the proof of the proposition. ∎

Let us recall the following result of Gromov.

Proposition 6.10 [Gro]

The group $\text{Symp}(\mathbb{C}P^2, \omega_{FS})$ is homotopy equivalent to $PU(3)$ and, in particular, connected.

Recall Proposition 3.12:

Proposition 6.11 (=Proposition 3.12)

There is only one connected component of $CK(\mathbb{C}P^2)$ compatible with ω_{FS} – it is the connected component of $CK(\mathbb{C}P^2)$ containing I_{st}. Any two complex structures in that connected component are isotopic.

Proof of Proposition 6.11 (= Proposition 3.12):

The proposition follows immediately from Corollary 6.5, Proposition 6.10. and part (II) of Theorem 6.2. ∎

Now we are ready to prove Theorem 3.15 – let us recall it here.

Theorem 6.12 (=Theorem 3.15)

A. Assume that $1 \leq k \leq 8$ and $r_1 \geq r_2 \geq \ldots \geq r_k > 0$.

Then for any proper (possibly empty) complex submanifold $\Sigma \subset (\mathbb{C}P^2, I_{st})$, any symplectic embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ is, in fact, of $[I_{st}]$-Kähler type: it is holomorphic with respect to a complex structure on $\mathbb{C}P^2$ that is compatible with ω_{FS} and isotopic to I_{st} by an isotopy fixing Σ (as a set).

Accordingly, by part A of Theorem 3.14, such a Kähler-type embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ exists if and only if the radii $r_1, \ldots, r_k$ satisfy the inequalities (v), (c1)-(c6) listed in part A of Theorem 3.14.

B. For any proper (possibly empty) complex submanifold $\Sigma \subset (\mathbb{C}P^2, I_{st})$ and any $k = l^2$, $l \in \mathbb{Z}_{>0}$, any symplectic embedding $\bigsqcup_{i=1}^{k} B^4(r) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ is, in fact, of $[I_{st}]$-Kähler type: it is holomorphic with respect to a complex structure on $\mathbb{C}P^2$ that is compatible with ω_{FS} and isotopic to I_{st} by an isotopy preserving Σ (as a set).

Accordingly, by part B of Theorem 3.14, such Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(r) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ are unobstructed: they exist if and only if $k\text{Vol}(B^4(r), \omega_0) < \text{Vol}(\mathbb{C}P^2, \omega_{FS})$.

C. Let $\Sigma \subset (\mathbb{C}P^2, \omega_{FS})$ be either of the following:

(1) the empty set;

(2) a finite union of closed compact symplectic submanifolds (without boundary) of real dimension 2 whose pairwise intersections (if they exist) are transverse and ω–orthogonal;

(3) a Lagrangian submanifold which is diffeomorphic to either $\mathbb{S}^2$ or $\mathbb{R}P^2$.

For any $k \in \mathbb{Z}_{>0}$ and $r_1, \ldots, r_k > 0$, any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ (if they exist!) lie in the same orbit of the $\text{Symp}_0^c(\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$-action, meaning that the space of Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2 \setminus \Sigma, \omega_{FS})$ is connected.

Proof of Theorem 6.12 (=Theorem 3.15):

As we already explained in Section 3.2, part B of Theorem 3.15 follows from Theorem 3.7 and from part (II) of Theorem 3.4, while part C of Theorem 3.15 follows directly from part (I) of Theorem 3.4.

Let us prove part A.

Assume $f : \bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2, \omega_{FS})$ is a symplectic embedding. Then, by Theorem 3.14, the radii $r_1, \ldots, r_k$ satisfy the conditions (v), (c1)-(c6). Consequently, Proposition 6.9 together with part (I) of Theorem 3.2 (=Theorem 5.9) yields that there exists an $[I_{st}]$-Kähler-type embedding $f' : \bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2, \omega_{FS})$. By part (II) of Theorem 3.4, this implies that f is of $[I_{st}]$-Kähler type too. This finishes the proof of part A and of the theorem. ∎

7 The case of $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$

Let $M = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$. As a smooth manifold, M will be identified with the complex blow-up $\widetilde{\mathbb{C}P}^2_{I_{st},x}$ of $(\mathbb{C}P^2, I_{st})$ at a point $x \in \mathbb{C}P^2$. Let J_1 be the complex structure on M identified in this way with the blow-up complex structure $\widetilde{I}_{st}$ on $\widetilde{\mathbb{C}P}^2_{I_{st},x}$.

Denote by $E \subset M = \widetilde{\mathbb{C}P}^2_{I_{st},x}$ the exceptional divisor. Let $[\mathbb{C}P^1] \in H_2(M;\mathbb{Z})$ be the homology class of a projective line $\mathbb{C}P^1 \subset M$. Denote by $h, e \in H^2(M;\mathbb{Z})$ the cohomology classes Poincaré-dual to the homology classes of $[\mathbb{C}P^1]$ and $[E]$.

Given $0 < \lambda < \pi$, assume that ω_λ is a Kähler form on the complex manifold (M, J_1) such that $\langle [\omega_\lambda], [\mathbb{C}P^1] \rangle = \pi$, $\langle [\omega_\lambda], [E] \rangle = \lambda$. Thus, $[\omega_\lambda] = \pi h - \lambda e$.

Further on in this section we fix

$$2 \leq k \leq 8,$$

$$R_1 \geq R_2 \geq \ldots \geq R_{k-1} > 0,$$

and set

$$r_1 := R_1 \geq \ldots \geq r_{i-1} := R_{i-1} \geq r_i := \sqrt{\lambda/\pi} \geq$$

$$\geq r_{i+1} := R_i \geq \ldots \geq r_k := R_{k-1}.$$

For each $(x_1, \ldots, x_{k-1}) \in \hat{M}^{k-1}$ and each complex structure I on M,

- $\widetilde{M}_{I,x_1,\ldots,x_{k-1}}$ will denote the complex blow-up of (M, I) at $x_1, \ldots, x_{k-1}$.

- $\widetilde{I}$ will denote the lift of I to $\widetilde{M}_{I,x_1,\ldots,x_{k-1}}$.

- $\Pi_M : \widetilde{M}_{I,x_1,\ldots,x_{k-1}} \to M$ denotes the natural projection.

- $e_{1,M}, \ldots, e_{k-1,M} \in H^2(\widetilde{M}_{I,x_1,\ldots,x_{k-1}};\mathbb{R})$ are the cohomology classes Poincaré-dual to the fundamental homology classes of the exceptional divisors.

We say that $k-1$ points $x_1, \ldots, x_{k-1} \in M$ are in **general position** if $x_1, \ldots, x_{k-1} \in M \setminus E \cong \mathbb{C}P^2 \setminus x$ and the k points $x_1, \ldots, x_{k-1}, x$ are in general position as points of $\mathbb{C}P^2$ (see Definition 6.7).

Proposition 7.1

The set of tuples $(x_1, \ldots, x_{k-1}) \in \hat{M}^{k-1}$ such that $x_1, \ldots, x_{k-1}$ are in general position is connected and dense in $\hat{M}^{k-1}$.

Proof of Proposition 7.1:

Arguing as in Proposition 6.8, we get that the set of $(k-1)$-tuples $(x_1, \ldots, x_{k-1}) \in \hat{M}^{k-1}$ such that $x_1, \ldots, x_k$ are in general position contains the complement of a proper analytic subset of $\hat{M}^{k-1}$. Thus, it is connected (since $\hat{M}^{k-1}$ is connected) and dense (and open) in $\hat{M}^{k-1}$. ∎

Proposition 7.2

Let $x_1, \ldots, x_{k-1} \in M$ be in general position.
Then the cohomology class

$$\Pi_M^*[\omega_\lambda] - \pi \sum_{i=1}^{k-1} R_i^2 e_{i,M} \in H^2(\widetilde{M}_{J_1, x_1, \ldots, x_{k-1}}; \mathbb{R})$$

is Kähler with respect to $\widetilde{J}_1$ if and only if the numbers $r_1, \ldots, r_k$ satisfy the inequalities (v), (c1)-(c6) listed in Proposition 6.9.

Proof of Proposition 7.2:

Since the points $x_1, \ldots, x_{k-1}$ lie in $M \setminus E \cong \mathbb{C}P^2 \setminus x$, we can view them as points of $\mathbb{C}P^2$. Denote by $\Pi_{\mathbb{C}P^2} : \widetilde{\mathbb{C}P^2}_{I_{st}, x_1, \ldots, x_{k-1}, x} \to \mathbb{C}P^2$ the blow-up projection and by $e_{1,\mathbb{C}P^2}, \ldots, e_{k-1,\mathbb{C}P^2} \in H^2(\widetilde{\mathbb{C}P^2}_{I_{st}, x_1, \ldots, x_{k-1}, x}; \mathbb{R})$ the cohomology classes Poincaré-dual to the fundamental homology classes of the exceptional divisors.

There exists a natural biholomorphism between $(\widetilde{M}_{J_1, x_1, \ldots, x_{k-1}}, \widetilde{J}_1)$ and $(\widetilde{\mathbb{C}P^2}_{I_{st}, x_1, \ldots, x_{k-1}, x}, \widetilde{I}_{st})$. Since $[\omega_\lambda] = \pi h - \lambda e$, this biholomorphism identifies the cohomology class

$$\Pi_M^*[\omega_\lambda] - \pi \sum_{i=1}^{k-1} R_i^2 e_{i,M} \in H^2(\widetilde{M}_{J_1, x_1, \ldots, x_{k-1}}; \mathbb{R})$$

with the cohomology class

$$\Pi_{\mathbb{C}P^2}^*(\pi h) - \lambda e - \pi \sum_{i=1}^{k-1} R_i^2 e_{i,\mathbb{C}P^2} \in H^2(\widetilde{\mathbb{C}P^2}_{I_{st}, x_1, \ldots, x_{k-1}, x}; \mathbb{R}).$$

(Here $e \in H^2(\widetilde{M}_{J_1, x_1, \ldots, x_{k-1}}; \mathbb{R})$ is identified with the cohomology class in $\widetilde{\mathbb{C}P^2}_{I_{st}, x_1, \ldots, x_{k-1}, x}$ associated to the exceptional fiber over x). Now the proposition follows from Proposition 6.9. ∎

Now we are ready to prove Theorem 3.18 – let us recall it here.

Theorem 7.3 (=Theorem 3.18)

Let $M = \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ and let ω_λ be a Kähler-type form on M as above.

Then the following claims hold:

A. Let $2 \leq k \leq 8$, $R_1 \geq R_2 \geq \ldots \geq R_{k-1} > 0$. Assume that $r_1 := R_1 \geq \ldots \geq r_{i-1} := R_{i-1} \geq r_i := \sqrt{\lambda/\pi} \geq r_{i+1} := R_i \geq \ldots \geq r_k := R_{k-1}$.

Then for any proper (possibly empty) complex submanifold $\Sigma \subset (M, J_1)$, the following conditions are equivalent:

- There exists a $[J_1]$-Kähler-type embedding $\bigsqcup_{i=1}^{k-1} B^4(R_i) \to (M \setminus \Sigma, \omega_\lambda)$ holomorphic with respect to a complex structure on M that is compatible with ω_λ and isotopic to J_1 by an isotopy preserving Σ (as a set).

- The numbers $r_1, \ldots, r_k$ satisfy the inequalities (v), (c1)-(c6) in part A of Theorem 3.14.

For $r_1, \ldots, r_k$ satisfying the inequalities (v), (c1)-(c6) in part A of Theorem 3.14, any symplectic embedding $\bigsqcup_{i=1}^{k-1} B^4(R_i) \to (M \setminus \Sigma, \omega_\lambda)$ is, in fact, of $[J_1]$-Kähler type.

B. For any $k \in \mathbb{Z}_{>0}$ and $R_1, \ldots, R_k > 0$, any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(R_i) \to (M, \omega_\lambda)$ (if they exist!) lie in the same orbit of the $\mathrm{Symp}_0(M, \omega_\lambda)$-action, meaning that the space of Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(R_i) \to (M, \omega_\lambda)$ is connected.

Proof of Theorem 7.3 (=Theorem 3.18):

As we already explained in Section 3.3, part B follows from part (I) of Theorem 3.4.

Let us prove part A.

Assume that the numbers $r_1, \ldots, r_k$ satisfy the conditions (v), (c1)-(c6). Then, by Proposition 7.2, combined with part (I) of Theorem 3.2, the union $\bigsqcup_{i=1}^{k-1} B^4(r_i)$ admits a Kähler-type embedding into (M, ω_λ) which is holomorphic with respect to a complex structure on M isotopic to J_1 and compatible with ω_λ, meaning that the embedding is of $[J_1]$-Kähler type. By part (II) of Theorem 3.4, this implies that any symplectic embedding $\bigsqcup_{i=1}^{k-1} B^4(r_i) \to (M, \omega_\lambda)$ is of $[J_1]$-Kähler type.

Conversely, assume there exists a Kähler-type embedding $f = \bigsqcup_{i=1}^{k-1} f_i : \bigsqcup_{i=1}^{k-1} B^4(r_i) \to (M, \omega_\lambda)$ which is holomorphic with respect to a complex structure I compatible with ω_λ and such that $I = \phi^* J_1$ for some $\phi \in \mathrm{Diff}_0(M)$. Set $x_i := f_i(0)$, $i = 1, \ldots, k-1$, and $x := (x_1, \ldots, x_{k-1})$, $\phi(x) := (\phi(x_1), \ldots, \phi(x_{k-1}))$.

Recall that a Kähler-type embedding of a closed ball is defined as a Kähler-type embedding of a larger open ball. Therefore using Proposition 6.8 and composing f, if necessary, with a small parallel translation of the balls in $\mathbb{R}^{2n}$, we may assume without loss of generality that the points $\phi(x_1), \ldots, \phi(x_{k-1}) \in M$ are in general position.

By part (I) of Theorem 3.2, the cohomology class $\widetilde{\alpha}_{\vec{r}} \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to $\widetilde{I}$. The diffeomorphism $\phi : M \to M$ induces a diffeomorphism $\widetilde{M}_{I,x} \to \widetilde{M}_{J_1, \phi(x)}$ that identifies $\widetilde{I}$ with $\widetilde{J}_1$ as well as the classes $\widetilde{\alpha}_{\vec{r}}$ in $H^2(\widetilde{M}_{I,x}; \mathbb{R})$ and in $H^2(\widetilde{M}_{J_1, \phi(x)}; \mathbb{R})$ (abusing the notation, we use the same letter for both classes). Therefore $\widetilde{\alpha}_{\vec{r}} \in H^2(\widetilde{M}_{J_1, \phi(x)}; \mathbb{R})$ is Kähler with respect to $\widetilde{J}_1$. Since $\phi(x_1), \ldots, \phi(x_{k-1})$ are in general position, Proposition 7.2 yields that the conditions (v), (c1)-(c6) are satisfied. This proves part A and finishes the proof of the theorem. ∎

8   The case of $\mathbb{C}P^1 \times \mathbb{C}P^1$

Let $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ and J_0 the standard product complex structure on M.

Let J_{2l}, $l \in \mathbb{Z}_{\geq 0}$, be a complex structure on M such that (M, J_{2l}) is biholomorphic to the Hirzebruch surface $\mathbb{F}_{2l}$. In particular, we assume that J_0 is the standard product complex structure on M. Let $b, f \in H^2(M; \mathbb{Z})$ be the cohomology classes Poincaré-dual, respectively, to $[\mathbb{C}P^1 \times pt]$ and $[F] = [pt \times \mathbb{C}P^1]$.

Let ω_0 be the Fubini-Study form on $\mathbb{C}P^1$ normalized so that $\int_{\mathbb{C}P^1} \omega_0 = \pi$. For each $\mu \geq 1$ define the symplectic form ω_μ on M as

$$\omega_\mu := \mu \omega_0 \oplus \omega_0.$$

Thus, $[\omega_\mu] = \pi(\mu b + f)$.

Further on in this section, we fix

$$2 \leq k \leq 8,$$

$$R_1, R_2, \ldots, R_{k-2} > 0, 0 < R_{k-1} < 1,$$

and define $r_1 \geq r_2 \geq \ldots \geq r_k$ as the numbers

$$\frac{R_1^2}{\mu + 1 - R_1^2}, \frac{R_2^2}{\mu + 1 - R_1^2}, \ldots, \frac{R_{k-2}^2}{\mu + 1 - R_1^2}, \frac{\mu - R_1^2}{\mu + 1 - R_1^2}, \frac{1 - R_1^2}{\mu + 1 - R_1^2},$$

sorted in the non-increasing order.

Abusing the notation, for a blow-up of M (respectively, of $\mathbb{C}P^2$) at any finite collection of points $x_1, x_2, \ldots$ (respectively, $y_1, y_2, \ldots$) we will use the same letter Π_M (respectively, $\Pi_{\mathbb{C}P^2}$) for the blow-up projection and the same letters $e_{1,M}, e_{2,M}, \ldots$ (respectively, $e_{1,\mathbb{C}P^2}, e_{2,\mathbb{C}P^2}, \ldots$) for the cohomology classes Poincaré-dual to the homology classes of the exceptional divisors. For a complex structure I on M we will denote by $\widetilde{I}$ the lift of I to any complex blow-up of (M, I), and similarly we will denote by $\widetilde{I}_{st}$ the lift of the standard complex structure I_{st} on $\mathbb{C}P^2$ to any complex blow-up of $\mathbb{C}P^2$.

Let $x_1, \ldots, x_{k-1} \in M$ be distinct points in M.

Pick $y_{k-1}, y_k \in \mathbb{C}P^2$. Consider the complex blow-ups $\widetilde{M}_{J_0, x_{k-1}}$ of (M, J_0) at x_{k-1} and $(\widetilde{\mathbb{C}P^2}_{I_{st}, y_{k-1}, y_k}, \widetilde{I}_{st})$ of $(\mathbb{C}P^2, I_{st})$ at y_{k-1}, y_k.

There exists a biholomorphism $\Psi : (\widetilde{M}_{J_0, x_{k-1}}, \widetilde{J}_0) \to (\widetilde{\mathbb{C}P^2}_{I_{st}, y_{k-1}, y_k}, \widetilde{I}_{st})$ such that the induced isomorphism

$$\Psi^* : H^2(\widetilde{\mathbb{C}P^2}_{I_{st}, y_{k-1}, y_k}; \mathbb{Z}) \to H^2(\widetilde{M}_{J_0, x_{k-1}}; \mathbb{Z})$$

identifies

$$\Pi_M^* b, \Pi_M^* f, e_{k-1, M} \in H^2(\widetilde{M}_{J_0, x_{k-1}}; \mathbb{Z})$$

respectively with

$$\Pi_{\mathbb{C}P^2}^* h - e_{k-1, \mathbb{C}P^2}, \Pi_{\mathbb{C}P^2}^* h - e_{k, \mathbb{C}P^2}, \Pi_{\mathbb{C}P^2}^* h - e_{k-1, \mathbb{C}P^2} - e_{k, \mathbb{C}P^2} \in H^2(\widetilde{\mathbb{C}P^2}_{I_{st}, y_{k-1}, y_k}; \mathbb{Z}).$$

Identify the complement of the exceptional divisor in $\widetilde{M}_{J_0, x_{k-1}}$ with $M \setminus x_{k-1}$ and the complement of the union of the exceptional divisors in $\widetilde{\mathbb{C}P^2}_{I_{st}, y_{k-1}, y_k}$ with $\mathbb{C}P^2 \setminus \{y_{k-1}, y_k\}$. The restriction of Ψ to the complements defines then a map $\psi : M \setminus x_{k-1} \to \mathbb{C}P^2 \setminus \{y_{k-1}, y_k\}$.

We say that the points $x_1, \ldots, x_{k-1} \in M \setminus x_{k-2}$ are **in general position** if the points $\psi(x_1), \ldots, \psi(x_{k-2})$, $y_{k-1}, y_k \in \mathbb{C}P^2$ are distinct and in general position as k points of $\mathbb{C}P^2$. It is easy to see that this definition does not depend on the choices of y_1, y_2 and Ψ.

Arguing as in the proofs of Proposition 6.8 and Proposition 7.1, and using the fact that any two points of M can be mapped into each other by a biholomorphism of (M, J_0), we get the following claim.

Proposition 8.1

The set of tuples $(x_1, \ldots, x_{k-2}, x_{k-1}) \in \hat{M}^{k-1}$ that are in general position is connected and dense in $\hat{M}^{k-1}$. ∎

Proposition 8.2

Assume $x_1, \ldots, x_{k-1} \in M$ are in general position.

Then the cohomology class

$$\Pi_M^*[\omega_\mu] - \pi \sum_{i=1}^{k-1} R_i^2 e_{i,M} \in H^2(\widetilde{M}_{J_0, x_1, \ldots, x_{k-1}}; \mathbb{R})$$

is Kähler with respect to $\widetilde{J}_0$ if and only if the numbers $r_1, \ldots, r_k$ satisfy the inequalities (v), (c1)-(c6) listed in Proposition 6.9.

Proof of Proposition 8.2:

The biholomorphism $\Psi: (\widetilde{M}_{J_0, x_{k-1}}, \widetilde{J}_0) \to (\widetilde{\mathbb{C}P^2}_{I_{st}, y_{k-1}, y_k}, \widetilde{I}_{st})$ induces a biholomorphism

$$\Upsilon: \left(\widetilde{M}_{J_0, x_1, \ldots, x_{k-1}}, \widetilde{J}_0 \right) \to \left(\widetilde{\mathbb{C}P^2}_{I_{st}, \psi(x_1), \ldots, \psi(x_{k-2}), y_{k-1}, y_k}, \widetilde{I}_{st} \right)$$

that identifies the cohomology class

$$\Pi_M^*[\omega_\mu] - \pi \sum_{i=1}^{k-1} R_i^2 e_{i,M} = \pi \left(\mu \Pi_M^* b + \Pi_M^* f - \sum_{i=1}^{k-1} R_i^2 e_{i,M} \right) \in H^2(\widetilde{M}_{J_0, x_1, \ldots, x_{k-1}}; \mathbb{R})$$

with

$$\pi \cdot \left(\mu \left(\Pi_{\mathbb{C}P^2}^* h - e_{k-1, \mathbb{C}P^2} \right) + \left(\Pi_{\mathbb{C}P^2}^* h - e_{k, \mathbb{C}P^2} \right) - \right.$$

$$\left. -R_{k-1}^2 \left(\Pi_{\mathbb{C}P^2}^* h - e_{k-1, \mathbb{C}P^2} - e_{k, \mathbb{C}P^2} \right) - \sum_{i=1}^{k-2} R_i^2 e_{i, \mathbb{C}P^2} \right) \in H^2 \left(\widetilde{\mathbb{C}P^2}_{I_{st}, \psi(x_1), \ldots, \psi(x_{k-2}), y_{k-1}, y_k}; \mathbb{R} \right).$$

Recalling the description of $\Psi^*: H^2(\widetilde{\mathbb{C}P^2}_{I_{st}, y_{k-1}, y_k}; \mathbb{Z}) \to H^2(\widetilde{M}_{J_0, x_{k-1}}; \mathbb{Z})$ above and the fact that $[\omega_{FS}] = \pi h \in H^2(\mathbb{C}P^2; \mathbb{R})$, we get that

$$\pi \cdot \left(\mu \left(\Pi_{\mathbb{C}P^2}^* h - e_{k-1, \mathbb{C}P^2} \right) + \left(\Pi_{\mathbb{C}P^2}^* h - e_{k, \mathbb{C}P^2} \right) - \right.$$

$$\left. -R_{k-1}^2 \left(\Pi_{\mathbb{C}P^2}^* h - e_{k-1, \mathbb{C}P^2} - e_{k, \mathbb{C}P^2} \right) - \sum_{i=1}^{k-2} R_i^2 e_{i, \mathbb{C}P^2} \right) =$$

$$\pi \cdot \left(\left(\mu + 1 - R_{k-1}^2 \right) \Pi_{\mathbb{C}P^2}^* h - \sum_{i=1}^{k-2} R_i^2 e_{i,\mathbb{C}P^2} - \right.$$

$$\left. - \left(\mu - R_{k-1}^2 \right) e_{k-1,\mathbb{C}P^2} - (1 - R_{k-1}^2) e_{k,\mathbb{C}P^2} \right) =$$

$$= \left(\mu + 1 - R_{k-1}^2 \right) \left(\Pi_{\mathbb{C}P^2}^*(\pi h) - \pi \cdot \sum_{i=1}^{k-2} \frac{R_i^2}{\mu + 1 - R_{k-1}^2} e_{i,\mathbb{C}P^2} - \right.$$

$$\left. - \frac{\mu - R_{k-1}^2}{\mu + 1 - R_{k-1}^2} e_{k-1,\mathbb{C}P^2} - \frac{1 - R_{k-1}^2}{\mu + 1 - R_{k-1}^2} e_{k,\mathbb{C}P^2} \right) =$$

$$= \left(\mu + 1 - R_{k-1}^2 \right) \left(\Pi_{\mathbb{C}P^2}^*[\omega_{FS}] - \pi \sum_{i=1}^{k-2} \frac{R_i^2}{\mu + 1 - R_{k-1}^2} e_{i,\mathbb{C}P^2} - \right.$$

$$\left. - \frac{\mu - R_{k-1}^2}{\mu + 1 - R_{k-1}^2} e_{k-1,\mathbb{C}P^2} - \frac{1 - R_{k-1}^2}{\mu + 1 - R_{k-1}^2} e_{k,\mathbb{C}P^2} \right)$$

Thus, the cohomology class

$$\Pi_M^*[\omega_\mu] - \pi \sum_{i=1}^{k-1} R_i^2 e_{i,M} = \pi \cdot \left(\mu \Pi_M^* b + \Pi_M^* f - \sum_{i=1}^{k-1} R_i^2 e_{i,M} \right) \in H^2(\widetilde{M}_{J_0,x_1,\ldots,x_{k-1}}; \mathbb{R})$$

is Kähler with respect to $\widetilde{J}_0$ if and only if the cohomology class

$$\Pi_{\mathbb{C}P^2}^*[\omega_{FS}] - \pi \sum_{i=1}^{k-2} \frac{R_i^2}{\mu + 1 - R_{k-1}^2} e_{i,\mathbb{C}P^2} - \frac{\mu - R_{k-1}^2}{\mu + 1 - R_{k-1}^2} e_{k-1,\mathbb{C}P^2} -$$

$$- \frac{1 - R_{k-1}^2}{\mu + 1 - R_{k-1}^2} e_{k,\mathbb{C}P^2} \in H^2\left(\widetilde{\mathbb{C}P^2}_{I_{st},\psi(x_1),\ldots,\psi(x_{k-2}),y_{k-1},y_k}; \mathbb{R} \right)$$

is Kähler with respect to $\widetilde{I}_{st}$. Since the numbers

$$\frac{R_1^2}{\mu + 1 - R_{k-1}^2}, \ldots, \frac{R_{k-2}^2}{\mu + 1 - R_{k-1}^2}, \frac{\mu - R_{k-1}^2}{\mu + 1 - R_{k-1}^2}, \frac{1 - R_{k-1}^2}{\mu + 1 - R_{k-1}^2},$$

sorted in non-increasing order, are exactly $r_1, \ldots, r_k$, and the points $\psi(x_1), \ldots, \psi(x_{k-2}), y_{k-1}, y_k \in \mathbb{C}P^2$ are in general position, the proposition follows from Proposition 6.9. ∎

Now we are ready to prove Theorem 3.20 – let us recall it here.

Theorem 8.3 (=Theorem 3.20)

Let $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ and let ω_μ be as above.

Let $\Sigma \subset (M, J_0)$ be either a proper (possibly empty) complex submanifold or $\Sigma = (\text{pt} \times \mathbb{C}P^1) \cup (\mathbb{C}P^1 \times \text{pt})$.

Then the following claims hold:

A. Assume $2 \le k \le 8$, $R_1, R_2, \ldots, R_{k-2} > 0$, $0 < R_{k-1} < 1$. Consider the numbers

$$\frac{R_1^2}{\mu + 1 - R_{k-1}^2}, \frac{R_2^2}{\mu + 1 - R_{k-1}^2}, \ldots, \frac{\mu - R_{k-1}^2}{\mu + 1 - R_{k-1}^2}, \frac{1 - R_{k-1}^2}{\mu + 1 - R_{k-1}^2},$$

sort them in the non-increasing order and denote the resulting k numbers by $r_1 \ge \ldots \ge r_k$.

Then the following conditions are equivalent:

- There exists a $[J_0]$-Kähler-type embedding $\bigsqcup_{i=1}^{k-1} B^4(R_i) \to (M \setminus \Sigma, \omega_\mu)$ holomorphic with respect to a complex structure on M that is compatible with ω_μ and isotopic to J_0 by an isotopy preserving Σ (as a set).

- The numbers $r_1, \ldots, r_k$ defined above satisfy the inequalities in part A of Theorem 3.14.

For $r_1, \ldots, r_k$ satisfying the inequalities (v), (c1)-(c6) in part A of Theorem 3.14, any symplectic embedding $\bigsqcup_{i=1}^{k-1} B^4(R_i) \to (M \setminus \Sigma, \omega_\mu)$ is, in fact, of $[J_0]$-Kähler type.

B. Assume that $\mu = 1$.

Then for any $l \in \mathbb{Z}_{>0}$, any symplectic embedding of a disjoint union of $2l^2$ equal balls into $(M \setminus \Sigma, \omega_1)$ is, in fact, of $[J_0]$-Kähler type, and such symplectic (or, equivalently, $[J_0]$-Kähler-type) embeddings are unobstructed.

More precisely, if $2l^2 \mathrm{Vol}(B^4(r), \omega_0) < \mathrm{Vol}(M, \omega_1)$, then there exists a Kähler-type embedding of $\bigsqcup_{i=1}^{2l^2} B^4(r)$ into $(M \setminus \Sigma, \omega_1)$ which is holomorphic with respect to a complex structure on M that is compatible with ω_1 and isotopic to J_0 by an isotopy preserving Σ (as a set).

C. For any $k \in \mathbb{Z}_{>0}$ and $R_1, \ldots, R_k > 0$, any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(R_i) \to (M \setminus \Sigma, \omega_\mu)$ (if they exist!) lie in the same orbit of the $\mathrm{Symp}_0^c(M \setminus \Sigma, \omega_\mu)$-action, meaning that the space of Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(R_i) \to (M \setminus \Sigma, \omega_\mu)$ is connected.

Proof of Theorem 8.3 (=Theorem 3.20):

As we already explained in Section 3.3, part B of the theorem follows from Theorem 3.7 and part C follows from part (I) of Theorem 3.4.

The proof of part A follows the proof of part A of Theorem 7.3, with Proposition 8.2 used instead of Proposition 7.2. ∎

9 Existence of Kähler-type embeddings of balls and Campana-simple complex structures

In this section we discuss Kähler-type embeddings of balls and other domains into symplectic manifolds admitting Kähler-type complex structures with "few" complex subvarieties. The relevant claims about

such complex structures on tori, K3 surfaces and IHS-hyperkähler manifolds will be discussed in Sections 10 and 12. The results of Sections 3.4, 3.5 on the existence of Kähler-type embeddings of disjoint unions of balls into tori, K3 surfaces and IHS-hyperkähler manifolds will then follow as particular cases of the general result (Theorem 9.3) stated further on in this section.

As before, let M, $\dim_{\mathbb{R}} = 2n$, be a closed connected manifold equipped with a Kähler-type symplectic form ω.

Let us recall the definition of a Campana-simple complex structure.

Assume I is a Kähler-type complex structure on M. Let $\mathfrak{U}$ be the union of all positive-dimensional proper complex subvarieties of (M,I). The set $\mathfrak{U}$ is either a countable union of proper analytic subvarieties of M (and hence has a dense connected complement) or the whole M (see e.g. [EV1, Remark 4.2]).

If $\mathfrak{U}$ is a countable union of proper analytic subvarieties of M, the complex structure I is called **Campana-simple** and the points of $M \setminus \mathfrak{U}$ are called **Campana-generic with respect to I**.

Examples of manifolds with a Campana-simple complex structure include the generic complex tori of complex dimension > 1 (see Proposition 10.10), as well as generic K3 surfaces (see Proposition 10.20), and generic deformations of hyperkähler manifolds [Ver4]. One can show that a torus equipped with a Campana-simple complex structure does not admit *any* positive-dimensional proper complex subvarieties (see Proposition 10.4), and the same holds for a generic complex K3 surface [Ver5].

Remark 9.1

1. If I is Campana-simple, then the complex manifold (M,I) is not projective, since otherwise it would have admitted a globally defined meromorphic function f and could have been represented as the union of zero divisors of the functions $f - a$, for all $a \in \mathbb{C}$ and the zero divisor of f^{-1}. Consequently, the symplectic form ω compatible with a Campana-simple complex structure cannot be rational – otherwise, for a Campana-simple complex structure I compatible with ω the complex manifold (M,I) would have been projective, by the Kodaira embedding theorem [Ko].

2. Conjecturally (see [Cam, Question 1.4], [CaDV, Conj. 1.1]), any closed connected manifold equipped with a Campana-simple complex structure is bimeromorphic to a hyperkähler orbifold or a finite quotient of a torus. For complex threefolds this was proved in [HöP].

Campana-simple complex structures on M form a $\mathrm{Diff}(M)$-invariant subset of the set of all Kähler-type complex structures on M.

Definition 9.2

Denote by $\mathrm{Teich}^s(M) \subset \mathrm{Teich}(M)$ the set of points of $\mathrm{Teich}(M)$ that can be represented by Campana-simple complex structures.

If $\mathcal{C}_0$ is a connected component of $CK(M)$, denote by $\mathrm{Teich}^s_{\mathcal{C}_0}(M) \subset \mathrm{Teich}_{\mathcal{C}_0}(M)$ the set of points of $\mathrm{Teich}_{\mathcal{C}_0}(M)$ that can be represented by Campana-simple complex structures. In other words,

$$\mathrm{Teich}^s_{\mathcal{C}_0}(M) = \mathrm{Teich}^s(M) \cap \mathrm{Teich}_{\mathcal{C}_0}(M).$$

Theorem 9.3

Let $\mathcal{C}_0$ be a connected component of $CK(M)$ compatible with ω. Let $k \in \mathbb{Z}_{>0}$, $\bar{r} = (r_1, \ldots, r_k) \in (\mathbb{R}_{>0})^k$.

Assume $\mathrm{Vol}\left(\bigsqcup_{i=1}^k B^{2n}(r_i), \omega_0 \right) < \mathrm{Vol}\,(M, \omega)$.

Then the following claims hold:

(I) Let M, $\dim_{\mathbb{R}} M = 2n$, be a closed manifold. Assume that ω is a symplectic form on M compatible with a Campana-simple complex structure I on M.

Then Kähler-type embeddings $\bigsqcup_{i=1}^k B^{2n}(r_i) \to (M, \omega)$ are unobstructed.

More precisely, if $\mathrm{Vol}\left(\bigsqcup_{i=1}^k B^{2n}(r_i), \omega_0 \right) < \mathrm{Vol}\,(M, \omega)$, then $\bigsqcup_{i=1}^k B^{2n}(r_i)$ admits an $[I]$-Kähler-type embedding into (M, ω).

(II) Assume that $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega) \cap \mathtt{Teich}_{\mathcal{C}_0}^s(M)$ is a dense connected subset of $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$.

Then any two Kähler-type embeddings $\bigsqcup_{i=1}^k B^{2n}(r_i) \to (M, \omega)$ favoring $\mathcal{C}_0$ lie in the same orbit of the $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$-action. In particular, there exists $[I] \in \mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$ such that both embeddings are of $[I]$-Kähler-type.

If, in addition, $\mathrm{Symp}_H(M)$ acts transitively on the set of connected components of $CK(M)$ compatible with ω, then any two Kähler-type embeddings $\bigsqcup_{i=1}^k B^{2n}(r_i) \to (M, \omega)$ lie in the same orbit of the $\mathrm{Symp}_H(M, \omega)$-action.

For the proof of Theorem 9.3 we need the following claim.

Proposition 9.4

Assume I is a Campana-simple complex structure on M, a cohomology class $\alpha \in H^2(M; \mathbb{R})$ is Kähler with respect to I, and the points $x_1, \ldots, x_k \in M$ are Campana-generic with respect to I. Set $x := (x_1, \ldots, x_k)$.

Then the cohomology class

$$\widetilde{\alpha}_{\bar{r}} := \Pi^* \alpha - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$$

is Kähler with respect to $\widetilde{I}$ if and only if $\widetilde{\alpha}_{\bar{r}}^n > 0$, or, equivalently, if and only if

$$\mathrm{Vol}\left(\bigsqcup_{i=1}^k B^{2n}(r_i), \omega_0 \right) < \langle \alpha^n, [M] \rangle.$$

Proof of Proposition 9.4:

This is proved in [EV1, Thm. 8.6] – the proof is a straightforward application of the Demailly-Paun theorem [DP, Thm. 0.1] describing the Kähler cone of an arbitrary closed manifold equipped with a Kähler-type complex structure. ∎

Now we can prove Theorem 9.3.

Proof of Theorem 9.3:

The complex structure I appearing in the statement of part (I) of the theorem is compatible with ω. Hence, the cohomology class $\alpha := [\omega]$ is Kähler with respect to I.

Pick points $x_1, \ldots, x_k \in M \setminus \Sigma$ that are Campana-generic with respect to I. It is possible to pick these points in the complement of Σ, because the set of Campana-generic points of (M, I) is dense in M (see above) and Σ is a proper complex submanifold of (M, I). Set $x := (x_1, \ldots, x_k)$.

Assume that

$$\text{Vol}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i), \omega_0\right) < \text{Vol}(M, \omega) = \langle \alpha^n, [M] \rangle.$$

Then Proposition 9.4 yields that $\widetilde{\alpha}_{\bar{r}} := \Pi^* \alpha - \pi \sum_{i=1}^{k} r_i^2 e_i \in H^2(\widetilde{M}_{I,x}; \mathbb{R})$ is Kähler with respect to $\widetilde{I}$. Now the claim of part (I) follows from part (I) of Theorem 3.2.

Let us prove part (II).

Assume that $\text{Teich}_{\mathcal{C}_0}(M, \omega) \cap \text{Teich}_{\mathcal{C}_0}^s(M)$ is a dense connected subset of $\text{Teich}_{\mathcal{C}_0}(M, \omega)$. The argument we have just used in the proof of part (I) shows that $I \in \mathsf{K}_{\mathcal{C}_0}(\bar{r})$. Therefore

$$\text{Teich}_{\mathcal{C}_0}(M, \omega) \cap \text{Teich}_{\mathcal{C}_0}^s(M) \subset \mathsf{K}_{\mathcal{C}_0}(\bar{r}).$$

Hence, $\mathsf{K}_{\mathcal{C}_0}(\bar{r})$ is also a dense connected subset of $\text{Teich}_{\mathcal{C}_0}(M, \omega)$. Now part (II) of Theorem 3.2 yields the needed claim.

This finishes the proof of Theorem 9.3. ∎

10   The case of tori and K3 surfaces

In this section we discuss Campana-simple complex structures on tori and K3 surfaces and show how to deduce Theorem 3.22 from Theorem 9.3.

10.1   Campana-simple complex structures on tori

Denote by $Lin \subset CK(\mathbb{T}^{2n})$ the set of linear complex structures on $\mathbb{T}^{2n}$ that are compatible with the standard orientation. It is naturally identified with the space of linear complex structures on $\mathbb{R}^{2n}$.

It is well-known (see e.g. [McDS, Prop. 2.5.2]) that the space Lin can be identified with $GL^+(2n, \mathbb{R})/GL(n, \mathbb{C})$ which is a connected smooth manifold. Here the group $GL(n, \mathbb{C})$ is embedded in $GL(2n, \mathbb{R})$ by the map

$$C \mapsto \begin{pmatrix} \text{Re } C & -\text{Im } C \\ \text{Im } C & \text{Re } C \end{pmatrix}.$$

The tangent space $T_J Lin$ of Lin at a point $J \in Lin$ is the space of all real $2n \times 2n$ matrices R satisfying $RJ + JR = 0$. The complex structure on $T_J Lin$ is given by $R \mapsto JR$ (since $GL^+(2n, \mathbb{R})/GL(n, \mathbb{C})$ is symmetric space, this almost complex structure on Lin is indeed integrable [Bes]). Thus, further on Lin will be viewed as a connected complex manifold.

The following claim is an easy exercise (cf. [EV1, Sec. 6]).

Proposition 10.1

The space $T^{2n} \times Lin$ can be equipped with a complex structure so that $T^{2n} \times Lin \to Lin$ is a complex-analytic deformation family such that for each $J \in Lin$ the complex structure on the fiber over J is J itself. ∎

For $l = 1, \ldots, n$ denote by Λ^{2l} be the space of exterior $2l$-forms on $\mathbb{R}^{2n}$ and by $\Lambda^{2l}_{\mathbb{Q}} \subset \Lambda^{2l}$ the space of exterior $2l$-forms on $\mathbb{R}^{2n}$ with rational coefficients.

Let ω be a linear symplectic form on $\mathbb{T}^{2n}$. Denote by $\overline{\omega} \in \Lambda^2$ its lift to $\mathbb{R}^{2n}$ – it is a linear symplectic form on $\mathbb{R}^{2n}$.

For a linear complex structure J on $\mathbb{T}^{2n}$ denote by $\overline{J}$ the linear complex structure on $\mathbb{R}^{2n}$ which is the lift of J.

Proposition 10.2

The cohomology class $[\omega] \in H^2(\mathbb{T}^{2n}; \mathbb{R})$ of the linear symplectic form ω is Kähler with respect to a linear complex structure J on $\mathbb{T}^{2n}$ if and only if ω is compatible with J.

Proof of Proposition 10.2:

If ω is compatible with J then, obviously, $[\omega] \in H^2(\mathbb{T}^{2n}; \mathbb{R})$ is Kähler with respect to J.

Assume $[\omega] \in H^2(\mathbb{T}^{2n}; \mathbb{R})$ is Kähler with respect to J and let us prove that ω is compatible with J – i.e., ω is Kähler on $(\mathbb{T}^{2n}, J)$. Since $[\omega] \in H^2(\mathbb{T}^{2n}; \mathbb{R})$ is Kähler with respect to J, there exists a Kähler form η on $(\mathbb{T}^{2n}, J)$ cohomologous to ω. Viewing $\mathbb{T}^{2n}$ as a compact Lie group and averaging η over $\mathbb{T}^{2n}$, we get a $\mathbb{T}^{2n}$-invariant – hence, linear – form ζ which is Kähler on $(\mathbb{T}^{2n}, J)$ and cohomologous to ω. Since there is only one linear form representing a given cohomology class, we get that $\zeta = \omega$, which implies that ω is Kähler on $(\mathbb{T}^{2n}, J)$.

This finishes the proof of the proposition. ∎

Denote by $Lin(\omega)$ the subset of Lin formed by the linear complex structures on $\mathbb{T}^{2n}$ compatible with the linear symplectic form ω. It is naturally identified with the space of the linear complex structures on $\mathbb{R}^{2n}$ compatible with $\overline{\omega}$.

For each $l = 1, \ldots, n$ and each $a \in H^{2l}(\mathbb{T}^{2n}; \mathbb{R})$, $a \neq 0$, denote by $\mathcal{T}_a \subset Lin$ the set of $J \in Lin$ such that $a \in H^{l,l}_J(\mathbb{T}^{2n})$. It can also be viewed as the Hodge locus of a for the complex-analytic deformation family $T^{2n} \times Lin \to Lin$ given by Proposition 10.1. Consequently, $\mathcal{T}_a$ is a complex subvariety of Lin – see e.g. [Voi, Vol. 2, Lem. 5.13].

Example 10.3

Let ω be a linear symplectic form on $\mathbb{T}^{2n}$.

Let $a = [\omega] \in H^2(\mathbb{T}^{2n}; \mathbb{R})$. Then the set $\mathcal{T}_a$ is the set of linear complex structures on $\mathbb{T}^{2n}$ preserving ω (because a real cohomology class of degree 2 is of type $(1,1)$ with respect to a complex structure if and only if it is preserved by it). In this case $\mathcal{T}_a$ is a smooth complex submanifold of Lin and this can be verified by elementary means.

Indeed, first of all, $\mathcal{T}_a$ is smooth (see e.g. [McDS, Lem. 2.5.5]). If $J \in \mathcal{T}_a$, then $T_J\mathcal{T}_a$ is formed by $2n \times 2n$ matrices R satisfying the two conditions: $RJ + JR = 0$ and $R^t AR = 0$, where A is the matrix of the linear symplectic form $\overline{\omega}$ on $\mathbb{R}^{2n}$. Since $J^t AJ = A$ (because $J \in \mathcal{T}_a$), JR satisfies the same two conditions meaning that $T_J\mathcal{T}_a$ is a complex vector subspace of $T_J Lin$. This means that $\mathcal{T}_a$ is a complex submanifold of Lin.

The set $Lin(\omega)$ is then an open contractible submanifold of $\mathcal{T}_a$ (see e.g. [McDS, Lem. 2.5.5]).

Proposition 10.4

Any Campana-simple complex structure on $\mathbb{T}^{2n}$ admits no proper positive-dimensional complex subvarieties.

Proof of Proposition 10.4:

A Campana-simple complex structure on $\mathbb{T}^{2n}$ is of Kähler type and therefore can be mapped by a diffeomorphism of $\mathbb{T}^{2n}$ into a linear complex structure (see [EV1, Prop. 6.1]), which is also Campana-simple. Thus, it suffices to prove the proposition for a Campana-simple linear complex structure J.

Assume, by contradiction, that such a J admits a positive-dimensional proper complex subvariety. The parallel translations of $\mathbb{T}^{2n}$ preserve J and therefore any point of $\mathbb{T}^{2n}$ is contained in the image of that subvariety under a parallel translation, which contradicts the assumption that J is Campana-simple. This finishes the proof of the proposition. ∎

Denote by $Lin^s \subset Lin$ the space of linear Campana-simple complex structures on $\mathbb{T}^{2n}$.

Proposition 10.5

Let ω be a linear symplectic form on $\mathbb{T}^{2n}$.

Then ω is compatible with a Campana-simple linear complex structure (i.e., $Lin^s \cap Lin(\omega) \neq \emptyset$) if and only if ω is irrational. Moreover, in this case $Lin^s \cap Lin(\omega)$ is a dense connected subset of $Lin(\omega)$.

Proof of Proposition 10.5:

Assume ω is rational. Then, by part 1 of Remark 9.1, there are no Campana-simple complex structures compatible with ω.

Now assume ω is irrational, or, equivalently, that $\overline{\omega}$ is not a real multiple of a form in $\Lambda^2_{\mathbb{Q}}$. Let us prove that $Lin^s \cap Lin(\omega)$ is a dense connected subset of $Lin(\omega)$.

We start with the following lemma.

Lemma 10.6

Assume that a linear symplectic form η is not a real multiple of a form in $\Lambda^2_{\mathbb{Q}}$.

Then for all $l = 1, \ldots, n-1$ the form η^l is not a real multiple of a form in $\Lambda^{2l}_{\mathbb{Q}}$.

Proof of Lemma 10.6:

Assume by contradiction that η^l is a real multiple of a form in $\Lambda^2_{\mathbb{Q}}$. Since η is symplectic, $\eta^l \neq 0$ and we can assume without loss of generality that $\eta^l \in \Lambda^{2l}_{\mathbb{Q}}$.

Consider the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ of $\mathbb{C}$ over $\mathbb{Q}$ (i.e., the group of field automorphisms of $\mathbb{C}$) and its action on the coefficients of η with respect to the standard basis of $\mathbb{R}^{2n}$. It is proved in [Lee] that if $\eta^l = \xi^l$ for some $\xi \in \Lambda^2$ and $l = 1, \dots, n-1$, then $\xi = \pm\eta$. This implies that the action of each element of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ either fixes all the coefficients of η or multiplies each of them by -1. The complex numbers preserved by the action of the whole group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ are exactly the rationals (a proof of this well-known fact can be easily extracted e.g. from [Yale]). Therefore the square of each coefficient of η lies in $\mathbb{Q}$. This easily implies that there exists a positive $r \in \mathbb{Q}$ such that the field $\mathbb{Q}[\sqrt{r}]$ contains all the coefficients of η.

If all elements $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ fix all the coefficients of η, we get that $\eta \in \Lambda_{\mathbb{Q}}^2$, in contradiction to the hypothesis of the lemma. Therefore there exists an element of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ whose action multiplies all the coefficients of η by -1, meaning that all these coefficients are rational multiples of $\sqrt{r}$. Therefore, η is a positive real multiple of a form in $\Lambda_{\mathbb{Q}}^2$, which contradicts again the hypothesis of the lemma. Thus, the assumption that η^l is a real multiple of a form in $\Lambda_{\mathbb{Q}}^2$ has led us to a contradiction, which means that it is false. This proves the lemma. ∎

Since $\overline{\omega}$ is not a real multiple of a form in $\Lambda_{\mathbb{Q}}^2$, Lemma 10.6 yields that for each $l = 1, \dots, n-1$ the form $\overline{\omega}^l \in \Lambda^{2l}$ is not a real multiple of a form in $\Lambda_{\mathbb{Q}}^{2l}$.

Let $a \in H^{2l}(\mathbb{T}^{2n}; \mathbb{Q})$, $a \neq 0$, $l = 1, \dots, n-1$, and let us view $a \in H^{2l}(\mathbb{T}^{2n}; \mathbb{Q})$ as an exterior $2l$-form on $\mathbb{R}^{2n}$ with rational coefficients: $a \in \Lambda_{\mathbb{Q}}^{2l}$. We also view $Lin(\omega)$ as the set of linear complex structures on $\mathbb{R}^{2n}$ compatible with $\overline{\omega}$.

The set $\mathcal{T}_a \cap Lin(\omega)$ is a complex subvariety of the complex manifold $Lin(\omega)$.

We claim that this complex subvariety is proper.

Indeed, assume, by contradiction, that it is not. Then, since $Lin(\omega)$ is connected, $\mathcal{T}_a \cap Lin(\omega) = Lin(\omega)$, meaning that any linear complex structure on $\mathbb{R}^{2n}$ compatible with $\overline{\omega}$ also preserves the exterior $2l$-form a.

Denote by $G \subset SL(2n, \mathbb{R})$ the group of linear automorphisms of $\mathbb{R}^{2n}$ preserving $\overline{\omega}$. It is isomorphic, as a real Lie group, to $Sp(2n, \mathbb{R})$. The set $Lin(\omega)$ is invariant under conjugations in G and thus generates a normal subgroup H of G. But $G \cong Sp(2n, \mathbb{R})$ is a simple Lie group and therefore each normal subgroup of G has to either lie in its center (equal to $\{Id, -Id\}$) or coincide with the whole G (see e.g. [Rag]). The former option is clearly not satisfied for H and therefore we get that $H = G$, meaning that a is preserved by the whole group G. By a theorem of Weyl [Weyl, Ch. VI, Sec. 1] (see e.g. [GV, Sect. 5.3.2] for a modern presentation), the only exterior $2l$-forms preserved by the whole G are the real multiples of $\overline{\omega}^{2l}$. Thus, $a \in \Lambda_{\mathbb{Q}}^{2l}$ is a real multiple of $\overline{\omega}^{2l}$ which, as we have seen above, is a not a real multiple of any form with rational coefficients. We have obtained a contradiction, and this proves the claim that $\mathcal{T}_a \cap Lin(\omega)$ is a proper complex subvariety of of the complex manifold $Lin(\omega)$.

The claim we have proved implies that the complement in the connected complex manifold $Lin(\omega)$ of the countable union of proper complex subvarieties $\mathcal{T}_a \cap Lin(\omega)$, $a \in H^{2l}(\mathbb{T}^{2n}; \mathbb{Q})$, $a \neq 0$, $l = 1, \dots, n-1$, is connected and dense in $Lin(\omega)$. Any complex structure J in this complement is Campana-simple. Indeed, the $2l$-dimensional complex subvarieties of $(\mathbb{T}^{2n}, J)$, $l = 1, \dots, n-1$, all have non-zero rational fundamental classes; the corresponding Poincaré-dual non-zero rational cohomology classes are of type (l, l) (see e.g. [Voi, Vol. 1, Sec. 11]), which is impossible since $J \notin \mathcal{T}_a$ for all $a \in H^{2l}(\mathbb{T}^{2n}; \mathbb{Q})$, $a \neq 0$. Therefore $(\mathbb{T}^{2n}, J)$ has no complex subvarieties of dimension $2l$, $l = 1, \dots, n-1$, meaning that J is

Campana-simple.

Thus, the set $Lin^s \cap Lin(\omega)$ of Campana-simple complex structures lying in $Lin(\omega)$ contains a subset which is connected and dense in $Lin(\omega)$. Hence, $Lin^s \cap Lin(\omega)$ is also connected and dense in $Lin(\omega)$.

This finishes the proof. ∎

Since Lin is connected, there is a connected component of $CK(\mathbb{T}^{2n})$ containing it – denote this connected component by $\mathcal{C}_l$.

Let $x = 0 \in \mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. Let $\mathrm{Diff}(\mathbb{T}^{2n}, x)$ be a subgroup of $\mathrm{Diff}(\mathbb{T}^{2n})$ formed by the diffeomorphisms of $\mathbb{T}^{2n}$ fixing x. Let $\mathrm{Diff}_0(\mathbb{T}^{2n}, x)$ be the identity component of $\mathrm{Diff}(\mathbb{T}^{2n}, x)$. The group $\mathrm{Diff}_0(\mathbb{T}^{2n}, x)$ lies in $\mathrm{Diff}_0(\mathbb{T}^{2n})$ and is, in fact, the identity component of the subgroup of $\mathrm{Diff}_0(\mathbb{T}^{2n})$ formed by the elements of $\mathrm{Diff}_0(\mathbb{T}^{2n})$ fixing x.

Let $\mathrm{Diff}_H(\mathbb{T}^{2n}, x) := \mathrm{Diff}_H(\mathbb{T}^{2n}) \cap \mathrm{Diff}(\mathbb{T}^{2n}, x)$. Clearly, $\mathrm{Diff}_0(\mathbb{T}^{2n}, x) \subset \mathrm{Diff}_H(\mathbb{T}^{2n}, x)$.

Proposition 10.7

Let $I \in CK(\mathbb{T}^{2n})$.

Then the following claims hold:

A. There exist unique $J_I \in Lin$ and $\phi_I \in \mathrm{Diff}_H(\mathbb{T}^{2n}, x)$ so that $J_I = \phi_I^* I$.

B. The complex structure J_I and the diffeomorphism ϕ_I depend continuously on I.

C. If $I \in \mathcal{C}_l$, then $\phi_I \in \mathrm{Diff}_0(\mathbb{T}^{2n}, x)$.

D. If $I \in \mathcal{C}_l$ and $\psi^* I \in Lin$ for some $\psi \in \mathrm{Diff}_0(\mathbb{T}^{2n})$, then $\psi^* I = I$.

Proof of Proposition 10.7:

Let us prove part A of the proposition.

The proof of the existence of ϕ_I and J_I goes as in the proof of [EV1, Prop. 6.1].

Namely, recall the construction of the Albanese map: Let $\Omega_I^1(\mathbb{T}^{2n})$ be the space of holomorphic complex-valued holomorphic 1-forms on the complex manifold $(\mathbb{T}^{2n}, I)$. It is a complex vector space of dimension n that can be identified with $H_I^{1,0}(\mathbb{T}^{2n}; \mathbb{C})$. Pick a basis $\varsigma_1, \ldots, \varsigma_n$ of $\Omega_I^1(\mathbb{T}^{2n})$. The Albanese map $\mathrm{Alb}_I : \mathbb{T}^{2n} \to \mathbb{C}^n/\Gamma_I$ is defined by the map $y \mapsto \left(\int_\gamma \varsigma_1, \ldots, \int_\gamma \varsigma_n \right)$, where γ is an arbitrary smooth path from $x = 0$ to y and Γ_I is a lattice in $\mathbb{C}^n$ formed by the vectors $\left(\int_\gamma \varsigma_1, \ldots, \int_\gamma \varsigma_n \right)$ for all the loops γ. The Albanese map Alb_I is a biholomorphism between $(\mathbb{T}^{2n}, I)$ and the complex torus $\mathbb{C}^n/\Gamma_I$ with the complex structure induced by the standard complex structure on $\mathbb{C}^n$ – see e.g. [Voi, Vol. 1, Def. 12.10 and Thm. 12.15].

Denote by $\pi_I : \mathbb{C}^n \to \mathbb{C}^n/\Gamma$ the natural projection and let $F_I : \mathbb{R}^{2n} \to \mathbb{C}^n$ be an $\mathbb{R}$-linear isomorphism of vector spaces mapping $\mathbb{Z}^{2n}$ to Γ_I. Then F_I covers a diffeomorphism $\bar{f}_I : T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n} \to \mathbb{C}^n/\Gamma_I$, i.e., $\bar{f}_I \circ \pi_I = \pi_I \circ F_I$. One easily sees that F_I can be chosen in such a way that $\bar{f}_I$ induces the same map $H^*(\mathbb{R}^{2n}/\mathbb{Z}^{2n}) \to H^*(\mathbb{C}^n/\Gamma)$ as Alb_I.

Clearly, $\bar{f}_I^* I$ is a linear complex structure on $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. Therefore $\phi_I := \bar{f}_I^{-1} \circ \mathrm{Alb}_I$ is a diffeomorphism mapping I to a linear complex structure J_I on T^{2n}, acting as identity on $H^*(T^{2n})$ and sending $x = 0$ to itself. Thus, $\phi_I \in \mathrm{Diff}_H(\mathbb{T}^{2n}, x)$. This proves the existence of J_I and ϕ_I.

Let us prove the uniqueness of J_I and ϕ_I. It suffices to show that if $J_0, J_1 \in Lin$ and $\phi^* J_0 = J_1$ for some $\phi \in \mathrm{Diff}_H(\mathbb{T}^{2n}, x)$, then $J_0 = J_1$ and $\phi = Id$.

Denote the lifts of J_0, J_1 to $\mathbb{R}^{2n}$ respectively by $\bar{J}_0, \bar{J}_1$. The diffeomorphism $\phi : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ lifts to a diffeomorphism $\bar{\phi} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\bar{\phi}^* \bar{J}_0 = \bar{J}_1$ and for each $v \in \mathbb{Z}^{2n}$ there exists $v' \in \mathbb{Z}^{2n}$ so that $\bar{\phi}(x+v) = \bar{\phi}(x) + v'$ for all $x \in \mathbb{Z}^{2n}$.

The group $GL^+(2n, \mathbb{R})$ acts transitively on *Lin* (see e.g. [McDS], Prop. 2.5.2). Therefore there exists $F \in GL^+(2n, \mathbb{R})$ so that $F^* \bar{J}_1 = \bar{J}_0$. Then $\Gamma := F(\mathbb{Z}^{2n})$ is a lattice in $\mathbb{R}^{2n}$ and $F\bar{\phi} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a diffeomorphism preserving $\hat{J}_0$ so that for each $v \in \mathbb{Z}^{2n}$ there exists $v'' \in \Gamma$ so that $F\bar{\phi}(x+v) = F\bar{\phi}(x) + v''$ for all $x \in \mathbb{Z}^{2n}$.

Using an appropriate linear change of coordinates on $\mathbb{R}^{2n}$, we can assume without loss of generality that $\bar{J}_0$ is the standard linear complex structure on $\mathbb{R}^{2n}$ coming from the identification $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and thus $F\bar{\phi} : \mathbb{C}^n \to \mathbb{C}^n$ is a holomorphic map. The partial derivatives of this map with respect to the complex coordinates on $\mathbb{C}^n$ are $\mathbb{Z}^{2n}$-invariant and holomorphic, hence descend to holomorphic maps on the standard complex torus $\mathbb{T}^{2n}$, hence are constant. Therefore $F\bar{\phi} : \mathbb{C}^n \to \mathbb{C}^n$ is a complex-linear affine map and consequently $\bar{\phi}$ is of the form $\bar{\phi}(x) = Ax + c$ for some $A \in SL(2n, \mathbb{Z})$, $c \in \mathbb{R}^{2n}$. Since ϕ acts trivially on $H_*(\mathbb{T}^{2n})$, the matrix A is just the identity and $\bar{\phi}$ is just a parallel translation by c. Hence, ϕ is a parallel translation too, and since ϕ preserves x, we have $\phi = Id$. Consequently $J_0 = J_1$.

This proves the uniqueness of J_I and ϕ_I and finishes the proof of part A.

Part B follows from Proposition A.1 and from the construction of J_I and ϕ_I in part A.

Let us prove part C. Assume $I \in \mathcal{C}_l$. Then it can be connected to some $J \in Lin$ by a continuous path $\{I_t\} \subset \mathcal{C}_l$. Considering the path $\{\phi_{I_t}\} \subset \text{Diff}_H(\mathbb{T}^{2n}, x)$, which is continuous by part B, and noting that $\phi_J = Id$, we immediately obtain that $\phi_I \in \text{Diff}_0(\mathbb{T}^{2n}, x)$. This proves part C.

Let us prove part D. If $I \in \mathcal{C}_l$ and $\psi^* I \in Lin$ for some $\psi \in \text{Diff}_0(M)$, then precomposing ψ with a parallel translation, which is a biholomorphism of $(\mathbb{T}^{2n}, I)$, we can assume without loss of generality that $\psi(x) = x$. Thus, $\psi \in \text{Diff}_H(\mathbb{T}^{2n}, x)$. By part A, this means that $\psi^* I = J_I = I$. This proves part D and finishes the proof of the proposition. ∎

Proposition 10.8

The restriction of the projection $\mathcal{C}_l \to \mathcal{C}_l / \text{Diff}_0(\mathbb{T}^{2n}) =: \texttt{Teich}_{\mathcal{C}_l}(\mathbb{T}^{2n})$ to *Lin* is a homeomorphism $Lin \to \mathcal{C}_l / \text{Diff}_0(\mathbb{T}^{2n}) = \texttt{Teich}_{\mathcal{C}_l}(\mathbb{T}^{2n})$ that identifies $Lin(\omega) \subset \mathcal{C}_l$ with $\texttt{Teich}_{\mathcal{C}_l}(\mathbb{T}^{2n}, \omega) \subset \texttt{Teich}_{\mathcal{C}_l}(\mathbb{T}^{2n})$.

Proof of Proposition 10.8:

Parts A and C of Proposition 10.7 imply that every orbit of the $\text{Diff}_0(\mathbb{T}^{2n})$-action on $\mathcal{C}_l$ intersects *Lin*. By part D of Proposition 10.7, it intersects *Lin* at exactly one point.

Thus, the map $Lin \to \mathcal{C}_l / \text{Diff}_0(\mathbb{T}^{2n})$, associating to each $I \in Lin$ its $\text{Diff}_0(\mathbb{T}^{2n})$-orbit, is well-defined and bijective. Clearly, it is also continuous.

By part B of Proposition 10.7, the map $\mathcal{C}_l \to Lin$, $I \mapsto J_I$, is continuous. This means that the inverse of the map $Lin \to \mathcal{C}_l / \text{Diff}_0(\mathbb{T}^{2n})$ is continuous too.

Thus, the projection $Lin \to \mathcal{C}_l / \text{Diff}_0(\mathbb{T}^{2n})$ is a homeomorphism. Clearly, the image of $Lin(\omega)$ under this homeomorphism lies in $\texttt{Teich}_{\mathcal{C}_l}(\mathbb{T}^{2n}, \omega)$. Let us show that it coincides with $\texttt{Teich}_{\mathcal{C}_l}(\mathbb{T}^{2n}, \omega)$, or, equivalently, that for any complex structure $I \in \mathcal{C}_l$ compatible with ω we have $J_I \in Lin(\omega)$. Indeed, since ω is compatible with I, the cohomology class $[\omega]$ is Kähler with respect to I, hence also with respect to J_I, because J_I obtained from I by the action of $\text{Diff}_0(\mathbb{T}^{2n})$. Now Proposition 10.2 implies that $J_I \in Lin(\omega)$. Consequently, the homeomorphism $Lin \to \mathcal{C}_l / \text{Diff}_0(\mathbb{T}^{2n})$ identifies $Lin(\omega)$ with $\texttt{Teich}_{\mathcal{C}_l}(\mathbb{T}^{2n}, \omega)$. ∎

Proposition 10.9

Let ω be a Kähler-type form on $\mathbb{T}^{2n}$.

Then $\mathrm{Symp}_H(\mathbb{T}^{2n}, \omega)$ acts transitively on the set of connected components of $CK(\mathbb{T}^{2n})$ compatible with ω.

Proof of Proposition 10.9:

By [EV1, Prop. 6.1], any Kähler-type form on $\mathbb{T}^{2n}$ can be mapped by a diffeomorphism of $\mathbb{T}^{2n}$ acting trivially on $H_*(\mathbb{T}^{2n})$ to a linear symplectic form. Thus, we can assume without loss of generality that our Kähler-type form ω is linear.

Let $I \in CK(\mathbb{T}^{2n})$ be an arbitrary complex structure compatible with ω. By [EV1, Prop. 6.1], there exists a diffeomorphism ϕ of $\mathbb{T}^{2n}$ acting trivially on $H_*(\mathbb{T}^{2n})$ so that the complex structure $\phi^* I$ is linear. The symplectic form $\phi^* \omega$ is then compatible with $\phi^* I$ and has the same cohomology class as ω. Thus, the cohomology class $[\omega]$ is Kähler with respect to the linear complex structure $\phi^* I$. Therefore, by Proposition 10.2, the linear form ω itself is Kähler with respect to $\phi^* I$.

We have obtained that $\phi^* \omega$ and ω are cohomologous and compatible with the same complex structure $\phi^* I$. Therefore the straight path in the space of 2-forms connecting these forms is formed by cohomologous symplectic forms. By Moser's theorem [Mos], this means that there exists $\psi \in \mathrm{Diff}_0(\mathbb{T}^{2n})$ such that $\psi^* \phi^* \omega = \omega$. Thus, $\phi\psi \in \mathrm{Symp}_H(\mathbb{T}^{2n}, \omega)$ and moreover $(\phi\psi)^* I = \psi^* \phi^* I$ lies in the same connected component of $CK(\mathbb{T}^{2n})$ as $\phi^* I$ – i.e., in $\mathcal{C}_l$.

We have proved that any complex structure $I \in CK(\mathbb{T}^{2n})$ compatible with ω can be mapped by an element of $\mathrm{Symp}_H(\mathbb{T}^{2n}, \omega)$ into a complex structure lying in $\mathcal{C}_l$. This readily implies the proposition. ∎

Proposition 10.10

Let ω be a Kähler-type symplectic form on $\mathbb{T}^{2n}$.

Then ω is compatible with a Campana-simple complex structure if and only if ω is irrational.

In this case for any connected component $\mathcal{C}_0$ of $CK(\mathbb{T}^{2n})$ compatible with ω the set $\mathrm{Teich}_{\mathcal{C}_0}(\mathbb{T}^{2n}, \omega) \cap \mathrm{Teich}^s(\mathbb{T}^{2n})$ is a dense connected subset of $\mathrm{Teich}_{\mathcal{C}_0}(\mathbb{T}^{2n}, \omega)$.

Proof of Proposition 10.10:

Assume that ω is rational. Then, by part 1 of Remark 9.1, there are no Campana-simple complex structures compatible with ω.

By [EV1, Prop. 6.1], any Kähler-type form on $\mathbb{T}^{2n}$ can be mapped by a diffeomorphism of $\mathbb{T}^{2n}$ to a linear symplectic form. Thus, we can assume without loss of generality that our Kähler-type form ω is linear. In view of Proposition 10.9, we can also assume without loss of generality that $\mathcal{C}_0 = \mathcal{C}_l$. Now the proposition follows from Proposition 10.5, Proposition 10.8. ∎

10.2 Campana-simple complex structures on K3 surfaces

Let M be a smooth manifold underlying a K3 surface.

The space $\mathrm{Teich}(M)$ has the structure of a complex manifold (see e.g. [Cat]). More precisely, a combination of the fundamental theorems of Kuranishi [Ku1] and Bogomolov-Tian-Todorov [Bo2, Tia,

Tod] shows that for any $J_0 \in CK(M)$ the Kuranishi space of (M, J_0) (the base of a certain universal complex-analytic local deformation family of a complex structure defined in [Ku1]) is a smooth complex manifold as long as the canonical bundle of (M, J_0) is trivial, which, of course, is true in the hyperkähler case, and in particular, for K3 surfaces. Moreover (see e.g. [Cat]), there exists a homeomorphism between a neighborhood $\mathtt{U}_{[J_0]}$ of $[J_0]$ in $\mathtt{Teich}(M)$ and an open subset of the Kuranishi space of (M, J_0). This homeomorphism maps each $[J] \in \mathtt{U}_{[J_0]}$ to the point x of the Kuranishi space corresponding to a complex structure on M representing $[J]$. The pullback of the universal complex-analytic local deformation family over the Kuranishi space under the homeomorphism, together with a smooth trivialization of the universal fibration over a neighborhood of x in the Kuranishi space, induces a complex-analytic deformation family $M \times \mathtt{U}_{[J_0]} \to \mathtt{U}_{[J_0]}$. The smooth and the complex structures on $\mathtt{U}_{[J_0]}$ are the pullbacks of the corresponding structures on the Kuranishi space.

Remark 10.11

Note that the complex manifold $\mathtt{Teich}(M)$ is not Hausdorff (see [At1], [BurR, p.238], cf. [Sal]).

The description above also implies that for any $J \in CK(M)$ any smooth path in $\mathtt{Teich}(M)$ starting at $[J]$ can be lifted to a smooth path in $CK(M)$ starting at J.

Let $q: H^2(M; \mathbb{C}) \times H^2(M; \mathbb{C}) \to \mathbb{C}$ be the intersection form. Its restriction to $H^2(M; \mathbb{R})$ is a real-valued symmetric form of signature $(3, 19)$. A subspace of $H^2(M; \mathbb{R})$ is called positive if the restriction of q to it is positive. We use $\perp$ to denote the orthogonal complement with respect to q in $H^2(M; \mathbb{R})$.

Let $I \in CK(M)$. Then

$$\dim_{\mathbb{C}} H_I^{2,0}(M; \mathbb{C}) = \dim_{\mathbb{C}} H_I^{0,2}(M; \mathbb{C}) = 1, \ \dim_{\mathbb{C}} H_I^{1,1}(M; \mathbb{C}) = 20$$

(see e.g. [Huyb]). The spaces $H_I^{2,0}(M; \mathbb{R}) \oplus H_I^{0,2}(M; \mathbb{R})$ and $H_I^{1,1}(M; \mathbb{R})$ are orthogonal with respect to q. If Ω is a holomorphic 2-form spanning $H_I^{2,0}(M; \mathbb{C})$, then $\left(H_I^{2,0}(M; \mathbb{C}) \oplus H_I^{0,2}(M; \mathbb{C}) \right) \cap H^2(M; \mathbb{R})$ is the 2-dimensional plane spanned by $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$. Denote this plane by W_I. The basis $\{\mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega\}$ defines an orientation of the plane W_I. We have $W_I^\perp = H_I^{1,1}(M; \mathbb{C}) \cap H^2(M; \mathbb{R})$. The restriction of q to W_I is positive and the restriction of q to its orthogonal complement $H_I^{1,1}(M; \mathbb{R})$ in $H^2(M; \mathbb{R})$ has signature $(1, 19)$. Consequently, the set $\{a \in H_I^{1,1}(M; \mathbb{R}) \mid q(a, a) > 0\}$ has two connected components: the one that contains the Kähler cone of (M, I) is called the **positive cone of** (M, I) and will be denoted $\mathrm{Pos}\,(M, I)$, and the other one is $-\mathrm{Pos}\,(M, I)$.

Let $Gr_{++}(H^2(M; \mathbb{R}))$ be the Grassmanian of positive oriented 2-dimensional planes in $H^2(M; \mathbb{R})$. There is a canonical diffeomorphism identifying $Gr_{++}(H^2(M; \mathbb{R}))$ with an open subset $\{l \in \mathbb{P}H^2(M; \mathbb{C}) \mid q(l, l) = 0, \ q(l, \bar{l}) > 0\}$ of the quadric $\{l \in \mathbb{P}H^2(M; \mathbb{C}) \mid q(l, l) = 0\}$ (see e.g. [Huyb, Ch.6, Prop. 1.5] and [Ver6, Claim 2.9]). Thus $Gr_{++}(H^2(M; \mathbb{R}))$ has the structure of a connected complex manifold.

Define the **period map** $\mathrm{Per} : \mathtt{Teich}(M) \to Gr_{++}(H^2(M; \mathbb{R}))$ by

$$\mathrm{Per}([I]) := W_I,$$

where $[I] \in \mathtt{Teich}(M)$ is the element of $\mathtt{Teich}(M)$ represented by I.

The global Torelli theorem for K3 surfaces (see [Huyb] and the references therein to the original papers) implies the following claim.

Theorem 10.12

A. There exists a subset $\mathcal{S} \subset \texttt{Teich}(M)$ so that for each connected component $\mathcal{C}_0$ of $CK(M)$ the intersection $\mathcal{S} \cap \texttt{Teich}_{\mathcal{C}_0}(M)$ is dense and connected in $\texttt{Teich}_{\mathcal{C}_0}(M)$ and the map

$$\mathsf{Per} : \texttt{Teich}_{\mathcal{C}_0}(M) \to Gr_{++}(H^2(M;\mathbb{R}))$$

is a surjective local biholomorphism which is injective on $\mathcal{S} \cap \texttt{Teich}_{\mathcal{C}_0}(M)$.

B. Assume that $I_0, I_1 \in CK(M)$, $[I_0], [I_1] \in \mathcal{S}$ and $\mathsf{Per}([I_0]) = \mathsf{Per}([I_1])$.

Then there exists a unique diffeomorphism $\phi \in \mathrm{Diff}_H(M)$ such that $\phi^* I_1 = I_0$. ∎

By a (-2)-**class** we will mean an integral cohomology class $b \in H^2(M;\mathbb{Z})$ such that $q(b,b) = -2$.

For each $I \in CK(M)$ denote by $S(M,I)$ the set of all (-2)-classes $b \in H^2(M;\mathbb{Z})$ that are of type $(1,1)$ with respect to I.

The Kähler cone of (M,I) admits the following description.

Theorem 10.13 (see e.g. [Huyb, Sec. 8, Thm. 5.2], cf. [AmV2, Thm. 6.2])

The Kähler cone of (M,I) is a connected component of the set

$$\mathrm{Pos}\,(M,I) \setminus \cup_{b \in S(M,I)} \left(b^\perp \cap \mathrm{Pos}\,(M,I) \right).$$

In particular, if $b^\perp \cap \mathrm{Pos}\,(M,I) = \emptyset$ for all $b \in S(M,I)$, then the Kähler cone of (M,I) is just $\mathrm{Pos}\,(M,I)$. ∎

The next result follows from the Calabi-Yau theorem (formerly the Calabi conjecture) [Yau2] combined with Bochner's technique – see [Bea1] (cf. e.g. [GHJ, Thm. 5.11]) for the proof of the existence and uniqueness; see [FuS, Thm. 6.3] for the proof of the smooth dependence.

Theorem 10.14

Let $I \in CK(M)$ and let $a \in H^2(M;\mathbb{R})$ be a Kähler class with respect to I.

Then there exists a unique $\eta \in SK(M)$ compatible with I such that $[\eta] = a$ and the Riemannian metric $\eta(\cdot, I\cdot)$ is Ricci-flat. The form η depends smoothly on the pair (I,a) (as I varies in a complex-analytic deformation family). ∎

Remark 10.15

For any Kähler structure (η, I) on M such that the Riemannian metric $\eta(\cdot, I\cdot)$ is Ricci-flat, the form η and the complex structure I can be included together in a hyperkähler structure $(\omega_1 := \eta, \omega_2, \omega_3, I_1 := I, I_2, I_3)$.

Proposition 10.16

Let $I \in CK(M)$ be compatible with ω.

Then there exists $J \in CK(M)$ isotopic to I, so that the Riemannian metric $\omega(\cdot, J\cdot)$ is Ricci-flat.

Proof of Proposition 10.16:

By Theorem 10.14, the complex structure I is a part of a (unique) Kähler structure (η, I) on M for which the corresponding Riemannian metric $\eta(\cdot, I\cdot)$ is Ricci-flat, and moreover $[\eta] = [\omega]$. The symplectic forms ω and η are cohomologous and compatible with the same I and therefore, by Moser's theorem [Mos], there exists $\phi \in \mathrm{Diff}_0(M)$ such that $\phi^*\eta = \omega$. Then $J := \phi^*I$ is compatible with ω and the Riemannian metric $\omega(\cdot, J\cdot)$ is Ricci-flat. ∎

Let ω be a Kähler-type symplectic form on M.

Define $D_\omega \subset Gr_{++}\left(H^2(M;\mathbb{R})\right)$ as the set of positive oriented 2-dimensional planes $W \subset H^2(M;\mathbb{R})$ such that $[\omega] \in W^\perp$ and $(\mathbb{R}[\omega] \oplus W)^\perp$ does not contain any (-2)-classes.

Proposition 10.17

The set D_ω is a dense connected subset of the submanifold $Gr_{++}\left([\omega]^\perp\right)$ of $Gr_{++}\left(H^2(M;\mathbb{R})\right)$.

Proof of Proposition 10.17:

We have $q([\omega], [\omega]) > 0$. The restriction of q to $[\omega]^\perp$ is a quadratic form of signature $(2, 19)$. The space $Gr_{++}([\omega]^\perp)$ of positive oriented 2-dimensional planes in $[\omega]^\perp$ is a smooth connected manifold diffeomorphic to $SO(2,19)/SO(2) \times SO(19)$.

If $a \in H^2(M;\mathbb{Z})$ is a (-2)-class, then a and $[\omega]$ are linearly independent and the restriction of q to $a^\perp \cap [\omega]^\perp$ has signature $(2, 18)$. The space $Gr_{++}\left(a^\perp \cap [\omega]^\perp\right)$ of positive oriented 2-dimensional planes in $a^\perp \cap [\omega]^\perp$ is a closed submanifold of $Gr_{++}\left([\omega]^\perp\right)$ diffeomorphic to $SO(2,18)/SO(2) \times SO(18)$ and thus has real codimension 2 in $Gr_{++}\left([\omega]^\perp\right)$.

The set D_ω is the complement in $Gr_{++}\left([\omega]^\perp\right)$ of the (at most) countable union of the closed codimension-2 submanifolds $Gr_{++}\left(a^\perp \cap [\omega]^\perp\right)$ for all (-2)-classes $a \in H^2(M;\mathbb{Z})$ such that $q(a, [\omega]) = 0$. Therefore D_ω is a dense connected subset of the submanifold $Gr_{++}\left([\omega]^\perp\right)$ of $Gr_{++}\left(H^2(M;\mathbb{R})\right)$. (The density follows from Baire's theorem; for a proof of the connectivity see e.g. [Ver6, Lem. 4.10].) ∎

Proposition 10.18

Assume that $\mathcal{C}_0$ is a connected component of $CK(M)$ compatible with the symplectic form ω (i.e., $\mathcal{C}_0$ contains a complex structure compatible with ω).

Then $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega) = \mathrm{Per}^{-1}(D_\omega) \cap \mathtt{Teich}_{\mathcal{C}_0}(M)$ and it is a dense connected subset of the closed submanifold $\mathrm{Per}^{-1}\left(Gr_{++}\left([\omega]^\perp\right)\right) \cap \mathtt{Teich}_{\mathcal{C}_0}(M)$ of $\mathtt{Teich}_{\mathcal{C}_0}(M)$.

Proof of Proposition 10.18:

Let us show that $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega) \subset \mathrm{Per}^{-1}(D_\omega) \cap \mathtt{Teich}_{\mathcal{C}_0}(M)$. Consider a complex structure $I \in \mathcal{C}_0$ compatible with ω. Then $[\omega] \in H_I^{1,1}(M;\mathbb{R})$. Therefore W_I is orthogonal to $[\omega]$. Moreover, $\mathbb{R}[\omega] \oplus W_I$ cannot contain any (-2)-class in its orthogonal complement: indeed, if $a \in H^2(M;\mathbb{Z})$ is orthogonal to $(H_I^{2,0}(M;\mathbb{C}) \oplus H_I^{0,2}(M;\mathbb{C})) \cap H^*(M;\mathbb{R})$, then $a \in H_I^{1,1}(M;\mathbb{R})$. However, by Theorem 10.13, a (-2)-class in $H_I^{1,1}(M;\mathbb{R})$ cannot be orthogonal to a Kähler class with respect to I. Thus, we have proved that $\mathrm{Per}([I]) = W_I$ belongs to D_ω. This yields that $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega) \subset \mathrm{Per}^{-1}(D_\omega) \cap \mathtt{Teich}_{\mathcal{C}_0}(M)$.

Now let us show that $\text{Teich}_{\mathcal{C}_0}(M,\omega) \supset \text{Per}^{-1}(D_\omega) \cap \text{Teich}_{\mathcal{C}_0}(M)$.

By the hypothesis of the proposition, there exists a complex structure $I \in \mathcal{C}_0$ compatible with ω. In view of Proposition 10.16, we may assume, without loss of generality, that the metric $\omega(\cdot, I\cdot)$ is Ricci-flat.

Let W be an arbitrary plane belonging to D_ω. By Proposition 10.17, it can be connected with W_I by a smooth path of planes in D_ω. Since Per is a local biholomorphism (by part A of Theorem 10.12) and by the above-mentioned path-lifting property of the $CK(M) \to \text{Teich}(M)$, this path can be lifted to a smooth path in $\mathcal{C}_0$ – i.e., there exists a smooth path of complex structures $\{I_t\}$, $0 \leq t \leq 1$, in $\mathcal{C}_0$ such that $I_0 = I$ and $W_{I_1} = W$ and $W_{I_t} \in D_\omega$ for each $t \in [0,1]$. Consequently, for each $t \in [0,1]$ the plane W_{I_t} is orthogonal to $[\omega]$ and $(\mathbb{R}[\omega] \oplus W)^\perp$ does not contain any (-2)-classes. Therefore, by Theorem 10.13, for each $t \in [0,1]$ either $[\omega]$ or $-[\omega]$ lies in the Kähler cone of (M, I_t).

We claim that it is always $[\omega]$, and not $-[\omega]$.

Indeed, consider the set A of $t \in [0,1]$ for which it is $[\omega]$. The set A is non-empty since $0 \in A$: the cohomology class $[\omega]$ is Kähler with respect to $I = I_0$. The set A is also open: if for some $t_0 \in [0,1]$ the class $[\omega]$ is Kähler with respect to I_{t_0}, then for any $t \in [0,1]$ sufficiently close to t_0, the class $[\omega]_{I_t}^{1,1} = [\omega]$ is Kähler with respect to I_t (by the corollary of the Kodaira-Spencer stability theorem [EV1, Thm. 5.6], see Section A.2). For the same reason the complement of A in $[0,1]$ (i.e., the set of $t \in [0,1]$ for which $-[\omega]$ is Kähler with respect to I_t) is also open. Thus, A is open, closed and non-empty, meaning that $A = [0,1]$, which proves the claim.

Thus, for each $t \in [0,1]$ the class $[\omega]$ is Kähler with respect to a Kähler-type complex structure I_t depending smoothly on t and for $t = 0$ the metric $\omega(\cdot, I_0\cdot)$ is Ricci-flat. By Theorem 10.14, there exists a smooth family of cohomologous symplectic forms ω_t, $0 \leq t \leq 1$, such that $\omega_0 = \omega$ and ω_t is compatible with I_t (and, moreover, the Riemannian metric $\omega_t(\cdot, I_t\cdot)$ is Ricci-flat) for each $t \in [0,1]$. Now Moser's theorem [Mos] implies that there exists $\phi \in \text{Diff}_0(M)$ such that $\phi^* \omega_1 = \omega$. Consequently, the complex structure $\phi^* I_1$ is compatible with ω, meaning that $[I_1] \in \text{Teich}_{\mathcal{C}_0}(M,\omega)$. Hence, $W = W_{I_1}$ lies in $\text{Per}(\text{Teich}_{\mathcal{C}_0}(M,\omega))$. This yields $\text{Teich}_{\mathcal{C}_0}(M,\omega) \supset \text{Per}^{-1}(D_\omega) \cap \text{Teich}_{\mathcal{C}_0}(M)$.

We have shown that $\text{Teich}_{\mathcal{C}_0}(M,\omega) = \text{Per}^{-1}(D_\omega) \cap \text{Teich}_{\mathcal{C}_0}(M)$. Since, by part A of Theorem 10.12, $\text{Per}: \text{Teich}_{\mathcal{C}_0}(M) \to Gr_{++}(H^2(M;\mathbb{R}))$ is a surjective local biholomorphism which is injective on a dense connected subset of $\text{Teich}_{\mathcal{C}_0}(M)$ and since, by Proposition 10.17, D_ω is a dense connected manifold of a closed submanifold of $Gr_{++}(H^2(M;\mathbb{R})) = \text{Per}(\text{Teich}_{\mathcal{C}_0}(M))$, we get that $\text{Teich}_{\mathcal{C}_0}(M,\omega)$ is a dense connected subset of the closed submanifold $\text{Per}^{-1}(Gr_{++}([\omega]^\perp)) \cap \text{Teich}_{\mathcal{C}_0}(M)$ of $\text{Teich}_{\mathcal{C}_0}(M)$. ∎

Proposition 10.19

The group $\text{Symp}_H(M,\omega)$ acts transitively on the set of connected components of $CK(M)$ compatible with ω.

Proof of Proposition 10.19:

Assume $\mathcal{C}_0$ and $\mathcal{C}_1$ are connected components of $CK(M)$ compatible with ω. We need to show that there exists a symplectomorphism in $\text{Symp}_H(M,\omega)$ mapping $\mathcal{C}_1$ into $\mathcal{C}_0$.

By Proposition 10.18,

$$\text{Per}(\text{Teich}_{\mathcal{C}_0}(M,\omega)) = \text{Per}(\text{Teich}_{\mathcal{C}_1}(M,\omega)) = D_\omega.$$

Pick $I_0 \in \text{Teich}_{\mathcal{C}_0}(M,\omega) \cap \mathcal{S}$, $I_1 \in \text{Teich}_{\mathcal{C}_1}(M,\omega) \cap \mathcal{S}$, so that $\text{Per}([I_0]) = \text{Per}([I_1])$. By part B of Theorem

10.12, there exists $\phi \in \mathrm{Diff}_H(M)$ such that $\phi^* I_1 = I_0$. Then $\phi^* \omega$ and ω are cohomologous and compatible with the same complex structure I_0. Therefore the straight path in the space of 2-forms connecting these forms is formed by cohomologous symplectic forms. By Moser's theorem [Mos], this means that there exists $\psi \in \mathrm{Diff}_0(M)$ such that $\psi^* \phi^* \omega = \omega$. Thus, $\phi \psi \in \mathrm{Symp}_H(\mathbb{T}^{2n}, \omega)$, and moreover $(\phi \psi)^* I_1 = \psi^* \phi^* I_1$ lies in the same connected component of $CK(M)$ as $I_0 = \phi^* I_1$ – i.e., in $\mathcal{C}_0$.

Thus, the symplectomorphism $\phi \psi \in \mathrm{Symp}_H(M, \omega)$ maps $\mathcal{C}_1$ into $\mathcal{C}_0$. ∎

Proposition 10.20

The Kähler-type symplectic form ω is compatible with a Campana-simple complex structure on M if and only if ω is irrational. Moreover, in this case for any connected component $\mathcal{C}_0$ of $CK(M)$ compatible with ω the set $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega) \cap \mathtt{Teich}^s(M)$ is a dense connected subset of $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$.

Proof of Proposition 10.20:

Assume ω is rational. Then, by part 1 of Remark 9.1, there are no Campana-simple complex structures compatible with ω.

Now assume ω is irrational. Let us show that the set of Campana-simple structures lying in $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$ is a dense connected subset of $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$.

Let $b \in H^2(M; \mathbb{Q})$, $b \neq 0$. Since $[\omega] \in H^2(M; \mathbb{R})$ is not a real multiple of a rational cohomology class (and, in particular, not a real multiple of b), $\dim_{\mathbb{R}} b^\perp \cap [\omega]^\perp = 20$. Consequently, the space $Gr_{++}\left(b^\perp \cap [\omega]^\perp\right)$ of positive oriented 2-dimensional planes in $b^\perp \cap [\omega]^\perp$ has real codimension 2 in $Gr_{++}\left([\omega]^\perp\right)$ if $q(b, b) < 0$ and is empty otherwise.

As we have seen in the proof of Proposition 10.17, D_ω is the complement in $Gr_{++}\left([\omega]^\perp\right)$ of an (at most) countable union of closed codimension-2 submanifolds $Gr_{++}\left(a^\perp \cap [\omega]^\perp\right)$ for all (-2)-classes $a \in H^2(M; \mathbb{Z})$ such that $q(a, [\omega]) = 0$. Therefore the complement in D_ω to the countable union of codimension-2 submanifolds $Gr_{++}\left(b^\perp \cap [\omega]^\perp\right)$, $b \in H^2(M; \mathbb{Q})$, $b \neq 0$, is a dense connected subset of D_ω. Denote the latter subset by A.

By part A of Theorem 10.12 and Proposition 10.18, $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega) \cap \mathrm{Per}^{-1}(A)$ is a dense connected subset of $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$. We claim that each I such that $[I] \in \mathrm{Per}^{-1}(A) \cap \mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$ is Campana-simple. Indeed, if (M, I) admits a proper positive-dimensional analytic subvariety – i.e., a (possibly singular) complex curve – then the cohomology class $b \in H^2(M; \mathbb{Q})$, $b \neq 0$, Poincaré-dual to the fundamental class of the subvariety is of type $(1, 1)$ with respect to I (see e.g. [Voi, Vol. 1, Sec. 11]). Therefore the 2-dimensional plane $\mathrm{Per}([I]) \in A$ is orthogonal to b in $H^2(M; \mathbb{R})$ which contradicts the definition of A.

Thus, $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega) \cap \mathrm{Per}^{-1}(A)$ is a dense connected subset of the set $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega) \cap \mathtt{Teich}^s(M)$. Hence, $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega) \cap \mathtt{Teich}^s(M)$ is a dense connected subset of $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$. ∎

10.3 Kähler-type embeddings of balls into tori and K3 surfaces – the proofs

Now we are ready to prove Theorem 3.22. For convenience we restate it here.

Theorem 10.21 (Theorem 3.22)

Let M, $\dim_{\mathbb{R}} M = 2n$, be either $\mathbb{T}^{2n}$ or a smooth manifold (of real dimension 4) underlying a complex K3 surface and let ω be a Kähler-type symplectic form on M.

Assume that ω is irrational.

Let $k \in \mathbb{Z}_{>0}$, $r_1, \ldots, r_k > 0$.

Then the following claims hold:

A. Kähler-type embeddings of $\bigsqcup_{i=1}^{k} B^{2n}(r_i)$ into (M, ω) are unobstructed.

More precisely, assume that $\mathrm{Vol}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i), \omega_0\right) < \mathrm{Vol}(M, \omega)$ and ω is compatible with a Campana-simple complex structure I.

Then there exists an $[I]$-Kähler-type embedding $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$.

B. The group $\mathrm{Symp}_H(M, \omega)$ acts transitively on the set of connected components of $CK(M)$ compatible with ω.

C. Any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ (if they exist!) lie in the same orbit of the $\mathrm{Symp}_H(M, \omega)$-action. They lie in the same orbit of the $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$-action if and only if they favor a common connected component of $CK(M)$. In the latter case there exists $[I] \in \mathtt{Teich}(M, \omega)$ such that both embeddings are of $[I]$-Kähler-type.

Proof of Theorem 10.21 (=Theorem 3.22):

Let us first consider the torus case. By Proposition 10.9, in this case we may assume without loss of generality that $\mathcal{C}_0 = \mathcal{C}_l$. Now part A of the theorem follows from part (I) of Theorem 9.3 and Proposition 10.10. Part B of the theorem is exactly Proposition 10.10. Part C of the theorem follows from part B and from part (II) of Theorem 9.3 and Proposition 10.10. This finishes the proof in the torus case.

Let us now prove the result in the K3 case. Here part A of the theorem follows from part (I) of Theorem 9.3 and Proposition 10.20. Part B is exactly Proposition 10.20. Part C of the theorem follows from part B and from part (II) of Theorem 9.3. This finishes the proof in the K3 case. ∎

11   Tame embeddings of arbitrary domains

In this section we will prove the results on tame embeddings into tori and K3 surfaces stated in Section 3.4.

Let M, $\dim_{\mathbb{R}} M = 2n$, be a closed connected oriented manifold admitting Kähler structures (compatible with the orientation).

Denote by $\Omega^m(M)$ the space of smooth differential m-forms on M, $m = 0, 1, \ldots, 2n$. Let $S(M)$ denote the space of symplectic structures on M and $C(M)$ the space of complex structures on M. We equip these spaces with the C^1-topologies.

Let $W_i \subset \mathbb{R}^{2n}$, $i = 1, \ldots, k$, be compact domains with piecewise-smooth boundary, so that their interiors contain the origin and $H^2(W_i; \mathbb{R}) = 0$. Denote

$$\mathbb{W} := \bigsqcup_{i=1}^{k} W_i.$$

Proposition 11.1

Assume ω is a symplectic form on M and $f : \mathbb{W} \to (M, \omega)$ is a tame embedding holomorphic with respect to a complex structure I on M tamed by ω. Let $\mathcal{V}$ be a sufficiently small neighborhood of I in $C(M)$ so that each $I' \in \mathcal{V}$ is tamed by ω.

Then there exists a neighborhood $\mathcal{U} = \mathcal{U}(\mathcal{V})$ of ω in $S(M)$ such that for each $\omega' \in \mathcal{U}$ there exists a tame embedding $f' : \mathbb{W} \to (M, \omega')$ holomorphic with respect to a complex structure $I' \in \mathcal{V}$ which is tamed by ω' and isotopic to I.

Proof of Proposition 11.1:

The following lemma is standard – it is proved by an easy modification of the proof of a similar result in [EV1, Lem. 9.3].

Lemma 11.2

Let $V \subset M$, $\dim_{\mathbb{R}} V = 2n$, be a compact submanifold with a piecewise-smooth boundary. Let $\mathcal{X}$ be a neighborhood of zero in the space $\Omega^1(M)$.

Then there exists a neighborhood $\mathcal{X}'$ of zero in the space $\Omega^2(M)$, so that for each $\eta \in \mathcal{X}'$, which is exact on V, one can choose $\sigma \in \mathcal{X}$ such that $d\sigma = \eta$ on V. ∎

Assume that the tame embedding $f := \bigsqcup_{i=1}^{k} f_i : \mathbb{W} \to (M, \omega)$ extends (as a tame embedding) to an embedding of an open neighborhood of $\mathbb{U} = \bigsqcup_{i=1}^{k} U_i$, where U_i, $i = 1, \dots, k$, is a compact domain with a piecewise-smooth boundary containing W_i in its interior and homotopy equivalent to W_i, so that f is holomorphic (on a neighborhood of $\mathbb{U}$) with respect to I.

Choose a neighborhood $\mathcal{U}_1$ of ω in $S(M)$ so that for each $\omega' \in \mathcal{U}_1$ the following conditions are satisfied:

- For each $t \in [0, 1]$ the form $\omega + t(\omega - \omega')$ is symplectic.

- ω' is tamed by any $I' \in \mathcal{V}$.

Choose a neighborhood $\mathcal{W}$ of Id in the group $\mathrm{Diff}_0(M)$ so that $\phi_* I \in \mathcal{V}$ for each $\phi \in \mathcal{W}$.

Choose a neighborhood $\mathcal{Y}$ of 0 in the space of time-dependent smooth vector fields on M, equipped with the C^1-topology, so that for each vector field $v \in \mathcal{Y}$ the time $[0, 1]$-flow of v generates a diffeomorphism lying in $\mathcal{W}$ and such that for each $t \in [0, 1]$ the time-$[0, t]$ flow of v maps each $f(W_i)$ into $f(Int\, U_i)$, $i = 1, \dots, k$.

Since for all $i = 1, \dots, k$ we have $H^2(W_i; \mathbb{R}) = H^2(U_i; \mathbb{R}) = 0$, for each $\omega' \in \mathcal{U}_1$ the restriction of the form $\omega - \omega'$ to $f(\mathbb{U})$ is exact. Each $\sigma \in \Omega^1(M)$ satisfying $d\sigma = \omega - \omega'$ defines a smooth time-dependent vector field v on M by the formula

$$\left(\omega + t(\omega - \omega') \right) (v_t, \cdot) = \sigma(\cdot), \ t \in [0, 1].$$

In view of Lemma 11.2, applied to $f(\mathbb{U})$, there exists a neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of ω in $SK(M)$ so that for each $\omega' \in \mathcal{U}_3$ one can choose σ as above so that the corresponding v lies in $\mathcal{Y}$.

Let $\omega' \in \mathcal{U}_2$ and let $\{\phi_t\} \subset \mathrm{Diff}_0(M)$, $t \in [0,1]$, be the time-$[0,t]$ flow of $v \in \mathcal{Y}$. Then there exists an open neighborhood $Z_i \subset Int\, U_i$ of each W_i, $i = 1,\dots,k$, so that $\phi_t(Z_i) \subset f(Int\, U_i)$ for all $t \in [0,1]$, $i = 1,\dots,k$. Moser's argument[3] [Mos] shows that

$$\phi_1^* \omega'|_{f(Z_i)} = \omega|_{f(Z_i)}, \; i = 1,\dots,k.$$

We claim that

$$g := \phi_1 \circ f = \bigsqcup_{i=1}^{k} \phi_1 \circ f_i : \mathbb{W} \to (M, \omega')$$

is a tame embedding.

Indeed, it is clearly a smooth embedding extending to a neighborhood $Z := \bigsqcup_{i=1}^{k} Z_i$ of $\mathbb{W}$ so that $g(Z) \subset Int\, f(\mathbb{U})$. It is also symplectic on $\mathbb{Z}$:

$$g^* \omega' = (\phi_1 \circ f)^* \omega' = f^* \phi_1^* \omega' = f^* \omega = \omega_0,$$

where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$. Note that since the embedding f is holomorphic with respect to the complex structure I, the embedding g is holomorphic with respect to the complex structure $J := (\phi_1)_* I$.

By our construction, $\phi_1 \in \mathcal{W}$, hence $J \in \mathcal{V}$. Since $\omega' \in \mathcal{U}_2 \subset \mathcal{U}_1$, the complex structure J is tamed by ω'. This shows that for $\mathcal{U} := \mathcal{U}_2$ and any $\omega' \in \mathcal{U}$ there exists a tame embedding $g : \mathbb{W} \to (M, \omega')$ holomorphic with respect to the complex structure J tamed by ω' and isotopic to I. ∎

Proposition 11.3

Let $\varepsilon > 0$.

Assume ω is a Kähler-type symplectic form on M and $f : \mathbb{W} \to (M, \omega)$ is an ε-tame embedding holomorphic with respect to a Kähler-type complex structure I on M tamed by ω.

Then there exists a neighborhood $\mathcal{U}$ of ω in $SK(M)$ such that for each $\omega' \in \mathcal{U}$ there exists an ε-tame embedding $f' : \mathbb{W} \to (M, \omega')$ holomorphic with respect to a complex structure which is tamed by ω' and isotopic to I.

Proof of Proposition 11.3:

Choose a neighborhood $\mathcal{V}$ of I in $CK(M)$.

Applying Proposition 11.1 to $\mathbb{W}$, we get a neighborhood $\mathcal{U}$ of ω in $S(M)$ such that for each $\omega' \in \mathcal{U}$ there exists a tame embedding $f' : \mathbb{W} \to (M, \omega')$ holomorphic with respect to a complex structure $I' \in \mathcal{V}$ tamed by ω' so that $I' = \phi_* I \in \mathcal{V}$ for some $\phi \in \mathrm{Diff}_0(M)$. In particular, for all $p, q \in \mathbb{Z}_{\geq 0}$,

$$[\omega']_{I'}^{p,q} = [\omega']_{\phi_* I}^{p,q} = [\omega']_I^{p,q}.$$

The cohomology class $[\omega']$ depends continuously on ω' and the Hodge decomposition (with respect to I) of a cohomology class depends continuously on that class. Also, the Kähler cone of (M, I) is an open subset of $H^{1,1}(M; \mathbb{R})$. In view of these facts, we can choose, if necessary, a smaller neighborhood $\mathcal{U}$ and assume without loss of generality that for any $\omega' \in \mathcal{U}$:

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[3]Let us note that in the proof of [EV1, Prop. 9.1] we applied Moser's argument to symplectic forms on the *domain of f*, while here we will apply it to symplectic forms on the *target* of f – i.e., on M.

- The class $[\omega']_{I'}^{1,1} = [\omega']_I^{1,1}$ is Kähler,

- For $a := [\omega]_{I'}^{2,0} + [\omega]_{I'}^{0,2} = [\omega']_I^{2,0} + [\omega']_I^{0,2}$, we have $|\langle a^n, [M] \rangle| < \varepsilon$.

This yields that $f' : \mathbb{W} \to (M, \omega')$ is an ε-tame symplectic embedding holomorphic with respect to the complex structure I' which is tamed by ω' and isotopic to I. ∎

Corollary 11.4

Assume ω is a Kähler-type symplectic form on M and $f : \mathbb{W} \to (M, \omega)$ is a Kähler-type embedding holomorphic with respect to a Kähler-type complex structure I on M compatible ω.

Then, for any $\varepsilon > 0$, there exists a neighborhood $\mathcal{U}$ of ω in $SK(M)$ such that for each $\omega' \in \mathcal{U}$ there exists an ε-tame embedding $f' : \mathbb{W} \to (M, \omega')$ holomorphic with respect to a complex structure which is tamed by ω' and isotopic to I. ∎

Proposition 11.5

Let $\varepsilon > 0$.

Then the function $\omega \mapsto v_{T,\varepsilon}(M, \omega, \mathbb{W})$ on $S(M)$ is lower semicontinuous with respect to the C^1-topology on $S(M)$.

Proof of Proposition 11.5:

Consider an arbitrary $\omega \in S(M)$. Let us prove that $v_{T,\varepsilon}$ is lower semicontinuous at ω.

Let $\delta, \lambda > 0$. Assume $f : \lambda \mathbb{W} \to (M, \omega)$ is an ε-tame embedding holomorphic with respect to a Kähler-type complex structure I on M tamed by ω.

By Proposition 11.3, there exists a neighborhood $\mathcal{U}$ of ω in $SK(M)$ such that for each $\omega' \in \mathcal{U}$ there exists an ε-tame embedding $f' : \mathbb{W} \to (M, \omega')$. Choosing, if necessary, a smaller neighborhood $\mathcal{U}$, we may assume without loss of generality that

$$\frac{\mathrm{Vol}\,(\lambda \mathbb{W}, \omega_0)}{\mathrm{Vol}\,(M, \omega')} > \frac{\mathrm{Vol}\,(\lambda \mathbb{W}, \omega_0)}{\mathrm{Vol}\,(M, \omega)} - \delta, \ \forall \omega' \in \mathcal{U}.$$

This shows that $v_{T,\varepsilon}$ is lower semicontinuous at ω.

Thus, $v_{T,\varepsilon}$ is lower semicontinuous at all $\omega \in S(M)$, meaning that it is a lower semicontinuous function. ∎

Let us now prove Theorem 3.27. For convenience, we restate it here.

Theorem 11.6 (= Theorem 3.27)

Let M be either a torus $\mathbb{T}^{2n}$ or a smooth manifold underlying a K3 surface.

Let ω_1, ω_2 be Kähler-type forms on M so that $\int_M \omega_1^n = \int_M \omega_2^n > 0$ and the forms ω_1, ω_2 are irrational.

Let $W_i \subset \mathbb{R}^{2n}$, $i = 1, \ldots, k$, be compact domains with piecewise-smooth boundary whose interiors contain the origin. Assume that $H^2(W_i; \mathbb{R}) = 0$ for all i. Set $\mathbb{W} := \bigsqcup_{i=1}^k W_i$.

Then, for any $\varepsilon > 0$,

$$v_{T,\varepsilon}(M, \omega_1, \mathbb{W}) = v_{T,\varepsilon}(M, \omega_2, \mathbb{W}).$$

Proof of Theorem 11.6 (= Theorem 3.27):

Without loss of generality, it is enough to proof the theorem in case $\mathrm{Vol}\,(M, \omega_1) = \mathrm{Vol}\,(M, \omega_2) = 1$.

Denote by $SK_1(M)$ the space of Kähler forms on M of total volume 1. Equip $SK_1(M)$ with the C^∞-topology. The group $\mathrm{Diff}^+(M)$ of orientation-preserving diffeomorphisms of M acts on $SK_1(M)$ and the function $\omega \mapsto v_{T,\varepsilon}(M, \omega, \mathbb{W})$ is clearly invariant under the action.

By [EV1, Thm. 9.2] in the torus case and [EV3, Thm. 1.7] in the K3 case, the $\mathrm{Diff}^+(M)$-orbits of ω_1 and ω_2 are dense in $SK_1(M)$. The function $v_{T,\varepsilon}(M, \cdot, \mathbb{W})$ is constant on both orbits. Together with the lower semicontinuity of $v_{T,\varepsilon}(M, \cdot, \mathbb{W})$ proved in Proposition 11.5, this implies that

$$v_{T,\varepsilon}(M, \omega_1, \mathbb{W}) \leq v_{T,\varepsilon}(M, \omega_2, \mathbb{W}),$$

and, by the same token,

$$v_{T,\varepsilon}(M, \omega_1, \mathbb{W}) \geq v_{T,\varepsilon}(M, \omega_2, \mathbb{W}).$$

Thus,

$$v_{T,\varepsilon}(M, \omega_1, \mathbb{W}) = v_{T,\varepsilon}(M, \omega_2, \mathbb{W}),$$

as required. This proves the theorem. ∎

Corollary 11.7

Let M be either a torus $\mathbb{T}^{2n}$ or a smooth manifold underlying a K3 surface. Let $W_i \subset \mathbb{R}^{2n}$, $i = 1, \ldots, k$, be compact domains with piecewise-smooth boundary whose interiors contain the origin.

Assume that $H^2(W_i; \mathbb{R}) = 0$ for all i. Set $\mathbb{W} := \bigsqcup_{i=1}^k W_i$. Let $\varepsilon > 0$.

Finally, assume that for any $\delta > 0$ there exists an irrational Kähler-type form ω_δ on M and $\lambda > 0$ so that $\lambda \mathbb{W}$ admits a Kähler-type embedding into (M, ω_δ) and $\mathrm{Vol}\,(\lambda \mathbb{W}, \omega_0)/\mathrm{Vol}\,(M, \omega_\delta) \geq 1 - \delta$.

Then the following claims hold:

A. If $\omega_\delta = \omega$ for all $\delta > 0$ and ω is irrational, then there exists a dense $\mathrm{Diff}^+(M)$-orbit – the $\mathrm{Diff}^+(M)$-orbit of ω – in the space of Kähler-type symplectic forms of volume $\mathrm{Vol}\,(M, \omega)$ on M such that for any ω' in this orbit we have $v_K(M, \omega', \mathbb{W}) = 1$.

B. There exists a $\mathrm{Diff}^+(M)$-invariant open dense set of Kähler-type symplectic forms on M, depending on $\mathbb{W}$ and ε and containing, in particular, all irrational Kähler-type symplectic forms on M, so that for each ω' in this set $v_{T,\varepsilon}(M, \omega', \mathbb{W}) = 1$.

Proof of Corollary 11.7:

Let us prove part A.

The hypothesis of part A implies that $v_K(M, \omega, \mathbb{W}) = 1$. Since ω is irrational, [EV1, Thm. 9.2] in the torus case and [EV3, Thm. 1.7] in the K3 case yield that the $\mathrm{Diff}^+(M)$-orbit of ω is dense in the space of Kähler-type symplectic forms of volume $\mathrm{Vol}\,(M, \omega)$ on M. Since the function $\omega' \mapsto v_K(M, \omega', \mathbb{W})$ is $\mathrm{Diff}^+(M)$-invariant and $v_K(M, \omega, \mathbb{W}) = 1$, we have $v_K(M, \omega', \mathbb{W}) = 1$ for any ω' in the $\mathrm{Diff}^+(M)$-orbit of ω.

This yields part A of the corollary.

Let us prove part B.

Rescaling, if necessary, the forms ω_δ, we may assume without loss of generality that they all have the same volume – say, volume 1. Since the forms ω_δ are all irrational, Theorem 11.6 implies then that $v_{T,\varepsilon}(M, \omega_\delta, \mathbb{W})$ does not depend on δ. On the other hand, the hypothesis of the corollary implies that $v_{T,\varepsilon}(M, \omega_\delta, \mathbb{W}) \geq 1 - \delta$ for all $\delta > 0$. Hence, $v_{T,\varepsilon}(M, \omega_\delta, \mathbb{W}) = 1$ for all $\delta > 0$. Using again Theorem 11.6 and rescaling the forms, we get that $v_{T,\varepsilon}(M, \omega', \mathbb{W}) = 1$ for any irrational Kähler-type form ω' on M, of any volume. By Proposition 11.3, we get $v_{T,\varepsilon}(M, \omega'', \mathbb{W}) = 1$ also for any Kähler-type form ω'' lying in a sufficiently small neighborhood $U(\omega')$ of any irrational form ω' in $SK(M)$. Let U be the $\mathrm{Diff}^+(M)$-orbit of $\bigcup_{\omega'} U(\omega')$, where the union is taken over all irrational $\omega' \in SK(M)$. Clearly, U is a $\mathrm{Diff}^+(M)$-invariant open dense subset of $SK(M)$, containing all the irrational forms in $SK(M)$, and $v_{T,\varepsilon}(M, \omega', \mathbb{W}) = 1$ holds for any $\omega' \in U$.

This finishes the proof of part B and of the corollary. ∎

Corollary 11.8 (= Corollary 3.29)

Assume:

- M is either $\mathbb{T}^{2n}$ or a smooth manifold underlying a K3 surface.

- $\mathbb{W} := \bigsqcup_{i=1}^{k} B^{2n}(r_i)$ is a disjoint union of k (possibly different) balls.

- $\varepsilon > 0$.

Then the following claims hold:

A. For any irrational Kähler-type symplectic form ω on M we have

$$v_K(M, \omega, \mathbb{W}) = 1,$$

meaning that Kähler-type embeddings of $\lambda \mathbb{W}$ into (M, ω) are unobstructed.

B. There exists a $\mathrm{Diff}^+(M)$-invariant open dense set of Kähler-type symplectic forms on M, depending on $\mathbb{W}$ and ε and containing, in particular, all irrational Kähler-type symplectic forms on M, so that for each ω in this set $v_{T,\varepsilon}(M, \omega, \mathbb{W}) = 1$ – meaning that ε-tame embeddings of $\lambda \mathbb{W}$ into (M, ω) are unobstructed.

Proof of Corollary 11.8 (=Corollary 3.29):

Assume without loss of generality that $n > 1$. (In the case $n = 1$ the result is known – see Remark 3.23).

Part A of the corollary is equivalent to the existence claim proved in Theorem 3.22.

Part B of the corollary follows then from part B of Corollary 11.7. ∎

Now let us prove Corollary 3.30. We recall it here.

Corollary 11.9 (= Corollary 3.30)

Assume:

- $M = \mathbb{T}^{2n}$.

- $\mathbb{W} := \bigsqcup_{i=1}^{k} W_i$ is either a disjoint union of k identical copies of a $2n$-dimensional polydisk

$$B^{2n_1}(R_1) \times \ldots \times B^{2n_l}(R_l),\ n_1 + \ldots + n_l = n,\ R_1, \ldots, R_l > 0, l > 1,$$

  or a disjoint union of k identical copies of a parallelepiped

$$P(e_1, \ldots, e_{2n}) := \left\{ \sum_{j=1}^{2n} s_j e_j, 0 \le s_j \le 1, j = 1, \ldots, 2n \right\},$$

  where $e_1, \ldots, e_{2n}$ is a basis of the vector space $\mathbb{R}^{2n}$.

- $\varepsilon > 0$.

Then the following claims hold:

A. For any positive volume there exists a dense $\mathrm{Diff}^+(\mathbb{T}^{2n})$-orbit (of an irrational Kähler-type symplectic form depending on $\mathbb{W}$) in the space of Kähler-type symplectic forms of that volume on $\mathbb{T}^{2n}$ such that for any ω' in this orbit we have $\nu_K(\mathbb{T}^{2n}, \omega', \mathbb{W}) = 1$ – or, in other words, Kähler-type embeddings $\lambda \mathbb{W} \to (\mathbb{T}^{2n}, \omega')$ are unobstructed.

B. There exists a $\mathrm{Diff}^+(\mathbb{T}^{2n})$-invariant open dense set of Kähler-type symplectic forms on $\mathbb{T}^{2n}$, depending on $\mathbb{W}$ and ε and containing, in particular, all irrational Kähler-type symplectic forms on $\mathbb{T}^{2n}$, so that for each ω' in this set $\nu_{T,\varepsilon}(\mathbb{T}^{2n}, \omega', \mathbb{W}) = 1$ – or, in other words, ε-tame embeddings $\lambda \mathbb{W} \to (\mathbb{T}^{2n}, \omega')$ are unobstructed.

Proof of Corollary 11.9 (=Corollary 3.30):

Assume without loss of generality that $n > 1$. (In the case $n = 1$ the result is known – see Remark 3.23).

Let us consider the case where $\mathbb{W}$ is the disjoint union of k identical copies of the polydisk $B^{2n_1}(R_1) \times \ldots \times B^{2n_l}(R_l)$. In this case the proof is the following straightforward modification of the proof of [EV1, Cor. 3.3].

Assume, without loss of generality, that $\mathrm{Vol}(\mathbb{W}, \omega_0) = 1$. For each $m \in \mathbb{Z}_{>0}$ denote by Ω_m the standard symplectic form on $\mathbb{R}^{2m}$. For each $i = 1, \ldots, l$ set

$$v_i := \mathrm{Vol}(B^{2n_i}(R_i), \Omega_{n_i}) = \pi^{n_i} n_i! R_i^{2n_i}.$$

Note that

$$\mathrm{Vol}(\mathbb{W}, \Omega_{2n}) = k\mathrm{Vol}(B^{2n_1}(R_1) \times \ldots B^{2n_l}(R_l), \Omega_{2n}) = kN \prod_{i=1}^{l} v_i,$$

where

$$N := \frac{n!}{n_1! \cdot \ldots \cdot n_l!}.$$

Since $\mathrm{Vol}(\mathbb{W}, \omega_0) = 1$, we have

$$kNv_1 \cdot \ldots \cdot v_l = 1.$$

For each $m \in \mathbb{Z}_{>0}$ and $w_1, \ldots, w_m \in \mathbb{R}_{>0}$, set

$$\bar{\mathbf{w}} := (w_1, \ldots, w_m) \in \mathbb{R}^m,$$

$$v_{\bar{\mathbf{w}}} := m! w_1 \cdot \ldots \cdot w_m,$$

and denote by $\omega_{\bar{\mathbf{w}}}$ the symplectic form

$$\omega_{\bar{\mathbf{w}}} := \sum_{i=1}^{m} w_i dp_i \wedge dq_i$$

on the torus $\mathbb{T}^{2m} = \mathbb{R}^{2m}/\mathbb{Z}^{2m}$. Note that $\mathrm{Vol}(\mathbb{T}^{2m}, \omega_{\bar{\mathbf{w}}}) = v_{\bar{\mathbf{w}}}$.

Set

$$k_1 := k, k_2 = \ldots = k_l := 1,$$

so that

$$k = k_1 \cdot \ldots \cdot k_l.$$

One can choose $\bar{\mathbf{w}}_i \in \mathbb{R}^{2n_i}$, $i = 1, \ldots, l$, so that the following conditions hold:

(i) $v_{\bar{\mathbf{w}}_i} = k_i v_i$ for all $i = 1, \ldots, l$.

(ii) The vector $\bar{\mathbf{w}} := (\bar{\mathbf{w}}_1, \ldots, \bar{\mathbf{w}}_k) \in \mathbb{R}^n$ is not proportional to a vector with rational coordinates.

(iii) For each $i = 1, \ldots, l$ such that $n_i > 1$ the vector $\bar{\mathbf{w}}_i \in \mathbb{R}^{n_i}$ is not proportional to a vector with rational coordinates.

Conditions (i) and (iii) can be achieved since, by our assumption, $n > 1$, and since for any $m > 1$ and $C > 0$ the subset of the set

$$\{(w_1, \ldots, w_m) \in \mathbb{R}^m \mid w_1, \ldots, w_m > 0, w_1 \cdot \ldots \cdot w_m = C\}$$

formed by the vectors that are not proportional to a vector with rational coordinates is dense.

Consider the symplectic form $\omega_{\bar{\mathbf{w}}}$ on $\mathbb{T}^{2n}$ – it is of Kähler type (since it is a linear symplectic form) and irrational (because of condition (ii)). Note that, by condition (i),

$$\int_{\mathbb{T}^{2n}} \omega_{\bar{\mathbf{w}}}^n = N \prod_{i=1}^{l} v_{\bar{\mathbf{w}}_i} = N \prod_{i=1}^{l} k_i v_i =$$

$$= Nk \prod_{i=1}^{l} k_i v_i = \int_{\mathbb{T}^{2n}} \omega^n = 1.$$

Also note that, by condition (ii), for any $0 < \lambda < 1$ for all i

$$\mathrm{Vol}(\mathbb{T}^{2n_i}, \omega_{\bar{\mathbf{w}}_i}) = v_{\bar{\mathbf{w}}_i} = k_i v_i > \lambda k_i v_i = \mathrm{Vol}\left(\lambda \bigsqcup_{k_i}(B^{2n_i}(R_i), \Omega_{n_i})\right),$$

where $\bigsqcup_{k_i}$ denotes the disjoint union of k_i equal copies of the ball. By condition (iii), for each $i = 1, \ldots, l$, such that $n_i > 1$, the form $\omega_{\bar{\mathbf{w}}_i}$ on $\mathbb{T}^{2n_i}$ is irrational. Therefore, by Theorem 3.22 and Remark 3.23, there exists a Kähler-type embedding

$$f_i : \bigsqcup_{k_i} \left(\lambda B^{2n_i}(R_i), \Omega_{n_i} \right) \to \left(\mathbb{T}^{2n_i}, \omega_{\bar{\mathbf{w}}_i} \right).$$

Accordingly, the direct product of all such embeddings f_i, $i = 1, \ldots, l$, is a Kähler-type embedding

$$f : (\lambda \mathbb{W}, \Omega_{2n}) \to (\mathbb{T}^{2n}, \omega_{\bar{\mathbf{w}}}),$$

where $\lambda \mathbb{W}$ is the disjoint union of $k_1 \cdot \ldots \cdot k_l = k$ copies of $\lambda \left(B^{2n_1}(R_1) \times \ldots \times B^{2n_l}(R_l) \right)$. Thus, we have found an irrational Kähler-type form $\omega_{\bar{\mathbf{w}}}$ of volume 1 on $\mathbb{T}^{2n}$ such that for any $0 < \lambda < 1$ there exists a Kähler-type embedding of $(\lambda \mathbb{W}, \Omega_{2n} = \omega_0)$ into $(\mathbb{T}^{2n}, \omega_{\bar{\mathbf{w}}})$. Now Corollary 11.7 yield the claims of parts A and B of Corollary 11.9 in the case where $\mathbb{W}$ is the disjoint union of k equal polydisks.

Now let us consider the case where $\mathbb{W}$ is the disjoint union of k identical copies of a parallelepiped

$$P(e_1, \ldots, e_{2n}) := \left\{ \sum_{j=1}^{2n} s_j e_j, 0 \le s_j \le 1, j = 1, \ldots, 2n \right\},$$

where $e_1, \ldots, e_{2n}$ is a basis of the vector space $\mathbb{R}^{2n}$.

Note that for any $0 < \lambda < 1$, putting k copies of $\lambda P(e_1, e_2, \ldots, e_{2n})$, alongside each other in the e_1-direction we get a Kähler-type embedding of the disjoint union of k identical copies of $\lambda P(e_1, e_2, \ldots, e_{2n})$ into $P(ke_1, e_2, \ldots, e_{2n})$. Considering the composition of this Kähler-type embedding with Kähler-type, or ε-tame, embeddings of $P(ke_1, e_2, \ldots, e_{2n})$ into $\mathbb{T}^{2n}$, we readily get that it suffices to prove the corollary for just one parallelepiped $P(e_1, \ldots, e_{2n})$ (i.e., for the case where $k = 1$).

Therefore let us prove the corollary for one parallelepiped $P(e_1, \ldots, e_{2n})$.

Consider the $2n \times 2n$-matrix $Z(e_1, e_2, \ldots, e_{2n})$ whose columns are the vectors $e_1, e_2, \ldots, e_{2n}$ spanning $P(e_1, \ldots, e_{2n})$.

For each $m \in \mathbb{Z}_{>0}$ let $Mat_m(\mathbb{R})$, respectively $Mat_m(\mathbb{Q})$, be the space of real, respectively rational, $m \times m$-matrices.

Let $S \subset Mat_{2n}(\mathbb{R})$ be the set of matrices $Z \in Mat_{2n}(\mathbb{R})$ such that for all $\lambda \in \mathbb{R}_{>0}$

$$\lambda Z^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} Z \notin Mat_{2n}(\mathbb{Q}).$$

(Here I_n is the identity $n \times n$-matrix.)

Lemma 11.10

The set S is dense in $Mat_{2n}(\mathbb{R})$.

Proof of Lemma 11.10:

If Z has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A, B, C, D \in Mat_n(\mathbb{R})$, then

$$\lambda Z^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} Z = \begin{pmatrix} A^t C - C^t A & A^t D - C^t B \\ B^t C - D^t A & B^t D - D^t B \end{pmatrix}.$$

Since $n > 1$, the set of $Z \in Mat_{2n}(\mathbb{R})$ such that $A, C \in Mat_n(\mathbb{Q})$ and $B^t D - D^t B \notin Mat_n(\mathbb{Q})$ is dense in $Mat_{2n}(\mathbb{R})$ and also lies in S. Thus, S is dense in $Mat_{2n}(\mathbb{R})$.

This proves the lemma. ∎

Since $S \subset Mat_{2n}(\mathbb{R})$ is a dense set, it suffices to prove the corollary only in the case when $Z(e_1, e_2, \ldots, e_{2n})$ lies in S.

Indeed, if $\mathrm{Vol}\left(\lambda P(e_1, \ldots, e_{2n}), \omega_0\right) < \mathrm{Vol}(\mathbb{T}^{2n}, \omega)$ for some $\lambda > 0$ and some Kähler-type form ω on $\mathbb{T}^{2n}$, then there exists a basis $e'_1, \ldots, e'_{2n}$ of $\mathbb{R}^{2n}$ such that $Z(e'_1, e'_2, \ldots, e'_{2n}) \in S$, the parallelepiped $P(e'_1, \ldots, e'_{2n})$ contains the parallelepiped $P(e_1, \ldots, e_{2n})$ and has only slightly bigger volume than it, so that $\mathrm{Vol}(\lambda P(e'_1, \ldots, e'_{2n}), \omega_0) < \mathrm{Vol}(\mathbb{T}^{2n}, \omega)$. If $\lambda P(e'_1, \ldots, e'_{2n})$ admits a Kähler-type, or respectively an ε-tame, embedding into $(\mathbb{T}^{2n}, \omega)$, then the composition of this embedding with the inclusion $\lambda P(e_1, \ldots, e_{2n}) \to \lambda P(e'_1, \ldots, e'_{2n})$ is a Kähler-type, or respectively an ε-tame, embedding of $\lambda P(e_1, \ldots, e_{2n})$ into $(\mathbb{T}^{2n}, \omega)$.

Now, in view of Corollary 11.7, it suffices to show that for any $P(e_1, \ldots, e_{2n})$, such that $Z(e_1, e_2, \ldots, e_{2n}) \in S$, there exists an irrational Kähler-type symplectic form ω on $\mathbb{T}^{2n}$ satisfying the following condition: for any $\lambda > 0$, such that $\mathrm{Vol}\left(\lambda P(e_1, \ldots, e_{2n})\right) < \mathrm{Vol}(\mathbb{T}^{2n}, \omega)$, there exists a Kähler-type embedding $P(e_1, \ldots, e_{2n}) \to (\mathbb{T}^{2n}, \omega)$.

Pick $P(e_1, \ldots, e_{2n})$, such that $Z(e_1, e_2, \ldots, e_{2n}) \in S$ and let us construct such an ω.

Without loss of generality, assume that $\mathrm{Vol}\left(\lambda P(e_1, \ldots, e_{2n}), \omega_0\right) = 1$.

Consider the integral lattice $\Lambda := \mathrm{Span}_{\mathbb{Z}}\{e_1, \ldots, e_{2n}\} \subset \mathbb{R}^{2n}$. The fundamental domain of Λ is exactly $P(e_1, \ldots, e_{2n})$. The space $\mathbb{R}^{2n}/\Lambda$ is a torus and the form ω_0 on $\mathbb{R}^{2n}$ induces a Kähler-type symplectic form on $\mathbb{R}^{2n}/\Lambda$ of total volume 1 that will be also denoted by ω_0. The matrix $Z(e_1, e_2, \ldots, e_{2n})$ induces a linear map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ mapping $\mathbb{Z}^{2n}$ to Λ. This map induces a diffeomorphism $F : \mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n} \to \mathbb{R}^{2n}/\Lambda$. Since $Z(e_1, e_2, \ldots, e_{2n}) \in S$, the form $\omega := F^*\omega_0$ is an irrational Kähler-type form on $\mathbb{T}^{2n}$ of total volume 1.

For any $0 < \lambda < 1$ the inclusion $\lambda P(e_1, \ldots, e_{2n}) \to (\mathbb{R}^{2n}/\Lambda, \omega_0)$ is a Kähler-type embedding. This embedding is holomorphic with respect to the complex structure on $\mathbb{R}^{2n}/\Lambda$ induced by the standard complex structure J_0 on $\mathbb{R}^{2n}$. The composition of this embedding with F^{-1} is then a Kähler-type embedding $\lambda P(e_1, \ldots, e_{2n}) \to (\mathbb{T}^{2n}, \omega)$ holomorphic with respect to the linear complex structure F^*J_0 on $\mathbb{T}^{2n}$.

Thus, for any $0 < \lambda < 1$ there exists a Kähler-type embedding $\lambda P(e_1, \ldots, e_{2n}) \to (\mathbb{R}^{2n}/\Lambda, \omega_0)$, provided $Z(e_1, e_2, \ldots, e_{2n}) \in S$.

This finishes the proof of the corollary in the case where $\mathbb{W}$ is the disjoint union of k identical copies of a parallelepiped. ∎

Let us prove Corollary 3.31. We recall it here.

Corollary 11.11 (= Corollary 3.31)
  Assume:

- *M* is a smooth manifold underlying a K3 surface.

- $\mathbb{W} := \bigsqcup_{i=1}^{k} W_i$ is a disjoint union of *k* identical copies of a parallelepiped

$$P(e_1, \ldots, e_4) := \left\{ \sum_{j=1}^{4} s_j e_j, 0 \leq s_j \leq 1, j = 1, \ldots, 4 \right\},$$

  where $e_1, \ldots, e_4$ is a basis of the vector space $\mathbb{R}^4$.

- $\varepsilon > 0$.

Then there exists a $\mathrm{Diff}^+(M)$-invariant open dense set of Kähler-type symplectic forms on *M*, depending on $\mathbb{W}$ and ε and containing, in particular, all irrational Kähler-type symplectic forms on *M*, so that for each ω' in this set $v_{T,\varepsilon}(M, \omega', \mathbb{W}) = 1$, meaning that ε-tame embeddings of $\lambda \mathbb{W}$ into (M, ω') are unobstructed.

Proof of Corollary 11.11 (=Corollary 3.31):
  Consider the integral lattice $\Lambda := \mathrm{Span}_{\mathbb{Z}}\{2e_1, 2e_2, 2e_3, 2e_4\} \subset \mathbb{R}^4$. The parallelepiped $P(e_1, \ldots, e_4)$ is the fundamental domain of the lattice $\Lambda/2$.
  Recall the following construction by Kummer of a K3 surface – see e.g. [Bea2, p.55].
  Consider the torus $\mathbb{T}^4 := \mathbb{R}^4/\Lambda$ and let $\pi : \mathbb{R}^4 \to \mathbb{T}^4$ be the projection. Equip the torus $\mathbb{T}^4$ with the complex structure J_0 and the symplectic structure ω_0 induced by the standard complex and symplectic structures on $\mathbb{R}^4$. The reflection with respect to the origin of $\mathbb{R}^4$ defines an involution of $\mathbb{R}^4$, which in turn defines a holomorphic and symplectic involution $\iota : (\mathbb{T}^4, J_0, \omega_0) \to (\mathbb{T}^4, J_0, \omega_0)$. The involution ι has 16 fixed points – these are exactly the points of $\mathbb{T}^4$ that are the images under π of the points in $\mathbb{R}^4$ all of whose coordinates with respect to the basis $\{2e_1, 2e_2, \ldots, 2e_{2n}\}$ are either 0 or 1/2. Let $(\tilde{\mathbb{T}}^4, \widetilde{J_0})$ be the complex blow-up of $(\mathbb{T}^4, J_0)$ at these 16 points. The involution ι lifts to a holomorphic involution $\tilde{\iota} : \tilde{\mathbb{T}}^4 \to \tilde{\mathbb{T}}^4$. Then $M := \tilde{\mathbb{T}}^4/\tilde{\iota}$, equipped with the complex structure induced by $\widetilde{J_0}$, is a complex manifold which is, in fact, a K3 surface (called a Kummer surface). Denote by I the complex structure on M induced by $\widetilde{J_0}$.
  The complex manifold (M, I) can also be obtained as follows: The space $\mathbb{T}^4/\iota$ is a complex orbifold with 16 isolated singular orbifold points $x_1, \ldots, x_{16}$. Denote by $\bar{J_0}$ the complex structure on $\mathbb{T}^4/\iota$ induced by J_0. Then (M, I) is the smooth complex manifold obtained by the complex blow-up of $(\mathbb{T}^4/\iota, \bar{J_0})$ at $x_1, \ldots, x_{16}$. Let $pr : \mathbb{T}^4 \to \mathbb{T}^4/\iota$ and $\Pi : M \to \mathbb{T}^4/\iota$ be the natural projections. The form ω_0 on $\mathbb{T}^4$ is ι-invariant and induces an orbifold Kähler form $\bar{\omega}_0$ on the complex orbifold $\mathbb{T}^4/\iota$ which is a smooth Kähler form outside the singularities.
  The points $x_1, \ldots, x_{16} \in \mathbb{T}^4/\iota$ lie on the boundary of $(pr \circ \pi)(P(e_1, \ldots, e_4))$. The interior $Int\, P(e_1, \ldots, e_4)$ of $P(e_1, \ldots, e_4)$ projects diffeomorphically onto its image under $pr \circ \pi$ and is an open dense subset of $\mathbb{T}^4/\iota$. Consequently, $pr \circ \pi : Int\, P(e_1, \ldots, e_4) \to \mathbb{T}^4/\iota$ is an embedding which is holomorphic with respect to $\bar{J_0}$ and symplectic with respect to $\bar{\omega}_0$. The map $F := \Pi^{-1} \circ pr \circ \pi : Int\, P(e_1, \ldots, e_4) \to M$ is then an I-holomorphic embedding.

Let $0 < \lambda < 1$. Move $\lambda P(e_1, \ldots, e_4)$ by a parallel translation into the interior of $P(e_1, \ldots, e_4)$ and then map it by F into M. Denote the resulting map by $f : \lambda P(e_1, \ldots, e_4) \to M$. It is an I-holomorphic embedding.

Following the proof of [EV2, Thm. 3.6] (or, alternatively, using Proposition 5.16), one can find rational cohomology classes $b_i \in H^2(\Pi^{-1}(x_i); \mathbb{Q})$, $i = 1, \ldots, 16$, so that for any $\varepsilon > 0$ and any sufficiently small $0 < \varepsilon_1, \ldots, \varepsilon_{16} < \varepsilon$, there exist a Kähler form ω_ε on (M, I) and sufficiently small neighborhoods $U_{x_i, \varepsilon, \lambda}$ of $\Pi^{-1}(x_i)$, $i = 1, \ldots, 16$, in M with the following properties:

- $U_{x_i, \varepsilon, \lambda} \cap f(\lambda P(e_1, \ldots, e_4)) = \emptyset$, $i = 1, \ldots, 16$.

- The form ω_ε coincides with $\Pi^* \overline{\omega}_0$ outside $\bigcup_{i=1}^{16} U_{x_i, \varepsilon, \lambda}$.

- $[\omega_\varepsilon] = \Pi^*[\overline{\omega}_0] + \sum_{i=1}^{16} \varepsilon_i b_i$.

- $\mathrm{Vol}(\mathbb{T}^4 / \iota, \overline{\omega}_0) > (1 - \varepsilon) \mathrm{Vol}(M, \omega_\varepsilon)$.

If the vector $(\varepsilon_1, \ldots, \varepsilon_{16}) \in \mathbb{R}^{16}$ is not a real multiple of a vector in $\mathbb{Q}^{16}$, the cohomology class

$$[\omega_\varepsilon] = \Pi^*[\overline{\omega}_0] + \sum_{i=1}^{16} \varepsilon_i b_i \in H^2(M; \mathbb{R})$$

is a not a real multiple of a rational cohomology class, meaning that ω_ε is irrational.

Our construction implies that $f : \lambda P(e_1, \ldots, e_4) \to (M, \omega_\varepsilon)$ is a Kähler-type embedding which is holomorphic with respect to I and

$$\mathrm{Vol}(\lambda P(e_1, \ldots, e_4), \omega_0) = \lambda^4 \mathrm{Vol}(P(e_1, \ldots, e_4), \omega_0) =$$

$$= \lambda^4 \mathrm{Vol}(\mathbb{T}^4 / \iota, \overline{\omega}_0) > \lambda^4 (1 - \varepsilon) \mathrm{Vol}(M, \omega_\varepsilon).$$

For any $\delta > 0$ we can choose $\lambda < 1$ sufficiently close to 1 and $\varepsilon > 0$ sufficiently small so that $\lambda^4 (1 - \varepsilon) > 1 - \delta$. Then

$$f : \lambda P(e_1, \ldots, e_4) \to (M, \omega_\varepsilon)$$

is a Kähler-type embedding and

$$\mathrm{Vol}(\lambda P(e_1, \ldots, e_4), \omega_0) / \mathrm{Vol}(M, \omega_\varepsilon) \geq 1 - \delta.$$

By Corollary 11.7, this implies the corollary. ∎

12 Hyperkähler case – proofs

As we mentioned in Section 3.5, the results of Section 3.4 generalize to the IHS-hyperkähler case. In this section we explain how the proofs of the results in the K3 case should be adjusted for general IHS-hyperkähler manifolds.

Let M be a closed connected and simply-connected manifold carrying IHS-hyperkähler structures.

Recall from Section 3.5 that in the IHS-hyperkähler case the words "Kähler-type" should be replaced everywhere by the words "IHS-hyperkähler-type". **Accordingly, in this section we will use the same notation as in Section 10.2 for IHS-hyperkähler-type objects as for Kähler-type objects. In particular, $CK(M)$ will denote the space of IHS-hyperkähler-type complex structures, $\mathrm{Teich}(M) = CK(M)/\mathrm{Diff}_0(M)$ the Teichmüller space of IHS-hyperkähler-type complex structures etc.; all the symplectic forms and complex structures will be assumed to be of IHS-hyperkähler-type.**

For general IHS-hyperkähler manifolds:

- The Teichmüller space $\mathrm{Teich}(M) = CK(M)/\mathrm{Diff}_0(M)$ has the structure of a (non-Hausdorff) complex manifold for the same reasons as in Section 10.2 (see [Cat]).

- Instead of the intersection form one should consider the Bogomolov-Beauville-Fujiki (BBF) symmetric bilinear form q on $H^2(M;\mathbb{R})$ (see [Bea1], [Fu]) – for K3 surfaces the intersection form and the BBF form coincide, up to a multiplicative factor. The BBF form is a primitive integral quadratic form of signature $(3, b_2 - 3)$, where $b_2 = \dim_{\mathbb{R}} H^2(M;\mathbb{R}) > 0$. The space $Gr_{++}(H^2(M;\mathbb{R}))$ is defined then using the BBF form q.

  We will use $\perp$ to denote orthogonal complements with respect to q.

- Part A of the Torelli theorem (Theorem 10.12) remains true for IHS-hyperkähler manifolds – see [Ver6].

- In the definition of $S(M, I)$, for $I \in CK(M)$, (-2)-classes should be replaced by the so-called MBM cohomology classes in $H^2(M;\mathbb{Z})$ defined in [AmV1]. With $S(M, I)$ defined in this way, Theorem 10.13 remains true for general IHS-hyperkähler manifolds.

- The analogues of Theorem 10.14 and Proposition 10.16 remain true for general IHS-hyperkähler manifolds.

Let ω be an IHS-hyperkähler-type symplectic form on M.

The set D_ω is then defined literally as in Section 10.2 with (-2)-classes being replaced by the MBM classes. Proposition 10.17 and Proposition 10.18 remain true in this case – in their proofs one should replace 19 by $b_2 - 3$, 18 by $b_2 - 4$, and (-2)-classes by MBM classes [AmV1].

Next, let us state the analogue of Proposition 10.20 in the IHS-hyperkähler case.

Proposition 12.1

The symplectic form ω is compatible with a Campana-simple IHS-hyperkähler-type complex structure on M if and only if ω is irrational. In this case, for any connected component $\mathcal{C}_0$ of $CK(M)$ compatible with ω the set $\mathrm{Teich}_{\mathcal{C}_0}(M, \omega) \cap \mathrm{Teich}^s(M)$ of Campana-simple structures lying in $\mathrm{Teich}_{\mathcal{C}_0}(M, \omega)$ is a dense connected subset of $\mathrm{Teich}_{\mathcal{C}_0}(M, \omega)$.

The proof of Proposition 12.1 that we are going to give will not be a direct generalization of the proof of Proposition 10.20 and requires a number of preliminaries.

Consider an IHS-hyperkähler-type structure $\mathfrak{h} = \{I_1, I_2, I_3, \omega_1, \omega_2, \omega_3\}$ on M. For any $a, b, c \in \mathbb{R}$, $a^2 + b^2 + c^2 = 1$, the tensor $aI_1 + bI_2 + cI_3$ is a complex structure of hyperkähler type on M. We will call

any such $aI_1 + bI_2 + cI_3$ **a complex structure induced by** $\mathfrak{h}$. Clearly, all complex structures induced by $\mathfrak{h}$ lie in the same connected component of $CK(M)$ – we will say that this connected component is **favored** by $\mathfrak{h}$.

Following [Ver2], we say that a closed subset $Z \subset M$ is **trianalytic** (with respect to $\mathfrak{h}$), if Z is a complex subvariety of (M, I) for each complex structure I induced by $\mathfrak{h}$.

Theorem 12.2 [Ver2, Thm. 4.1]

Let $a \in H^*(M; \mathbb{Q})$, $a \neq 0$.

Then a is Poincaré-dual to the fundamental class $[Z]$ of a trianalytic $Z \subset M$ if and only if a is invariant with respect to the action of any induced complex structure on $H^*(M; \mathbb{Q})$. ∎

Let $\mathcal{C}_0$ be a connected component of $CK(M)$.

Let Z be a closed subvariety of M for a certain (IHS-hyperkähler-type) complex structure on M lying in $\mathcal{C}_0$. Following [Kur, SolV], we say that Z is **absolutely trianalytic** (with respect to $\mathcal{C}_0$) if the cohomology class Poincaré-dual to the fundamental class of Z is of type (p, p) for all complex structures $I \in \mathcal{C}_0$. By Theorem 12.2, such a manifold is trianalytic with respect to any IHS-hyperkähler structure favoring $\mathcal{C}_0$.

Given $a \in H^*(M; \mathbb{Q})$, $a \neq 0$, denote by $\mathtt{T}_{\mathcal{C}_0, a} \subset \mathtt{Teich}_{\mathcal{C}_0}(M)$ the set of all $[I] \in \mathtt{Teich}_{\mathcal{C}_0}(M)$ such that $a \in H^{1,1}_I(M; \mathbb{R})$. It follows e.g. from the Torelli theorem for hyperkähler manifolds [Ver6] that $\mathtt{T}_{\mathcal{C}_0, a}$ is an analytic submanifold of $\mathtt{Teich}_{\mathcal{C}_0}(M)$ and $\mathsf{Per}(\mathtt{T}_{\mathcal{C}_0, a}) = Gr_{++}(a^\perp)$.

For a real vector space W equipped with a non-degenerate indefinite quadratic form we denote by $SO^+(W)$ the identity component of $SO(W)$. Similarly, $SO^+(p, q)$ stands for the identity component of the Lie group $SO(p, q)$.

For the proof of Proposition 12.1 we will need the following algebraic lemma.

Lemma 12.3

Let W be a real vector space equipped with a non-degenerate quadratic form q of signature $(2, l)$.

Consider the set $\mathcal{F}(W)$ of $F \in SO^+(W)$ fixing pointwise a linear subspace $U \subset W$, $\dim_{\mathbb{R}} U = \dim_{\mathbb{R}} W - 2$ whose orthogonal complement is a positive 2-dimensional plane in W.

Then the set $\mathcal{F}(W)$ generates $SO^+(W)$.

Proof of Lemma 12.3:

Let N be the subgroup of $SO^+(W)$ generated by $\mathcal{F}(W)$. Clearly, N is normal.

Assume $l \neq 2$. Then the group $SO^+(W)$ is a connected simple Lie group (in the sense of Lie theory). Therefore it follows from [Rag] that any normal subgroup of $SO^+(W)$ is either the whole $SO^+(W)$ or is contained in the center of $SO^+(W)$ (which is discrete). Hence, $N = SO^+(W)$, meaning that $\mathcal{F}(W)$ generates $SO^+(W)$ for $l \neq 2$.

Let us consider the case $l = 2$ – i.e., the case when q has signature $(2, 2)$. Identify (W, q) with the space $Mat_2(\mathbb{R})$ of real 2×2-matrices equipped with the bilinear form $\langle A, B \rangle = -tr(AJBJ)$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Consider the following action of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ on $W \cong Mat_2(\mathbb{R})$: each $h_1 \times h_2 \in SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ acts on $A \in Mat_2(\mathbb{R})$ by $A \mapsto h_1 A h_2^{-1}$. This action preserves the bilinear form and

induces a homomorphism $P : SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \to SO^+(W)$ whose kernel is $\{\pm Id\}$. The subgroup $P^{-1}(N) \subset SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ is normal. The Lie group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ is a connected semi-simple Lie group. Therefore it follows from [Rag] that any normal subgroup of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ which is not the whole $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ is contained either in $SL(2,\mathbb{R}) \times \{\pm Id\}$ or in $\{\pm Id\} \times SL(2,\mathbb{R})$.

Let us show that $P^{-1}(\mathcal{F}(W)) \not\subseteq SL(2,\mathbb{R}) \times \{\pm Id\}$ and consequently $P^{-1}(N) \not\subseteq SL(2,\mathbb{R}) \times \{\pm Id\}$ (the claim $P^{-1}(N) \not\subseteq \{\pm Id\} \times SL(2,\mathbb{R})$ is proved similarly). Indeed, if an element $h \times Id \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ is sent by P into $\mathcal{F}(W)$, then its action on $Mat_2(\mathbb{R})$ fixes a two-dimensional subspace of $Mat_2(\mathbb{R})$. This easily implies that $h \in SL(2,\mathbb{R})$ has a two-dimensional eigenspace for the eigenvalue 1, meaning that $h = Id$. Hence, $P^{-1}(\mathcal{F}(W)) \not\subseteq SL(2,\mathbb{R}) \times \{\pm Id\}$ and consequently $P^{-1}(N) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ and $N = SO^+(W)$, meaning that $\mathcal{F}(W)$ generates $SO^+(W)$ also for $l = 2$.

This finishes the proof of the lemma. ∎

Now we are ready to prove Proposition 12.1.

Proof of Proposition 12.1:

If the cohomology class $[\omega]$ is rational, then by part 1 of Remark 9.1, there are no Campana-simple complex structures compatible with ω.

Assume that ω is irrational and $\mathcal{C}_0$ is a connected component of $CK(M)$ compatible with ω.

The analogue of Proposition 10.18 in the IHS-hyperkähler case implies that $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$ is a dense connected subset of the analytic subvariety $\mathtt{T}_{\mathcal{C}_0,[\omega]}$ of $\mathtt{Teich}_{\mathcal{C}_0}(M)$. Consequently, for each $a \in H^*(M;\mathbb{Q})$, $a \neq 0$, there are only two possible options:

(1) $\mathtt{T}_{\mathcal{C}_0,a} \cap \mathtt{T}_{\mathcal{C}_0,[\omega]} = \mathtt{T}_{\mathcal{C}_0,[\omega]}$, and consequently $\mathtt{T}_{\mathcal{C}_0,a} \cap \mathtt{Teich}_{\mathcal{C}_0}(M, \omega) = \mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$,

(2) $\mathtt{T}_{\mathcal{C}_0,a} \cap \mathtt{T}_{\mathcal{C}_0,[\omega]}$ is a proper (possibly empty) complex subvariety of $\mathtt{T}_{\mathcal{C}_0,[\omega]}$ and consequently $\mathtt{T}_{\mathcal{C}_0,a} \cap \mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$ is a proper complex subvariety of $\mathtt{Teich}_{\mathcal{C}_0}(M, \omega)$.

Lemma 12.4

Assume $Z \subset M$ is a complex subvariety of (M,I) (some complex structure $I \in \mathcal{C}_0$) which is not absolutely trianalytic with respect to $\mathcal{C}_0$. Let $a \in H^{even}(M;\mathbb{Q})$ be the cohomology class Poincaré-dual to $[Z]$.

Then $\mathtt{T}_{\mathcal{C}_0,a} \cap \mathtt{T}_{\mathcal{C}_0,[\omega]} \not\subseteq \mathtt{T}_{\mathcal{C}_0,[\omega]}$ (meaning that option (2) holds for a).

Proof of Lemma 12.4:

We start with a number of preparations.

Given $I \in \mathcal{C}_0$ and $t \in \mathbb{R}$, consider the Hodge rotation map $H^*(M;\mathbb{C}) \to H^*(M;\mathbb{C})$ that acts on each $H_I^{p,q}(M;\mathbb{C})$ as multiplication by $e^{\sqrt{-1}(p-q)t}$. Such a map preserves $H^*(M;\mathbb{R})$ and its fixed point set is exactly the union of $H_I^{p,p}(M;\mathbb{C})$ for all p. We will view the action of this map on $H^*(M;\mathbb{R})$ as an element of $GL(H^*(M;\mathbb{R}))$ and denote it by $\rho_I(t) : H^*(M;\mathbb{R}) \to H^*(M;\mathbb{R})$. The restriction of each $\rho_I(t)$ to $H^2(M;\mathbb{R})$ preserves the BBF form q and lies in $SO^+(H^2(M;\mathbb{R}))$. The maps $\rho_I(t)|_{H^2(M;\mathbb{R})}, t \in \mathbb{R}$, are exactly the maps that act trivially on the subspace $H_I^{1,1}(M;\mathbb{R})$ and rotate its orthogonal complement, which is the positive 2-dimensional plane $(H_I^{2,0}(M;\mathbb{C}) \oplus H_I^{0,2}(M;\mathbb{C})) \cap H^2(M;\mathbb{R})$.

Denote by $G \subset GL(H^{even}(M;\mathbb{R}))$ the subgroup generated by $\rho_I(t)|_{H^{even}(M;\mathbb{R})}$ for all $I \in \mathcal{C}_0$ and $t \in \mathbb{R}$. The restriction to $H^2(M;\mathbb{R})$ defines a homomorphism $G \to SO^+(H^2(M;\mathbb{R}))$. It follows from [Ver3, Thm.

2.2] and [Ver1, Corollary 8.2] that this homomorphism is, in fact, an isomorphism. (On $H^{odd}(M)$ the group generated by $\rho_I(t)$, for all $I \in \mathcal{C}_0$ and $t \in \mathbb{R}$, acts as the identity component of $\mathrm{Spin}(H^2(M;\mathbb{R}))$, with the center of the spinor group acting as $-\mathrm{Id}$.)

Denote by $G_{[\omega]} \subset G$ the subgroup generated by all Hodge rotation maps $\rho_I(t)$, $t \in \mathbb{R}$, $I \in \mathcal{C}_0$, $[I] \in \mathrm{T}_{\mathcal{C}_0,[\omega]}$. Each such $\rho_I(t)$ fixes $[\omega]$ because $[\omega] \in H_I^{1,1}(M;\mathbb{R})$. Hence, $[\omega]$ is fixed by $G_{[\omega]}$. Thus, $G_{[\omega]}$ lies in the stabilizer of $[\omega]$ in $G \cong SO^+\left(H^2(M;\mathbb{R})\right)$. The latter stabilizer is isomorphic to $SO^+([\omega]^\perp)$. Note that the restriction of the BBF form q to $[\omega]^\perp \subset H^2(M;\mathbb{R})$ has signature $(2, b_2 - 3)$. Now Lemma 12.3 yields that the maps $\rho_I(t)|_{H^2(M;\mathbb{R})}$, $t \in \mathbb{R}$, $[I] \in \mathrm{T}_{\mathcal{C}_0,[\omega]}$, generate the stabilizer. In other words, $G_{[\omega]}$ coincides with the stabilizer of $[\omega]$ in G.

For the class $a \in H^{even}(M;\mathbb{Q})$ given in the hypothesis of the lemma, denote by G_a the stabilizer of a in G.

Now we are ready to prove the lemma.

Assume, by contradiction, that $\mathrm{T}_{\mathcal{C}_0,[\omega]} \subset \mathrm{T}_{\mathcal{C}_0,a}$, meaning that option (1) above, and not (2), holds for a. Then a is of type (p,p) for all complex structures $I \in \mathcal{C}_0$ such that $[I] \in \mathrm{T}_{\mathcal{C}_0,[\omega]}$. In particular, a is fixed by $\rho_I(t)$ for all such I and all $t \in \mathbb{R}$. Thus, $G_{[\omega]}$ fixes a or, in other words,

$$SO^+(2, b_2 - 3) \cong G_{[\omega]} \subset G_a \subset G \cong SO^+(3, b_2 - 3).$$

By [EV1, Lem. 9.9], any intermediate subgroup $G_{[\omega]} \subset H \subset G$ is equal either to $G_{[\omega]}$ or to G. This implies that G_a equals either $G_{[\omega]}$ or G. Since $[\omega]$ is not rational, [EV1, Lem. 9.10] yields that $G_a \neq G_{[\omega]}$. Since Z is not absolutely trianalytic, there exists $I \in \mathcal{C}_0$ such that $a \notin H_I^{1,1}(M;\mathbb{R})$ and consequently a is not fixed by $\rho_I(t) \in G$ for $t \neq 0$. Therefore $\rho_I(t) \notin G_a$ and $G_a \neq G$. This yields a contradiction and finishes the proof of the lemma. ∎

Now let us finish the proof of Proposition 12.1.

Let $\mathcal{A} \subset H^*(M;\mathbb{Q}) \setminus \{0\}$ be the set of all $a \in H^*(M;\mathbb{Q})$, $a \neq 0$, for which $\mathrm{T}_{\mathcal{C}_0,a} \cap \mathrm{T}_{\mathcal{C}_0,[\omega]}$ is a proper (possibly empty) complex subvariety of $\mathrm{T}_{\mathcal{C}_0,[\omega]}$ and consequently $\mathrm{T}_{\mathcal{C}_0,a} \cap \mathrm{Teich}_{\mathcal{C}_0}(M,\omega)$ is a proper complex subvariety of $\mathrm{Teich}_{\mathcal{C}_0}(M,\omega)$. By Lemma 12.4, $\mathcal{A}$ includes, in particular, all $a \in H^2(M;\mathbb{Q})$ that are Poincaré-dual to the fundamental classes of not absolutely trianalytic subvarieties (with respect to $\mathcal{C}_0$).

The set
$$S := \mathrm{Teich}_{\mathcal{C}_0}(M,\omega) \setminus \bigcup_{a \in \mathcal{A}} \left(\mathrm{T}_{\mathcal{C}_0,a} \cap \mathrm{Teich}_{\mathcal{C}_0}(M,\omega) \right)$$

is connected and dense in $\mathrm{Teich}_{\mathcal{C}_0}(M,\omega)$. For any $I \in \mathcal{C}_0$ such that $[I] \in S$, the complex manifold (M,I) can contain only absolutely trianalytic subvarieties. By [EV1, Lem. 7.9], the union of all such subvarieties of (M,I) is of measure zero, meaning that I is Campana-simple and $[I] \in \mathrm{Teich}_{\mathcal{C}_0}(M,\omega) \cap \mathrm{Teich}^s(M)$. Hence, $S \subset \mathrm{Teich}_{\mathcal{C}_0}(M,\omega) \cap \mathrm{Teich}^s(M) \subset \mathrm{Teich}_{\mathcal{C}_0}(M,\omega)$. Since S is connected and dense in $\mathrm{Teich}_{\mathcal{C}_0}(M,\omega)$, so is $\mathrm{Teich}_{\mathcal{C}_0}(M,\omega) \cap \mathrm{Teich}^s(M)$.

This finishes the proof of the proposition. ∎

We are now ready to discuss the analogue of Theorem 3.22 in the IHS-hyperkähler case, as described in Section 3.5.

The proof of the claims of Theorem 3.22 in the IHS-hyperkähler case (apart from the claim on the transitivity of the $\mathrm{Symp}_H(M,\omega)$-action) is the same as in the K3 case – one just needs to use the

generalizations of the results of Section 10.2 to the IHS-hyperkähler case discussed above (instead of the corresponding results for the K3 surfaces) and use Theorem 9.3 in the simply-connected case involving Campana-simple complex structures.

As for the transitivity of the action of $\mathrm{Symp}_H(M,\omega)$, we cannot say anything in the IHS-hyperkähler case, as we have no analogue of part B of Theorem 10.12 for IHS-hyperkähler manifolds.

Finally, let us discuss the results on tame embeddings into (M,ω) for an IHS-hyperkähler-type form ω.

Recall from Section 3.5 that there are two possibilities to define tame embeddings into such an (M,ω). One possibility is to use Kähler-type complex structures tamed by ω. Another possibility is to use instead IHS-hyperkähler-type complex structures tamed by ω.

The claim $v_{T,\varepsilon}(M,\omega_1,\mathbb{W}) = v_{T,\varepsilon}(M,\omega_2,\mathbb{W})$ of Theorem 3.27 holds for both definitions of $v_{T,\varepsilon}$, as long as ω_1 and ω_2 lie in the same connected component of the space of IHS-hyperkähler-type symplectic forms on M. Indeed, it is easy to see that Proposition 11.5 holds for both definitions of $v_{T,\varepsilon}$. The relevant claim about the density of the $\mathrm{Diff}^+(M)$-orbits of the form ω_1, ω_2 in the space of forms of the same volume in the same connected component of the space of IHS-hyperkähler-type symplectic forms on M follows from [EV1, Thm. 9.2].

A   Appendix

In this section we discuss various well-known facts concerning the dependence of the Hodge decomposition on the complex structure, the deformations of complex structures, and the Moser method, that we used in the paper and for most of which we could not find a direct reference in the literature.

A.1   Dependence of the Hodge decomposition on the complex structure

In this section we prove two auxiliary facts about Kähler manifolds. These facts must have been known to the experts but we have not been able to find a reference to them.

Assume M, $2n = \dim_{\mathbb{R}} M$, is a closed manifold admitting Kähler structures – i.e., $CK(M) \neq \emptyset$. Recall that we use the C^∞-topology on $CK(M)$.

For each $l = 0, 1, \ldots, 2n$ denote by $A^l(M)$ the space of smooth complex-valued differential l-forms on M. For each $I \in CK(M)$ let

$$A^l(M) = \oplus_{p+q=l} A_I^{p,q}(M)$$

be the (p,q)-decomposition of $A^l(M)$.

Let $\Omega_I^{1,0}(M)$ be the space of I-holomorphic 1-forms on M, with respect to I.

By Hodge theory, $\dim_{\mathbb{C}} H^1(M;\mathbb{C}) = 2\dim_{\mathbb{C}} H_I^{1,0}(M) = 2\dim_{\mathbb{C}} H^0(\Omega_I^1(M))$. Let $m := \dim_{\mathbb{C}} H^1(M;\mathbb{C})/2$.

Proposition A.1

Let $J \in CK(M)$.

Then there exists a neighborhood $\mathcal{W}$ of J in $CK(M)$ with the following property:

For each $I \in \mathcal{W}$ one can choose a basis $\theta_1(I), \ldots, \theta_m(I)$ of $H^0(\Omega_I^1(M))$ continuously with respect to I.

Proof of Proposition A.1:

Pick a J-invariant Riemannian metric g on M. Then there exists a neighborhood $\mathcal{W}$ of J in $CK(M)$ such that for any $I \in \mathcal{W}$ sufficiently close to J we have an I-invariant Riemannian metric $(g + I^*g)/2$ that depends continuously on I and coincides with g for $I = J$. Together with $I \in \mathcal{W}$ such an I-invariant Riemannian metric defines a Hermitian metric h_I on (M, I) that depends continuously on I.

For each $I \in \mathcal{W}$ and each $l = 0, 1, \ldots, 2n$ the Hermitian metric h_I allows to define the Laplace-Beltrami operator on $A^l(M)$ that will be denoted $\Delta_I^l : A^l(M) \to A^l(M)$. This is an elliptic operator. Denote the kernel of Δ_I^l (i.e., the space of harmonic l-forms with respect to Δ_I^l) by $\mathcal{H}_I^l(M)$. Then $\dim_{\mathbb{C}} \mathcal{H}_I^l(M) = \dim_{\mathbb{C}} H^l(M; \mathbb{C})$. Also, $\mathcal{H}_I^1(M) \cap A_I^{1,0}(M) = \Omega_I^{1,0}(M)$. These results follow from the classical Hodge theory (see e.g. [Wel, Sec. IV.4, IV.5]).

Thus we have obtained that for each l the operators Δ_I^l form a family of elliptic operators depending continuously on $I \in \mathcal{W}$ so that the kernels of these operators all have the same dimension. Then these kernels – i.e., the spaces $\mathcal{H}_I^l(M)$, $I \in \mathcal{W}$ – form a continuous vector bundle over $\mathcal{W}$. This follows from a standard argument essentially contained in [At2, Lem. 2.1]. Namely, one passes to a Hilbert completion of $A^l(M)$ and extends the continuous family of elliptic operators Δ_I^l, $I \in \mathcal{W}$, to a continuous family of Fredholm operators that, by elliptic regularity, have the same kernels, all of the same dimension. An easy functional analysis argument implies then that the kernels of these Fredholm operators form a continuous vector bundle (in fact, we only need the operators to be bounded with a closed image and these conditions are satisfied by any Fredholm operator).

Since the spaces $\mathcal{H}_I^1(M)$, $I \in \mathcal{W}$, form a continuous vector bundle over $\mathcal{W}$, so do their I-invariant parts – i.e., the spaces $\Omega_I^{1,0}(M)$ – which are all of the same dimension m. This readily implies the proposition. ∎

A.2 Deformation families of complex structures

Let (M, I) be a complex manifold.

Assume $\mathcal{X}, \mathcal{B}$ are connected (not necessarily Hausdorff) smooth manifolds. Let $s_0 \in \mathcal{B}$ be a marked point. Let $\mathcal{X} \to \mathcal{B}$ be a smooth map.

If $\mathcal{X} \to \mathcal{B}$ is a proper fiber bundle whose fibers are equipped with complex structures depending smoothly on the fiber and the fiber over s_0 is (M, I), we say that $\mathcal{X} \to (\mathcal{B}, s_0)$ is a **smooth deformation family**, or a **smooth deformation of** (M, I).

If $\mathcal{X}, \mathcal{B}$ are complex manifolds and the map $\mathcal{X} \to \mathcal{B}$ is a proper holomorphic submersion, then it is a fiber bundle (by Ehresmann's lemma, the fibration admits *smooth* local trivializations). The fibers of $\mathcal{X} \to \mathcal{B}$ are then closed complex submanifolds of $\mathcal{X}$ that are diffeomorphic to M. Denote by M_s the fiber over $s \in \mathcal{B}$ and denote by I_s the complex structure on M_s induced by the complex structure on $\mathcal{X}$. Assume that $I_{s_0} = I$. In this case we say that $\mathcal{X} \to (\mathcal{B}, s_0)$ is a **complex-analytic deformation family**, or a **complex-analytic deformation of** (M, I).

Given a complex structure I on M, a smoothly trivial smooth (respectively complex-analytic) deformation family $M \times U \to U$ of (M, I) over an open set $U \subset \mathbb{C}^m$ (respectively $U \subset \mathbb{R}^m$), $s_0 = 0 \in U$, is called a **local smooth (respectively complex-analytic) deformation** of I. It can be viewed as a smooth

(respectively complex analytic) family $\{I_s\}_{s \in U}$, $I_0 = I$.

Assume $\{I_s\}_{s \in U}$, $I_0 = I$, is a smooth local deformation of a Kähler-type complex structure I on M. The Kodaira-Spencer stability theorem [KoS2] says that for any symplectic form ω on M compatible with I there exists a neighborhood U' of 0 in U and a smooth family $\{\omega_s\}_{s \in U'}$, $\omega_0 = \omega$, of symplectic forms ω_s on M compatible with I_s for each $s \in U'$; in particular, each complex structure I_s, $s \in U'$, is of Kähler type. It follows from the Kodaira-Spencer stability theorem (see [EV1, Thm. 5.6]) that there exists a neighborhood $U'' \subset U'$ of 0 so that for each $s \in U''$ the cohomology class $[\omega]_{I_s}^{1,1}$ is Kähler with respect to I_s.

Proposition A.2

The space $CK(M)$ is locally C^∞-path-connected.

Proof of Proposition A.2:

Kuranishi's theorem [Ku1], [Ku2], [Ku3] implies that the $\mathrm{Diff}_0(M)$-action on the space of complex structures on M admits local slices that are finite-dimensional analytic spaces. This implies that the space of complex structures on M is locally path-connected. Therefore, since $\mathrm{Diff}_0(M)$ acts trivially on the cohomology and preserves $CK(M)$, the Kodaira-Spencer stability theorem yields that the space $CK(M)$ is locally C^∞-path-connected. This proves the proposition. ∎

A.3 Relative version of Moser's method

Here we recall the following (well-known) relative version of Moser's method [Mos].

Proposition A.3

Let N be a closed manifold.

I. Assume that $X \subset N$ is a closed domain with a piecewise-smooth boundary, such that $H^1(X; \mathbb{R}) = 0$.

Let $\{\Omega_t\}$, $t \in [0, 1]$, be a smooth family of cohomologous symplectic forms on N, so that all the forms Ω_t, $t \in [0, 1]$, coincide on a neighborhood of X.

Then there exists an isotopy $\{\phi_t : N \to N\}_{0 \le t \le 1}$, $\phi_0 = Id$, such that for each $t \in [0, 1]$ the diffeomorphism ϕ_t is identity on a neighborhood of X and $\phi_t^* \Omega_0 = \Omega_t$.

II. Assume that $X \subset N$ is a proper submanifold. Let $\{\Omega_t\}$, $t \in [0, 1]$, be a smooth family of cohomologous symplectic forms on N, so that all the forms $\Omega_t|_X$, $t \in [0, 1]$, are non-degenerate.

Then there exists an isotopy $\{\phi_t : N \to N\}_{0 \le t \le 1}$, $\phi_0 = Id$, such that $\phi_t(X) = X$ and $\phi_t^* \Omega_0 = \Omega_t$ for each $t \in [0, 1]$.

III. Assume that $N = \mathbb{C}P^n \times \mathbb{C}P^1$, $X_1 := \mathbb{C}P^{n-1} \times \mathbb{C}P^1 \subset N$, $X_2 := \mathbb{C}P^n \times \mathrm{pt} \subset N$. Let $\{\Omega_t\}$, $t \in [0, 1]$, be a smooth family of cohomologous symplectic forms on N, so that all the forms $\Omega_t|_{X_i}$, $i = 1, 2$, $t \in [0, 1]$, are non-degenerate.

Then for every open neighborhood $\mathcal{U}$ of $X_1 \cap X_2$ there exists an isotopy $\{\phi_t : N \to N\}_{0 \le t \le 1}$, $\phi_0 = Id$, such that

- $\phi_t(X_1 \cup X_2) = X_1 \cup X_2$,

- $\phi_t^* \Omega_0 = \Omega_t$ outside $\mathcal{U}$ for each $t \in [0,1]$.

Proof of Proposition A.3:

Let us prove part I of the proposition.

The proof follows the lines of the proof of the standard, absolute, version of the same result, as long as one can find a smooth family $\{\lambda_t\}$, $t \in [0,1]$, of 1-forms on N so that for all $t \in [0,1]$

$$d\lambda_t = \frac{d}{dt}\Omega_t,$$

and all the forms λ_t, $0 \leq t \leq 1$, vanish on a neighborhood of X.

In order to find such a family, pick first a smooth family $\{\mu_t\}$, $t \in [0,1]$, of 1-forms on N satisfying

$$d\mu_t = \frac{d}{dt}\Omega_t$$

for all $t \in [0,1]$ (this can be done e.g. using Hodge theory – see e.g. [McDS, Thm. 3.2.4]). Note that all the forms $d\mu_t$, $0 \leq t \leq 1$, vanish on a neighborhood of X. Now, since $H^1(X;\mathbb{R}) = 0$, it is easy to show that there exists a smooth family $\{F_t\}$, $t \in [0,1]$, of functions defined on a neighborhood U of X so that $dF_t = \mu_t$ on U for all $t \in [0,1]$. Extend these functions from possibly a smaller neighborhood $U' \subset U$ of the set X to a smooth family of functions on N. By a slight abuse of notation, denote the latter family of functions on N also by $\{F_t\}$, $t \in [0,1]$. Then $\{\lambda_t := \mu_t - dF_t\}$, $t \in [0,1]$, is the wanted family of 1-forms on N.

This finishes the proof of part I.

Part II is proved in [McDP, Cor. 4.1.B].

Part III is proved in [McDP, Prop. 4.1.C].

This finishes the proof of the proposition. ∎

A.4 Alexander's trick

Let $W \subset \mathbb{R}^{2n}$ be a compact domain with piecewise-smooth boundary that is starshaped with respect to the origin.

Proposition A.4

Assume $f : W \to \mathbb{R}^{2n}$ is an embedding. Let $\iota : W \to \mathbb{R}^{2n}$ be the inclusion map.

The following claims hold:

I. Let $U \subset \mathbb{R}^{2n}$ be an open neighborhood W in $\mathbb{R}^{2n}$. Assume that the embedding f is symplectic.

If f is sufficiently C^1-close to ι, then there exists a smooth family of symplectic embeddings $\{f_t : W \to U\}_{0 \leq t \leq 1}$ so that $f_0 = \iota$, $f_1 = f$. Moreover, all f_t, $t \in [0,1]$, can be made arbitrarily C^∞-close to ι, provided f is sufficiently C^∞-close to ι.

II. Assume that the embedding f is holomorphic, $f(0) = 0$, and $f(W) \subset W$. Then there exists a smooth family of holomorphic embeddings $\{f_t : W \to W\}_{0 \leq t \leq 1}$ so that $f_0 = \iota$, $f_1 = f$ and $f_t(0) = 0$ for all $t \in [0, 1]$.

Proof of Proposition A.4:

Let us prove part I.

If f is sufficiently C^1-close to ι, then, composing f, if necessary, with a family of small affine symplectomorphisms staring at Id, we may assume without loss of generality, that $f(0) = 0$ and $df(0) = Id$.

Consider the following family of maps $\{g_t\}_{0 \leq t \leq 1}$ defined on W:

$$g_t(x) := g(tx)/t, \text{ if } t \in (0, 1],$$

$$g_0 = \iota.$$

If f is sufficiently C^1-close to ι, then all the maps g_t are symplectic embeddings taking values in U. The family $\{g_t\}_{0 \leq t \leq 1}$ may not be smooth at $t = 0$ but for an appropriate smooth function $\chi : [0, 1] \to [0, 1]$, $\chi(0) = 0$, $\chi(1) = 1$, the family $\{g_{\chi(t)}\}_{0 \leq t \leq 1}$ is a smooth family of symplectic embeddings $W \to U$ connecting f and ι. The construction clearly shows that all f_t, $t \in [0, 1]$, can be made arbitrarily C^∞-close to ι, provided f is sufficiently C^∞-close to ι. This finishes the proof of part I.

Let us prove part II. For $C > 0$ consider the following family of maps $\{g_t\}_{0 \leq t \leq 1}$ defined on W:

$$g_t(z) := f(tz)/Ct, \text{ if } t \in (0, 1],$$

$$g_0 := df(0)/C.$$

For a sufficiently large $C > 0$ we have $\mathrm{Im}\, g_t \subset W$ for all $t \in [0, 1]$. The family $\{g_t\}_{0 \leq t \leq 1}$ may not be smooth at $t = 0$ but for an appropriate smooth function $\chi : [0, 1] \to [0, 1]$, $\chi(0) = 0$, $\chi(1) = 1$, the family $\{g_{\chi(t)}\}_{0 \leq t \leq 1}$ is a smooth family of holomorphic embeddings $(W, 0) \to (W, 0)$ connecting g_0 and $g_1 = f/C$. If $C > 0$ is sufficiently large, the linear holomorphic embedding g_0 can be easily connected by a smooth family of holomorphic embeddings $(W, 0) \to (W, 0)$ to Id/C and the latter can be connected to Id. ∎

Corollary A.5

Let M^{2n} be a manifold and ω a Kähler-type symplectic form on M. Let $f : W \to (M, \omega)$ be a Kähler-type/tame/ε-tame embedding (in the latter case we assume that M is closed).

Then any symplectic embedding $f' : W \to (M, \omega)$ sufficiently C^1-close to f is of Kähler-type/tame/ε-tame, and there exists $\phi \in \mathrm{Symp}_0(M)$ such that $\phi \circ f = f'$. The symplectomorphism ϕ can be made arbitrarily C^∞-close to Id provided f' is sufficiently C^∞-close to f.

Proof of Corollary A.5:

Assume f extends to an I-holomorphic symplectic embedding $f : U \to (M, \omega)$, where U is a neighborhood of W in $\mathbb{R}^{2n}$ and I is a complex structure on M with the relevant properties as needed for a Kähler-type/tame/ε-tame embedding.

Let $f' : W \to (M, \omega)$ be a symplectic embedding (extending to a symplectic embedding of an open neighborhood of W). If f' is sufficiently C^1-close to f, then $f'(W) \subset f(U)$ and $f^{-1} \circ f' : W \to U$ is a

symplectic embedding that can be made arbitrarily C^1-close to the identity. By part I of Proposition A.4, it can be connected to the inclusion $\iota : W \to U$ by a smooth family of symplectic embeddings $W \to U$ arbitrarily C^∞-close to the identity, provided f' is sufficiently C^∞-close to f. Composing these embeddings with f, we get a smooth family of symplectic embeddings $f_t : W \to (M, \omega)$, $0 \leq t \leq 1$, connecting f and f'. These embeddings can be made arbitrarily C^∞-close to f, provided f' is sufficiently C^∞-close to f.

By a standard result concerning extension of symplectic isotopies (proved as [McDS, Thm. 3.3.1]), there exists $\phi \in \mathrm{Symp}_0(M, \omega)$ such that $\phi \circ f = f'$. The symplectomorphism ϕ can be chosen to be arbitrarily C^∞-close to the identity, if the symplectic embeddings f_t, $0 \leq t \leq 1$, are sufficiently C^∞-close to f, which is true if f' is sufficiently C^∞-close to f. The symplectic embedding f' is of Kähler-type/tame/ε-tame because it is holomorphic with respect to the complex structure $\phi_* I$ on M satisfying the relevant properties.

This finishes the proof of the corollary. ∎

Acknowledgments

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