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Zeta statistics at the square-root barrier for cubes

Victor Y. Wang * 

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Abstract:

Hooley and Heath-Brown bounded, optimally up to X^ε , the number of integral zeros of $\sum_{i=1}^6 x_i^3$ in $[-X, X]^6$, assuming the Riemann Hypothesis for geometric L -functions. We attribute the ε to several factors in harmonic analysis and algebraic geometry. We then remove the ε assuming, mainly, the Ratios Conjecture for geometric families of L -functions. We give applications to sums of three cubes.

Key words and phrases: Cubic form, circle method, rational points, Hasse–Weil L -functions, correlations

1 Introduction

Let $F_0 = F_0(x, y, z) := x^3 + y^3 + z^3$ and $F_0(S^3) := \{x^3 + y^3 + z^3 : x, y, z \in S\}$. For each nonzero $a \in \mathbb{Z}$, the cubic surface $F_0 = a$ is known to have Zariski-dense sets of $\mathbb{Q}$ -points [Seg43] and $\mathbb{Z}[a^{1/3}]$ -points [Leh56, Beu99, HT01]. In contrast, Mordell suggests that producing large $\mathbb{Z}$ -points on $F_0 = 3$ could be as hard as “finding when an assigned sequence, e.g. 123456789, occurs in the decimal expansion of π ” [Mor53, p. 505]. Despite the recent progress [BS21] of Booker and Sutherland in this case ($a = 3$), the general case $F_0 = a$ remains wide open.

Heath-Brown [HB92, p. 623] has conjectured that $F_0 = a$ should have infinitely many solutions $(x, y, z) \in \mathbb{Z}^3$ for any fixed $a \not\equiv \pm 4 \pmod{9}$. To represent all $a \not\equiv \pm 4 \pmod{9}$ even once, one must allow both positive and negative values of x, y, z . The set $F_0(\mathbb{Z}_{>0}^3)$ has upper density $\leq \Gamma(4/3)^3/6 = 0.1186788\dots$ in $\mathbb{Z}_{\geq 0}$ [Dav39]; Deshouillers, Hennecart, and Landreau conjecture a precise density of $0.0999425\dots$

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[DHL06]. Wooley has shown, unconditionally, that $F_0(\mathbb{Z}_{\geq 0}^3)$ contains $\gg A^{0.91709477}$ integers $a \in [0, A]$ for reals $A \geq 1$ [Woo95, Woo00, Woo15]. We now recall a result of Hooley [Hoo86b, Hoo97] and Heath-Brown [HB98]:

Theorem 1.1 (Hooley; Heath-Brown). *For certain Hasse–Weil L -functions, assume automorphy and GRH. Then $\gg_\varepsilon A^{1-\varepsilon}$ integers $a \in [0, A]$ lie in $F_0(\mathbb{Z}_{\geq 0}^3)$, for any $\varepsilon > 0$.*

Let $\Delta^* = \prod (c_1^{3/2} \pm c_2^{3/2} \pm \cdots \pm c_6^{3/2})$. In the preprint [Wan23b], we proved the following:

Theorem 1.2. *For certain Hasse–Weil L -functions, assume automorphy, GRH, and the Ratios Conjecture. Assume the Square-free Sieve Conjecture for $\Delta^* \in \mathbb{Z}[c_1, \dots, c_6]$. Then $F_0(\mathbb{Z}_{\geq 0}^3)$ has lower density > 0 , and $F_0(\mathbb{Z}^3)$ has density 1, in $\{a \in \mathbb{Z} : a \not\equiv \pm 4 \pmod{9}\}$.*

To ease the exposition, we have decided to split [Wan23b] into two papers: the present, proving the density statement on $F_0(\mathbb{Z}_{\geq 0}^3)$, and a sequel, on $F_0(\mathbb{Z}^3)$. This gives us more space to describe our key new ideas, most of which appear already for $F_0(\mathbb{Z}_{\geq 0}^3)$. We build on [Wan23a, Wan24b], but any reference to other work of ours is purely for exposition.

Informally, one can interpret Theorems 1.1 and 1.2 as passing from *sparse* statistics of integer cubes to *rich* statistics of L -functions and divisors. In a function-field setting, [BV15, GH25] have made [HB98] unconditional, and one can simplify our hypotheses significantly (since automorphy, GRH, and the Square-free Sieve Conjecture are known, and homological stability following [BDPW23, MPPRW24, Wan24a] might yield enough progress on the Ratios Conjecture for our purposes). Indeed, our work [BGW24, BGW25] with Browning and Glas, on F_0 over $\mathbb{F}_q[t]$ for $\text{char}(\mathbb{F}_q) > 3$, is conditional only on the Ratios Conjecture.

For a typical a , the integer solutions to $F_0 = a$ are expected to be at least exponentially sparse, if they in fact exist. Also, for the analogous equation $5x^3 + 12y^3 + 9z^3 = a$, the Hasse principle is known to fail in general [GS22, p. 691, footnote 3], and the paper [LMU24] gives many more examples of the same diagonal cubic type. Given these subtleties, and the difficulty of studying $F_0(\mathbb{Z}^3)$ algebraically, it is natural to concentrate on statistics.

To study $F_0(\mathbb{Z}_{\geq 0}^3)$, it is customary to use second moments. The fibers of $F_0 : \mathbb{Z}_{\geq 0}^3 \rightarrow \mathbb{Z}_{\geq 0}$ have average size $\asymp 1$, so if the mean-square fiber size over $a \leq A$ is bounded as $A \rightarrow \infty$, then the lower density of $F_0(\mathbb{Z}_{\geq 0}^3)$ is positive by Cauchy–Schwarz. This mean square features in [Hoo86a, Conjecture 2 for $l = 3$], a difficult open question on point counts in $3 + 3 = 6$ variables, which lies just beyond the square-root barrier in the classical Hardy–Littlewood circle method. We attack this question under standard number-theoretic hypotheses, primarily regarding L -function statistics of Random Matrix Theory (RMT) type.

What complicates matters for us is that the relevant L -functions are mixed in with other factors from harmonic analysis and algebraic geometry. Such difficulties already appeared to some extent in [Hoo86b, Hoo97, HB98]. Indeed, GRH is only one of several ingredients in [Hoo86b, Hoo97, HB98]. The present paper likewise requires a diverse toolkit.

We now set up the paper precisely. Fix a cubic form $F \in \mathbb{Z}[x_1, \dots, x_m]$ in $m \geq 4$ variables with nonzero discriminant. Let V be the hypersurface $F = 0$ in $\mathbb{P}_{\mathbb{Q}}^{m-1}$. Let Υ be the set of $\lfloor m/2 \rfloor$ -dimensional $\mathbb{Q}$ -vector spaces $L \subseteq \mathbb{Q}^m$ on which F vanishes.

Let $\mathbf{x} := (x_1, \dots, x_m)$. Given a real $X \geq 1$, and a function $w \in C_c^\infty(\mathbb{R}^m)$, let

$$N_{F,w}(X) := \sum_{\mathbf{x} \in \mathbb{Z}^m : F(\mathbf{x})=0} w(\mathbf{x}/X), \quad N_F(X) := \sum_{\mathbf{x} \in [-X, X]^m : F(\mathbf{x})=0} 1. \quad (1.1)$$

In our main case of interest, $F = \sum_{i=1}^6 x_i^3$, we have $m = 6$ and $\lfloor m/2 \rfloor = 3$. If $m = 6$, let

$$E_{F,w}(X) := N_{F,w}(X) - \mathfrak{S}_F \cdot \sigma_{\infty,F,w} \cdot X^3 - \sum_{L \in \mathcal{L}} \sum_{\mathbf{x} \in L \cap \mathbb{Z}^6} w(\mathbf{x}/X), \quad (1.2)$$

where $\mathfrak{S}_F$ is the familiar singular series defined in [Wan24b, (6.4)], and where

$$\sigma_{\infty,F,w} := \lim_{\varepsilon \rightarrow 0} (2\varepsilon)^{-1} \int_{|F(\mathbf{x})| \leq \varepsilon} w(\mathbf{x}) d\mathbf{x}.$$

For $m = 6$, one could attribute to Hooley [Hoo86a, Conjecture 2], Manin (see e.g. [FMT89]), Vaughan–Wooley [VW95, Appendix], Peyre [Pey95], et al. the following conjecture:

$$\lim_{X \rightarrow \infty} X^{-3} E_{F,w}(X) = 0. \quad (1.3)$$

Unconditionally, [Hua38] (resp. [Vau20]) proved $N_F(X) \ll_{\varepsilon} X^{3.5+\varepsilon}$ (resp. $X^{3.5}(\log X)^{\varepsilon-2.5}$) for $X \geq 2$, when $m = 6$ and F is diagonal. Conditionally, near-optimal bounds are available for the same F [Hoo97, HB98]. For each $\mathbf{c} = (c_1, \dots, c_m) \in \mathbb{Z}^m$, let $V_{\mathbf{c}}$ be the intersection $F(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} = 0$ in $\mathbb{P}_{\mathbb{Q}}^{m-1}$, where $\mathbf{c} \cdot \mathbf{x} := \sum_{i=1}^m c_i x_i$. Let $F^{\vee} \in \mathbb{Z}[c_1, \dots, c_m]$ define the dual hypersurface of V ; we can take $F^{\vee} = \Delta^*$ if $F = \sum_{i=1}^6 x_i^3$ [Hoo86b, (34)]. Let

$$\mathcal{S}_0 := \{\mathbf{c} \in \mathbb{Z}^m : F^{\vee}(\mathbf{c}) = 0\}, \quad \mathcal{S}_1 := \{\mathbf{c} \in \mathbb{Z}^m : F^{\vee}(\mathbf{c}) \neq 0\}. \quad (1.4)$$

For each $\mathbf{c} \in \mathcal{S}_1$, one can package local data on $V_{\mathbf{c}}$ into a Hasse–Weil L -function $L(s, V_{\mathbf{c}})$, defined in §3 along with the rest of the list (1.5).

Conjecture 1.3 (HW2). *Let $\mathbf{c} \in \mathcal{S}_1$. Let $L(s)$ be one of the Hasse–Weil L -functions*

$$L(s, V_{\mathbf{c}}), L(s, V_{\mathbf{c}}, \otimes^2), L(s, V_{\mathbf{c}}, \text{Sym}^2), L(s, V_{\mathbf{c}}, \wedge^2), \zeta(s), L(s, V). \quad (1.5)$$

Let $L_v(s)$ be the local factors associated to $L(s)$, where $v \leq \infty$ runs over all places of $\mathbb{Q}$.

1. *There exist an integer $d \geq 1$, and an isobaric automorphic representation Π of $\text{GL}_d(\mathbf{A}_{\mathbb{Q}})$, such that $L_v(s) = L_v(s, \Pi)$ at all places $v \leq \infty$.*
2. *$L(s, \Pi)$ has no zeros in the half-plane $\text{Re}(s) > \frac{1}{2}$.*

(For relevant definitions and background on L -functions, see §3.)

Conjecture 1.3 concerns automorphy and the Grand Riemann Hypothesis (GRH). It refines Hypothesis HW of [Hoo86b, HB98]. However, Conjecture 1.3 includes automorphy, instead of Selberg-type axioms like analyticity. The automorphic perspective is cleaner for us, yet essentially equivalent to the analytic perspective [BK16, FPRS19].

Theorem 1.4 ([Hoo86b, Hoo97]; [HB98]). *Assume $m = 6$ and F is diagonal. For each $\mathbf{c} \in \mathcal{S}_1$, assume Conjecture 1.3 for $L(s, V_{\mathbf{c}})$. Then $N_F(X) \ll_{\varepsilon} X^{3+\varepsilon}$ for all $\varepsilon > 0$.*

This implies Theorem 1.1. Theorem 1.4 uses GRH on $1/L(s, V_{\mathbf{c}})$, coupled with algebro-geometric factors of a different nature. See §2 for details. The L -function ingredients extend directly to general F , but some other ingredients have yet to be generalized.

GRH gives a pointwise bound on $1/L(s, V_{\mathbf{c}})$ for $\operatorname{Re}(s) > \frac{1}{2}$, sufficient for Theorem 1.4. Going past the critical line (in mean value over $\mathbf{c}$) requires new ideas, both with L -functions and with other factors. In terms of L -functions, we mainly use $L(s, V_{\mathbf{c}})$ for $\operatorname{Re}(s) \geq \frac{1}{2} - \delta$, as well as $L(s, V_{\mathbf{c}}, \wedge^2)$, $\zeta(s)$, and $L(s, V)$ for $\operatorname{Re}(s) \geq 1 - \delta$. It will be convenient to assume Conjecture 1.3 in full, but average versions might also suffice.

We use mean-value RMT-type predictions given by the Moments and Ratios Conjectures of [CFK⁺05, CS07, CFZ08]. See also [DGH03, Čec24], and references within, for another important perspective on such conjectures. The Hasse–Weil L -functions $L(s, V_{\mathbf{c}})$ form a geometric family, in the sense of [SST16]. For each $\mathbf{c} \in \mathcal{S}_1$, define the Dirichlet series

$$\Phi^{\mathbf{c},1}(s) := \zeta(2s)^{-1} L(s + \tfrac{1}{2}, V)^{-1} L(s, V_{\mathbf{c}})^{-1}. \quad (1.6)$$

Note that $\zeta(2s)$ and $L(s + \frac{1}{2}, V)$ are independent of $\mathbf{c}$. Let $\sigma(Z) := \frac{1}{2} + \frac{1}{\log Z}$ for $Z \geq 2$. In §6.3.2, we will define an Euler product $A_{F,2}(s_1, s_2)$ absolutely convergent for $\operatorname{Re}(s_1), \operatorname{Re}(s_2) > \frac{1}{3}$, and use the Ratios Recipe of [CFZ08] to derive the following Ratios Conjecture:

Conjecture 1.5 (R2). *Write $s_j = \sigma(Z) + it_j$. If $2 \mid m$ and Conjecture 1.3 holds, then there exists a real $\hbar > 0$ such that uniformly over reals $Z \geq 2$ and $t_1, t_2 \in [-Z^{\hbar}, Z^{\hbar}]$, we have*

$$\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \Phi^{\mathbf{c},1}(s_1) \Phi^{\mathbf{c},1}(s_2) = \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} (\zeta(s_1 + s_2) + O(1)) \cdot A_{F,2}(s_1, s_2). \quad (1.7)$$

We emphasize that $A_{F,2}$ is independent of $\mathbf{c}$, so the right-hand side of (1.7) may be equivalently written in the more transparent form

$$\zeta(s_1 + s_2) A_{F,2}(s_1, s_2) |\mathcal{S}_1 \cap [-Z, Z]^m| + O(Z^m).$$

The Ratios Conjecture can be stated with a power saving, but the softer asymptotic formula (1.7) suffices for us. The line $\operatorname{Re}(s) = \sigma(Z)$ lies in the standard ranges for the Ratios Conjecture; see [CS07, (2.11b)]. We could work farther from the critical line if we assumed a better error term. But (1.7) is clean, and easy to compare with the existing literature.

For our main results, we also need the Square-free Sieve Conjecture (cf. [Mil04, p. 956] in one variable and [Poo03] in more) for the polynomial $F^{\vee}$, restricted to a certain range $1 \leq P \leq Z^{3/2}$. This hypothesis concerns “unlikely” square divisors of the outputs of $F^{\vee}$. Such hypotheses can be made unconditional over function fields; see e.g. [Poo03, Lemma 7.1]. For progress over number fields, see [Gra98, Bha14, Vuk23, KV24, KV25] and references within.

Conjecture 1.6 (SF). *There exists $\eta_0 \in \mathbb{R}_{>0}$ such that if $Z, P \in \mathbb{R}_{\geq 1}$ and $P \leq Z^{3/2}$, then $\#\{\mathbf{c} \in \mathbb{Z}^m \cap [-Z, Z]^m : \exists \text{ a prime } p \in [P, 2P) \text{ with } p^2 \mid F^{\vee}(\mathbf{c})\} \ll_{F^{\vee}} Z^m P^{-\eta_0}$.*

This conjecture is used to confront some new geometric issues. We use it to prove, for diagonal F , a geometric relative (Conjecture 9.8) of the automorphic Sarnak–Xue Density Hypothesis ([SX91,

Conjecture 1], concerning the extent to which a naive generalization of the Ramanujan Conjecture fails). Conjecture 1.6 is trivial when $P \leq Z^{1/2}$.

We now come to the main self-contained result of the present paper:

Theorem 1.7. *Suppose $F = \sum_{i=1}^6 x_i^3$. Assume Conjectures 1.3, 1.5, and 1.6. Then*

$$N_F(X) \ll X^3. \quad (1.8)$$

Let $S \subseteq \mathbb{Z}_{\geq 0}$. If S has positive lower density in $\mathbb{Z}_{\geq 0}$, then so does $F_0(S^3)$.

This implies the first half of Theorem 1.2, concerning $F_0(\mathbb{Z}_{\geq 0}^3)$. In future work, we will upgrade (1.8) to an asymptotic (1.3), and complete the proof of Theorem 1.2, assuming a natural first-moment Ratios Conjecture (RA1) in adelic boxes of vectors $\mathbf{c}$.

1.1 Proof overview

Throughout §1.1, we assume $F = \sum_{i=1}^6 x_i^3$. Like [HB98], our work on $N_F(X)$ opens with the *delta method* of [DFI93, HB96]. This in principle allows for cancellation over the difficult minor arcs in the classical circle method. For certain exponential sums $S_{\mathbf{c}}(n)$ and integrals $I_{\mathbf{c}}(n)$, recalled in §2, that can be interpreted as weighted averages over circular regions $\Omega \in \{(\mathbb{Z}/n\mathbb{Z})^\times, \mathbb{R}\}$, we need to estimate sums like

$$\mathcal{S} := \sum_{\mathbf{c} \in \mathcal{S}_1} \frac{I_{\mathbf{c}}(N_0 N_1)}{X^3 (N_0 N_1)^6} \sum_{n_0 \in [N_0, 2N_0]: n_0 | (3F^\vee(\mathbf{c}))^\infty} S_{\mathbf{c}}(n_0) \sum_{n_1 \in [N_1, 2N_1]: \gcd(n_1, 3F^\vee(\mathbf{c}))=1} S_{\mathbf{c}}(n_1),$$

where $N_0 N_1 \ll X^{3/2}$. The analogous sums over $\mathbf{c} \in \mathcal{S}_0$ were handled unconditionally in our previous work [Wan24b], accounting for the special subvarieties $\mathbf{L} \in \mathcal{Y}$. As for $\mathcal{S}_1$, it turns out that we must decompose $\mathcal{S}$ into various new pieces relative to [Hoo97, HB98]; these pieces are defined in terms of $F^\vee$ and some factorizations of Euler products.

The sum $\mathcal{S}$ contains several factors of differing natures. Here [Hoo97, HB98] would control $I_{\mathbf{c}}(\cdot)$ and $S_{\mathbf{c}}(n_0)$ pointwise over Ω , and use GRH over n_1 , to get $\mathcal{S} \ll_\varepsilon X^{3+\varepsilon}$. We prove $\mathcal{S} \ll X^3 N_0^{-\delta} (X^{3/2}/N_0 N_1)^{-\delta}$, which is ε -free in X and decaying in both N_0 and $X^{3/2}/N_0 N_1$. When N_0 and $X^{3/2}/N_0 N_1$ are tiny, the Ratios Conjecture controls $\mathcal{S}$ well; here, a variant of Chebyshev’s bias [RS94] for the Dirichlet series $\sum_{n_1 \geq 1: \gcd(n_1, 3F^\vee(\mathbf{c}))=1} S_{\mathbf{c}}(n_1) n_1^{-s}$ ensures that we lose no powers of $\log X$. In general, we combine L -function techniques with other new inputs, identifying cancellation over Ω based roughly on the size and factorization of $F^\vee(\mathbf{c})$, to handle $I_{\mathbf{c}}(\cdot)$ and $S_{\mathbf{c}}(n_0)$.

§2 gives background on discriminants and the delta method. The delta method connects $N_{F,w}(X)$ to the local behavior of $V_{\mathbf{c}}$ over $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{R}$, and other rings, as $\mathbf{c} \in \mathbb{Z}^m$ and p vary. We highlight several distinct sources of epsilon in the work of Hooley and Heath-Brown within this method, and sketch “cures” thereof.

§3 provides background on Hasse–Weil L -functions and automorphic L -functions.

§4 gives some local control on polynomials and L -factors. We need this for some local estimates and calculations. The control on L -factors is based on [Kis99].

§5 gives a useful smooth framework for dyadic decomposition and separation of variables. This lets us break certain key sums throughout the paper into more manageable pieces.

§6 derives some L -function statistics over $\mathbf{c} \in \mathcal{S}_1$. We first do local calculations connected by [SST16, Universality Conjecture] to RMT Symmetry Types. We then state and build on some instances of the Ratios Conjecture for $L(s, V_{\mathbf{c}})$.

§7 begins to connect the pure L -function statistics from §6 to the delta method. We approximate certain Dirichlet series past the critical line, in a reasonably simple and uniform way over $\mathbf{c} \in \mathcal{S}_1$, and control the resulting approximation errors on average.

The “mollified” series $\Phi^{\mathbf{c},1}(s)$ from (1.6) not only simplifies the formulas in §6, but also holds significance in the delta method, as we will see in §7. Roughly speaking, the structure of (1.6) removes biases of the type encountered in [RS94, Lemma 2.1].

§8 proves new bounds on $I_{\mathbf{c}}(\cdot)$ sensitive to some real geometry involving discriminants, via cancellation over weighted arcs in critical ranges. Our approach shares some important features with [Hua20], a beautiful recent paper on *approximate* integral points.

§9, like §8, gives some discriminant-sensitive estimates, but on $S_{\mathbf{c}}(\cdot)$ instead of $I_{\mathbf{c}}(\cdot)$. Finite-field geometry and [Eke91, Bha14]’s geometric sieve both play an important role here, as does a useful result of [BJ14] on discriminants. It is crucial in §9 that $\deg F \geq 3$; see Remark 9.2. This reflects a randomness not present for quadrics with $m \equiv 0 \pmod{2}$.

§10 ties everything together, using Hölder’s inequality over $\mathbf{c} \in \mathcal{S}_1$ to separate the different aspects studied in previous sections, in order to prove Theorem 1.7. This entirely positive Hölder argument does not detect any cancellation over $\mathbf{c}$ that would go beyond a log-free Mertens-type heuristic on average (cf. [Ng04, Theorem 1(iii)]) over $\mathbf{c}$. Despite the decoupling convenience and power of Hölder, we must therefore be careful to obtain ε -free bounds.

1.2 Conventions

The reader may assume $F = \sum_{i=1}^6 x_i^3$, but we work generally *when this causes no complications*. By default, we assume $F \in \mathbb{Z}[x_1, \dots, x_m]$ has nonzero discriminant, and $m \geq 4$. Additional hypotheses are specified when appropriate, as in Theorems 2.5 and 10.5. We call F *diagonal* if $F = \sum_{i=1}^m F_i x_i^3$ for some nonzero integers $F_1, \dots, F_m$.

We write $f \ll_S g$, or $g \gg_S f$, to mean $|f| \leq Cg$ for some $C = C(S) > 0$. We let $O_S(g)$ denote a quantity that is $\ll_S g$. We write $f \asymp_S g$ to mean $f \ll_S g \ll_S f$. We let $o_{S;X \rightarrow \infty}(g)$ denote a quantity f such that for every $\varepsilon > 0$, there exists $X_0 = X_0(\varepsilon, S) > 0$ such that $|f| \leq \varepsilon g$ holds for all $X \geq X_0$.

We let $\mathbf{1}_E := 1$ if E holds, and $\mathbf{1}_E := 0$ otherwise. We let $\mathbb{Z}_{\geq 0} := \{a \in \mathbb{Z} : a \geq 0\}$, and similarly define sets like $\mathbb{Z}_{\neq 0}$, $\mathbb{R}_{>1}$, $\mathbb{R}_{\geq 2}$. For $n \in \mathbb{Z}_{\geq 1}$, we let $\varphi(n)$ denote the totient function, and $\text{rad}(n)$ the radical of n . We write $n \mid R^{\infty}$ to mean $\text{rad}(n) \mid R$.

We let $C_c^{\infty}(\mathbb{R}^s)$ (resp. $C_c^{\infty}(\mathbb{R}^s) \otimes \mathbb{C}$) denote the set of smooth compactly supported functions $\mathbb{R}^s \rightarrow \mathbb{R}$ (resp. $\mathbb{R}^s \rightarrow \mathbb{C}$). For any function $f = f(\mathbf{u})$, we let $\text{Supp } f$ denote the closure of $\{\mathbf{u} : f(\mathbf{u}) \neq 0\}$ in the domain of f . In integrals, we use notation analogous to summation notation. For instance, we write $\int_X dx f(x)$ to mean $\int_X f(x) dx$.

We need concise notation for L^p -norms and ℓ^p -norms. If f is a quantity depending on a scalar or vector variable t , and possibly also on other variables, we write $\|f\|_{L_t^p(S)} := (\int_{t \in S} dt |f|^p)^{1/p}$ or $\|f\|_{\ell_t^p(S)} := (\sum_{t \in S} |f|^p)^{1/p}$ to denote the p -norm of f over $t \in S$, according as t is a continuous or discrete variable, respectively. If the variable t is clear from context, we may omit it.

We let $\partial_u := \partial/\partial u$ for $u \in \mathbb{R}$. When doing calculus in d dimensions, a *multi-index* is a tuple of d nonnegative integers, where d will always be clear from context. Given a multi-index $\alpha \geq 0$, we let $|\alpha|$ denote the sum of the coordinates of α . For a vector $\mathbf{u} \in \mathbb{R}^s$, we let $\|\mathbf{u}\| := \max_i(|u_i|)$, and write $d\mathbf{u} := du_1 \cdots du_s$ and $\partial_{\mathbf{u}}^{\alpha} := \partial_{u_1}^{\alpha_1} \cdots \partial_{u_s}^{\alpha_s}$. For example, if $f = f(\mathbf{u}) \in C_c^{\infty}(\mathbb{R}^s) \otimes \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$, then under our conventions,

$$\max_{|\alpha| \leq k} \max_{\mathbf{u} \in \mathbb{R}^s} |\partial_{\mathbf{u}}^{\alpha} f| = \max_{\alpha_1, \dots, \alpha_s \geq 0: \alpha_1 + \dots + \alpha_s \leq k} \max_{u_1, \dots, u_s \in \mathbb{R}} |\partial_{u_1}^{\alpha_1} \cdots \partial_{u_s}^{\alpha_s} f|.$$

For repeated later use, we fix a function $v_0 \in C_c^{\infty}(\mathbb{R}^m)$, supported on $[-2, 2]^m$, with $0 \leq v_0 \leq 1$ everywhere, $v_0 = 1$ on $[-\frac{1}{2}, \frac{1}{2}]^m$, and $\int_{\mathbb{R}^m} d\mathbf{x} v_0(\mathbf{x}) = 1$. We also fix a radial¹ function $v_1 \in C_c^{\infty}(\mathbb{R}^m)$, supported on the annulus $m \leq \sum_{i=1}^m x_i^2 \leq m^2$, such that $\int_{\lambda > 0} \frac{d\lambda}{\lambda} v_1(\mathbf{x}/\lambda) = 1$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$. In particular, $v_1|_{[-1, 1]^m} = 0$ and $\text{Supp } v_1 \subseteq [-m, m]^m$.

2 Background on discriminants and delta

As in §1, let V be the projective hypersurface over $\mathbb{Q}$ defined by F , and let $F^{\vee} \in \mathbb{Z}[c_1, \dots, c_m]$ be an arbitrary polynomial defining the dual variety of V . Let $\nabla F := (\partial_{x_1} F, \dots, \partial_{x_m} F)$. Let $\text{disc}(F)$ be the *divided* discriminant of F , as in [Ter18, Definition 1.13] or [BJ14, Definition 4.6]. Let $\text{disc}(F, \mathbf{c})$ be the discriminant associated in [Ter18, §1.1] to the pair $(F(\mathbf{x}), \mathbf{c} \cdot \mathbf{x})$; it is a bi-homogeneous polynomial expression in $(F, \mathbf{c})$. Let

$$\Delta(\mathbf{c}) := \text{disc}(F) \cdot \text{disc}(F, \mathbf{c}) \in \mathbb{Z}[c_1, \dots, c_m]. \quad (2.1)$$

Then $\Delta/F^{\vee} \in \mathbb{Q}^{\times}$ by, say, [Wan23a, Proposition 4.4]. Also, by [Wan23a, Proposition 3.2],

$$\text{disc}(F, \mathbf{c}) = \pm \text{disc}(F(x_1, \dots, x_{m-1}, -\sum_{i=1}^{m-1} c_i x_i / c_m)) \cdot c_m^{3 \cdot 2^{m-2}}, \quad (2.2)$$

with the discriminant on the right being for a cubic form in $m-1$ variables.

The equality (2.2) lets us bring into play (for §9) a beautiful result of [BJ14]:

Theorem 2.1 ([BJ14, Corollary 4.30]). *Let R be a ring and let $f \in R[x_1, \dots, x_{m-1}]$ be a homogeneous polynomial. If $1 \leq a \leq m-1$, then there exists $N \geq 0$ such that $x_a^N \text{disc}(f)$ lies in the homogeneous ideal of $R[x_1, \dots, x_{m-1}]$ generated by*

$$f, \{(\partial_{x_i} f)(\partial_{x_j} f) : 1 \leq i, j \leq m-1\}.$$

Corollary 2.2. *If $1 \leq a, b \leq m$, then there exists $N \geq 0$ such that $x_a^N c_b^N \Delta(\mathbf{c})$ lies in the homogeneous ideal of $\mathbb{Z}[x_1, \dots, x_m, c_1, \dots, c_m]$ generated by*

$$F(\mathbf{x}), \mathbf{c} \cdot \mathbf{x}, \{(c_k \cdot \partial_{x_i} F - c_i \cdot \partial_{x_k} F)(c_k \cdot \partial_{x_j} F - c_j \cdot \partial_{x_k} F) : 1 \leq i, j, k \leq m\}.$$

Proof. Assume $b = m$. Then use (2.1), (2.2), Theorem 2.1 with $R = \mathbb{Z}[c_1/c_m, \dots, c_{m-1}/c_m]$, and the congruence $-\sum_{i=1}^{m-1} c_i x_i / c_m \equiv x_m \pmod{(\mathbf{c}/c_m) \cdot \mathbf{x}}$ in $R[x_1, \dots, x_m]$. $\square$

¹in the Euclidean sense (so that the value of $v_1(\mathbf{x})$ depends only on $\sum_{i=1}^m x_i^2$)

By the definition of $F^\vee$, we have $F(\mathbf{x}) \mid F^\vee(\nabla F(\mathbf{x}))$. Since $\Delta/F^\vee \in \mathbb{Q}^\times$, it follows that

$$\Delta(\nabla F(\mathbf{x}))/F(\mathbf{x}) \in \mathbb{Q}[x_1, \dots, x_m]. \quad (2.3)$$

Also, $\Delta(\mathbf{c})$ is irreducible in $\overline{\mathbb{Q}}[c_1, \dots, c_m]$, and has total degree $3 \cdot 2^{m-2}$ in $c_1, \dots, c_m$. In particular, $\deg \Delta \geq 12$, so by [CCDN20, Theorem 1], we have the convenient bound

$$|\mathcal{S}_0 \cap [-Z, Z]^m| \ll_m Z^{m-2} \quad (2.4)$$

for $Z \geq 1$. (For diagonal F , a stronger bound $|\mathcal{S}_0 \cap [-Z, Z]^m| \ll_{m,\varepsilon} Z^{m/2} \mathbf{1}_{2|m} + Z^{(m-1)/2+\varepsilon}$ can be proven along the lines of [HB98, p. 687] or [Wan25, Lemma 7.5], since for any partition $\sum_{k \geq 1} \lambda_k = m$ with some part $\lambda_k \notin \{0, 2\}$, we have $\sum_{k \geq 1} e_k \leq m-1$, where $e_k := 2 \cdot \mathbf{1}_{2 \leq \lambda_k \leq 4} + (\lambda_k - 2) \cdot \mathbf{1}_{\lambda_k \geq 5}$. This is implicit in [Wan24b, Proposition 8.1] for $m \in \{4, 6\}$, and a weaker $Z^{m/2+\varepsilon}$ bound is recorded in [Wan25, display after (7.11)] for all $m \geq 4$.)

For any $\mathbf{c} \in \mathbb{Z}^m$ and prime p with $p \nmid \Delta(\mathbf{c})$, the varieties $F(\mathbf{x}) = 0$ and $F(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} = 0$ in $\mathbb{P}_{\mathbb{F}_p}^{m-1}$ are both smooth. Now recall $\mathcal{S}_0, \mathcal{S}_1$ from (1.4). For each $\mathbf{c} \in \mathcal{S}_1$, let

$$\mathcal{N}^c := \{n \geq 1 : p \mid n \Rightarrow p \nmid \Delta(\mathbf{c})\}, \quad \mathcal{N}_c := \{n \geq 1 : p \mid n \Rightarrow p \mid \Delta(\mathbf{c})\}. \quad (2.5)$$

Lemma 2.3. *Let $N, R \geq 1$ be integers. Then $|\{N \leq n < 2N : n \mid R^\infty\}| \ll_\varepsilon (RN)^\varepsilon$. In particular, if $\mathbf{c} \in \mathcal{S}_1$, then $|\{N \leq n < 2N : n \in \mathcal{N}_c\}| \ll_\varepsilon \|\mathbf{c}\|^\varepsilon N^\varepsilon$.*

Proof. See e.g. [HB98, antepenultimate display of p. 683]. $\square$

Let $\text{Hess} F$ be the Hessian matrix of F . Fix $w \in C_c^\infty(\mathbb{R}^m)$ satisfying

$$\text{Supp } w \subseteq \{\mathbf{x} \in \mathbb{R}^m : \det(\text{Hess} F(\mathbf{x})) \neq 0\}. \quad (2.6)$$

We now recall the delta method for the quantity $N_{F,w}(X)$ in (1.1). Let $h: \mathbb{R}_{>0} \times \mathbb{R} \rightarrow \mathbb{R}$ be the smooth function given by [HB98, (2.3)]; we will need and recall the full definition later, in §8. For each real $X \geq 2$, let $Y := X^{3/2}$. Let $e(t) := e^{2\pi i t}$ and $e_r(t) := e(t/r)$. Let

$$I_c(n) := \int_{\mathbf{x} \in \mathbb{R}^m} d\mathbf{x} w(\mathbf{x}/X) h(n/Y, F(\mathbf{x})/Y^2) e(-\mathbf{c} \cdot \mathbf{x}/n), \quad (2.7)$$

$$S_c(n) := \sum_{1 \leq a \leq n: \gcd(a,n)=1} \sum_{1 \leq \mathbf{x} \leq n} e_n(aF(\mathbf{x}) + \mathbf{c} \cdot \mathbf{x}), \quad (2.8)$$

$$S_c^\natural(n) := n^{-(m+1)/2} S_c(n), \quad J_{c,X}(n) := X^{-m} I_c(n). \quad (2.9)$$

A simple rearrangement of [HB98, (2.1)] gives, for all $A > 0$,

$$(1 + O_A(Y^{-A})) \cdot N_{F,w}(X)/X^{m-3} = \sum_{n \geq 1} \sum_{\mathbf{c} \in \mathbb{Z}^m} n^{(1-m)/2} S_c^\natural(n) J_{c,X}(n). \quad (2.10)$$

The infinite sums in (2.10) are essentially finite, by the following standard result:

Proposition 2.4 ([HB98, (2.3)–(3.7)]). *For some constant $A_0 > 0$, we have $J_{c,X}(n) = 0$ for all $n \geq A_0 Y$. Also, $J_{c,X}(n) \ll_{\varepsilon,A} X^{-A}$ holds whenever $\varepsilon, A > 0$ and $\|\mathbf{c}\| \geq X^{1/2+\varepsilon}$.*

Let $m_* := m - 3$. We now recall some background [Hoo86b, §§5–6] on the sums $S_{\mathbf{c}}(n)$. It is known that $S_{\mathbf{c}}(n)$, and thus $S_{\mathbf{c}}^{\natural}(n)$ too, is multiplicative in n . The Dirichlet series

$$\Phi(\mathbf{c}, s) := \sum_{n \geq 1} n^{-s} S_{\mathbf{c}}^{\natural}(n) \quad (2.11)$$

thus has an Euler product. One can approximate Φ in terms of Hasse–Weil zeta functions. Given a prime power q , let $\mathcal{V}(\mathbb{F}_q)$ (resp. $\mathcal{V}_{\mathbf{c}}(\mathbb{F}_q)$) be the set of $\mathbb{F}_q$ -points on the variety $F(\mathbf{x}) = 0$ (resp. $F(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} = 0$) in $\mathbb{P}_{\mathbb{F}_q}^{m-1}$, and let

$$E_F(q) := |\mathcal{V}(\mathbb{F}_q)| - |\mathbb{P}^{1+m_*}(\mathbb{F}_q)|, \quad E_{\mathbf{c}}(q) := |\mathcal{V}_{\mathbf{c}}(\mathbb{F}_q)| - |\mathbb{P}^{m_*}(\mathbb{F}_q)|,$$

where $|\mathbb{P}^d(\mathbb{F}_q)| = (q^{d+1} - 1)/(q - 1)$. Then let

$$E_F^{\natural}(q) := q^{-(1+m_*)/2} E_F(q), \quad E_{\mathbf{c}}^{\natural}(q) := q^{-m_*/2} E_{\mathbf{c}}(q). \quad (2.12)$$

If $p \nmid \mathbf{c}$, then by [Hoo14, Lemma 7], we have

$$S_{\mathbf{c}}^{\natural}(p) = E_{\mathbf{c}}^{\natural}(p) - p^{-1/2} E_F^{\natural}(p). \quad (2.13)$$

If $p \nmid \Delta(\mathbf{c})$ and $l \geq 2$, then by [Hoo14, Lemma 10], we have

$$S_{\mathbf{c}}^{\natural}(p^l) = 0. \quad (2.14)$$

Recall $\mathcal{S}_0, \mathcal{S}_1$ from (1.4). For any set $\mathcal{S} \subseteq \mathbb{Z}^m$, let

$$\Sigma^{\natural}(X, \mathcal{S}) := \sum_{n \geq 1} \sum_{\mathbf{c} \in \mathcal{S}} n^{(1-m)/2} S_{\mathbf{c}}^{\natural}(n) J_{\mathbf{c}, X}(n), \quad \Sigma(X, \mathcal{S}) := X^{m-3} \Sigma^{\natural}(X, \mathcal{S}). \quad (2.15)$$

If $\mathbf{c} \in \mathcal{S}_0$, then $\Phi(\mathbf{c}, s)$ resembles $\zeta(s - \frac{1}{2})$ on average, leading to the following result:

Theorem 2.5 ([Wan24b]). *Suppose $m = 6$ and F is diagonal. Then*

$$\Sigma(X, \mathcal{S}_0) = \mathfrak{S}_F \cdot \sigma_{\infty, F, w} \cdot X^3 + \sum_{\mathbf{L} \in \mathbf{Y}} \sum_{\mathbf{x} \in \mathbf{L} \cap \mathbb{Z}^6} w(\mathbf{x}/X) + O_{F, w, \varepsilon}(X^{2.75+\varepsilon}) \quad (2.16)$$

unconditionally. In particular, $\Sigma(X, \mathcal{S}_0) \ll_{F, w} X^3$.

Proof. This follows from [Wan24b, Corollary 1.2 and (1.7)], even if we relax the condition (2.6) to $\mathbf{0} \notin \text{Supp } w$. The earlier paper [HB98] had proved $\Sigma(X, \mathcal{S}_0) \ll_{F, w, \varepsilon} X^{3+\varepsilon}$. $\square$

Since $N_{F, w}(X) = \Sigma(X, \mathcal{S}_0) + \Sigma(X, \mathcal{S}_1) + O_A(X^{-A})$, we may thus concentrate on $\Sigma(X, \mathcal{S}_1)$. The rest of §2 provides some technical context for our work (in comparison with Theorem 1.4 due to [Hoo97, HB98]), but is not logically necessary for the paper.

If $\mathbf{c} \in \mathcal{S}_1$, then by (2.13), (2.14), and the Grothendieck–Lefschetz trace formula, one might expect $\Phi(\mathbf{c}, s)$ to resemble $L(s, V_{\mathbf{c}})^{(-1)^{m_*}}$, up to a factor absolutely convergent for $\text{Re}(s) > \frac{1}{2}$. With this in mind, we now recall a proof that (as is key for Theorem 1.4 when $m = 6$)

$$\Sigma^{\natural}(X, \mathcal{S}_1) \ll_{\varepsilon} X^{(6-m)/4+\varepsilon} \quad (2.17)$$

under Conjecture 1.3 for $L(s, V_{\mathbf{c}})$, when m is even and F is diagonal.

Conditional proof sketch for (2.17). For this proof sketch only, let $Z := X^{1/2+\varepsilon_0}$. Let

$$\Psi^{c,1}(s) := 1/L(s, V_c), \quad \Psi^{c,2}(s) := \Phi(c, s)L(s, V_c), \quad (2.18)$$

for $c \in \mathcal{S}_1$; then $\Phi = \Psi^{c,1}\Psi^{c,2}$. Let $\mu_c(n)$, $a_c^0(n)$ be the n th coefficients of the Dirichlet series $\Psi^{c,1}$, $\Psi^{c,2}$, respectively. Then the following holds (see e.g. [Wan22, [B1']] in §4.1):

[B1'] For $N \leq Z^3$, we have $\sum_{c \in \mathcal{S}_1 \cap [-Z, Z]^m} \sum_{N \leq n < 2N} |a_c^0(n)| \ll_\varepsilon Z^{m+\varepsilon} N^{1/2}$.

The arguments in [Hoo86b, Hoo97] and [HB98] can then loosely be interpreted as

(H1) using partial summation over $n \in [N, 2N]$ to “factor out” $J_{c,X}(n)$ from the sum over n in (2.15) for $\mathcal{S} = \mathcal{S}_1$, and then bounding the J -contribution in $\ell_n^\infty([N, 2N])$;

(H2) expanding $S_c^\natural = \mu_c * a_c^0$ using $\Phi = \Psi^{c,1}\Psi^{c,2}$;

(H3) using GRH to bound the $\Psi^{c,1}$ -contribution in $\ell_c^\infty(\mathcal{S}_1 \cap [-Z, Z]^m)$; and

(H4) using [B1'] afterwards, to bound the $\Psi^{c,2}$ -contribution in $\ell_c^1(\mathcal{S}_1 \cap [-Z, Z]^m)$.

Upon dyadic summation over $1 \ll N \ll Y$, one gets $\Sigma^\natural(X, \mathcal{S}_1) \ll_{\varepsilon_0} X^{(6-m)/4+O(\varepsilon_0)}$. □

We now diagnose, and sketch “cures” for, some key sources of ε above:

1. In (H3), the *pointwise* GRH bound $\sum_{N \leq n < 2N} \mu_c(n) \ll_\varepsilon \|c\|^\varepsilon N^{1/2+\varepsilon}$. (*Cure*: RMT-type predictions such as Conjecture 1.5.)
2. In (H4), the bound [B1']. In fact, the proof of [B1'] has *two* sources of ε :
 - (a) Prime factors $p \nmid \Delta(c)$ of n , via the factor $\Psi^{c,2}$. (*Cure*: Replacing $\Psi^{c,1}$ with a better approximation of Φ .)
 - (b) Primes $p \mid \gcd(\Delta(c), n)$, via the *sometimes large* failure of square-root cancellation in *individual* sums of the form $S_c(p^l)$. (*Cure*: New and old pointwise bounds on $|S_c(p^l)|$, and a new saving on average via Conjecture 1.6.)
3. The fact that over $1 \ll N \ll Y$, each dyadic range $N \leq n < 2N$ contributes *roughly equally* to the final bound (2.17); cf. [Wan22, Remark 4.1.1]. If unaddressed, then a fatal $\log X$ factor would arise in our proof of Theorem 1.7, via summation over N . (*Cure*: New integral bounds that decay, as $n \rightarrow 0$, fairly uniformly over c .)

3 Background on individual L -functions

3.1 Geometric background

We need to give a precise meaning to Conjecture 1.3. We first define the necessary Hasse–Weil L -functions and their local factors, following [Ser70, Kah20]. Let $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s}\Gamma(s)$.

The case of the Riemann zeta function $\zeta(s)$ in (1.5) is familiar, with $\zeta_p(s) = (1 - p^{-s})^{-1}$ and $\zeta_\infty(s) = \Gamma_{\mathbb{R}}(s)$.

Now let $\mathbf{c} \in \mathcal{S}_1$. For any perfect field K , let $G_K := \text{Gal}(\bar{K}/K)$. Let ℓ be a prime, and consider the ℓ -adic Galois representations

$$\rho_V: G_{\mathbb{Q}} \rightarrow \frac{H^{1+m_*}(V \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_{\ell})}{H^{1+m_*}(\mathbb{P}_{\mathbb{Q}}^{m-1}, \mathbb{Q}_{\ell})}, \quad \rho_{V_{\mathbf{c}}}: G_{\mathbb{Q}} \rightarrow \frac{H^{m_*}(V_{\mathbf{c}} \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_{\ell})}{H^{m_*}(\mathbb{P}_{\mathbb{Q}}^{m-1}, \mathbb{Q}_{\ell})}.$$

It is known that the representations $\rho_V, \rho_{V_{\mathbf{c}}}, \otimes^2 \rho_{V_{\mathbf{c}}}, \text{Sym}^2 \rho_{V_{\mathbf{c}}}, \wedge^2 \rho_{V_{\mathbf{c}}}$ of $G_{\mathbb{Q}}$ have dimensions depending only on m , and are pure of weight $1 + m_*, m_*, 2m_*, 2m_*, 2m_*$, respectively.

Let $\rho: G_{\mathbb{Q}} \rightarrow M$ be one of these five representations. Let d_{ρ}, w_{ρ} be the dimension and weight of ρ , respectively. Define $\Gamma_{\rho}(s)$ using Hodge theory, following [Ser70, §3.2], after passing from M to a singular cohomology group $M'/\mathbb{C}$ independent of ℓ . Then for certain $h_{\rho}(+), h_{\rho}(-), h_{\rho}(a, b) \in \mathbb{Z}_{\geq 0}$ with $h_{\rho}(+) + h_{\rho}(-) + 2 \sum_{0 \leq a < b: a+b=w_{\rho}} h_{\rho}(a, b) = d_{\rho}$, we have

$$\Gamma_{\rho}(s) = \Gamma_{\mathbb{R}}(s - \frac{w_{\rho}}{2})^{h_{\rho}(+)} \Gamma_{\mathbb{R}}(s - \frac{w_{\rho}}{2} + 1)^{h_{\rho}(-)} \prod_{0 \leq a < b: a+b=w_{\rho}} \Gamma_{\mathbb{C}}(s - a)^{h_{\rho}(a, b)}. \quad (3.1)$$

Let $L_{\infty}(s, \rho) := \Gamma_{\rho}(s + \frac{w_{\rho}}{2})$; then $L_{\infty}(s, \rho)$ is holomorphic on the half-plane $\text{Re}(s) > 0$. Note that if we let $\mathbf{c}, \rho$ vary, the number of possible functions $L_{\infty}(s, \rho)$ could be is $\ll_m 1$.

For each prime $p \neq \ell$, consider the restriction of ρ to $G_{\mathbb{Q}_p}$. Following [Ser70, §2.2], let $P_p(T) := \det(1 - \pi_p T) \in \mathbb{Q}_{\ell}[T]$ be the reverse characteristic polynomial of the geometric Frobenius element π_p on $M^{I_{\mathbb{Q}_p}}$, the inertia invariants of M . Clearly $\deg P_p \leq d_{\rho}$, with equality if and only if ρ is unramified at p . Write $P_p(T) = \prod_{1 \leq j \leq \deg P_p} (1 - \alpha_{\rho, j}(p) T)$, and let

$$\tilde{\alpha}_{\rho, j}(p) := p^{-w_{\rho}/2} \alpha_{\rho, j}(p), \quad L_p(s, \rho) := \prod_{1 \leq j \leq \deg P_p} (1 - \tilde{\alpha}_{\rho, j}(p) p^{-s})^{-1} = P_p(p^{-s-w_{\rho}/2})^{-1}. \quad (3.2)$$

Because $V, V_{\mathbf{c}}$ are complete intersections in $\mathbb{P}_{\mathbb{Q}}^{m-1}$, it is known [Sai03, Sch12, Las17] that $P_p(T)$ lies in $\mathbb{Q}[T]$ and is independent of ℓ , and that for any embedding of $\alpha_{\rho, j}(p)$ into $\mathbb{C}$,

$$|\alpha_{\rho, j}(p)| \leq p^{w_{\rho}/2}, \quad |\tilde{\alpha}_{\rho, j}(p)| \leq 1. \quad (3.3)$$

Let $L_{\infty}(s, V) := L_{\infty}(s, \rho_V)$ for any ℓ . Given a prime p , let $L_p(s, V) := L_p(s, \rho_V)$ for any $\ell \neq p$, and let $\tilde{\alpha}_{V, j}(p) := \tilde{\alpha}_{\rho_V, j}(p)$ and $d_{V, p} := \deg P_p$. Let

$$L(s, V) := \prod_{p < \infty} L_p(s, V) =: \sum_{n \geq 1} \lambda_V^{\natural}(n) n^{-s},$$

so that $\lambda_V^{\natural}(p) = \sum_{1 \leq j \leq d_{V, p}} \tilde{\alpha}_{V, j}(p)$ for all p . Let $\deg L(s, V) := \max_p d_{V, p} = d_{\rho_V}$. Make analogous definitions for $L(s, V_{\mathbf{c}})$ and its tensor squares in (1.5), in terms of $\rho_{V_{\mathbf{c}}}$ and its tensor squares. If $p \nmid \Delta(\mathbf{c})$ and $d = \deg L(s, V_{\mathbf{c}})$, then by (2.12), (3.2), smooth proper base change, and the Grothendieck–Lefschetz trace

formula, we have

$$\lambda_V^{\natural}(p) = \sum_{1 \leq j \leq \deg L(s, V)} \tilde{\alpha}_{V, j}(p) = (-1)^{1+m_*} E_F^{\natural}(p), \quad (3.4)$$

$$\lambda_{V_c}^{\natural}(p) = \sum_{1 \leq j \leq d} \tilde{\alpha}_{V_c, j}(p) = (-1)^{m_*} E_c^{\natural}(p), \quad (3.5)$$

$$\lambda_{V_c, \text{Sym}^2}^{\natural}(p) = \sum_{1 \leq i \leq j \leq d} \tilde{\alpha}_{V_c, i}(p) \tilde{\alpha}_{V_c, j}(p) = \lambda_{V_c}^{\natural}(p^2), \quad (3.6)$$

$$\lambda_{V_c, \wedge^2}^{\natural}(p) = \sum_{1 \leq i < j \leq d} \tilde{\alpha}_{V_c, i}(p) \tilde{\alpha}_{V_c, j}(p) = \lambda_{V_c}^{\natural}(p)^2 - \lambda_{V_c}^{\natural}(p^2), \quad (3.7)$$

$$(-1)^{m_*} E_c^{\natural}(p^2) = \sum_{1 \leq j \leq d} \tilde{\alpha}_{V_c, j}(p)^2 = E_c^{\natural}(p)^2 - 2\lambda_{V_c, \wedge^2}^{\natural}(p), \quad (3.8)$$

where for all j , we have by Deligne's resolution of the Weil Conjectures, since $p \nmid \Delta(\mathbf{c})$,

$$|\tilde{\alpha}_{V, j}(p)| = 1, \quad |\tilde{\alpha}_{V_c, j}(p)| = 1. \quad (3.9)$$

Remark 3.1. Let $n \geq 1$ be an integer. By working locally from the beginning, one can define the quantities $\prod_{p|n} L_p(s, V_c)$ and $\lambda_{V_c}^{\natural}(n)$ on all of $\{\mathbf{c} \in \prod_{p|n} \mathbb{Z}_p^m : \Delta(\mathbf{c}) \neq 0\}$. Extended definitions like this will be convenient for local calculations in §6.1.

3.2 Automorphic background

We need some background on automorphic representations Π of $\text{GL}_d(\mathbb{A}_{\mathbb{Q}})$ for $d \geq 1$. We will only work with *cuspidal* Π 's, or more generally, *isobaric* Π 's. These Π 's have well-defined L -functions $L(s, \Pi)$, and good formal properties (due to Rankin, Selberg, Langlands, Godement, Jacquet, Shalika, and others):

1. If Π is cuspidal, then $L(s, \Pi)$ is *primitive* in the sense of [FPRS19, Lemma 2.4], and has certain familiar analytic properties [FPRS19, Theorem 3.1].
2. For each isobaric Π , there is a *unique* multiset $\{\Pi_1, \dots, \Pi_r\}$, consisting of *cuspidals*, such that $L(s, \Pi) = L(s, \Pi_1) \cdots L(s, \Pi_r)$. We call the Π_i 's *cuspidal constituents* of Π .
3. *Strong multiplicity one*: If Π, Π' are isobaric, and $L_p(s, \Pi) = L_p(s, \Pi')$ for all but finitely many primes p , then $L(s, \Pi) = L(s, \Pi')$.

Conjecture 1.3 has a host of standard consequences, which may be treated as a black box.

Proposition 3.2. *Let $\mathbf{c} \in \mathcal{S}_1$. Let $L(s)$ be one of the Hasse–Weil L -functions in (1.5). Assume Conjecture 1.3 for $L(s)$ holds for some d, Π . Then the following hold:*

1. $d = \deg L(s)$; in particular, $d \ll_m 1$.
2. The conductor $q(\Pi) \in \mathbb{Z}_{\geq 1}$ of Π satisfies $q(\Pi) \mid \Delta(\mathbf{c})^{O_m(1)}$.
3. Each cuspidal constituent of Π has unitary, finite-order central character.

4. $L(s)$ is holomorphic on $\mathbb{C}$, except possibly for poles at $s = 1$ corresponding to trivial constituents of Π .
5. $(s-1)^d L(s)$ is an entire function of order 1.
6. $L(s)$ has a standard functional equation, with critical line $\operatorname{Re}(s) = \frac{1}{2}$, involving $q(\Pi)$ and some root number $\varepsilon(\Pi) \in \{z \in \mathbb{C} : |z| = 1\}$.
7. $L(s)$ has real coefficients, Π is self-dual, and $\varepsilon(\Pi) \in \{-1, +1\}$.
8. Let ψ be a cuspidal or isobaric constituent of Π . Then $1/L(s, \psi) \ll_{m, \varepsilon} \|c\|^\varepsilon (1 + |s|)^\varepsilon$ for $\operatorname{Re}(s) \geq \frac{1}{2} + \varepsilon$. If $\mu_\psi(n)$ denotes the n th coefficient of the Dirichlet series $1/L(s, \psi)$, then $\sum_{1 \leq n \leq N} \mu_\psi(n) n^{-it} \ll_{m, \varepsilon} \|c\|^\varepsilon (1 + |t|)^\varepsilon N^{1/2+\varepsilon}$ for all $t \in \mathbb{R}$ and $N \in \mathbb{R}_{>0}$.

Proof. If p is a prime, then $L_p(s, \Pi)$ has degree $\leq d$, with equality if and only if $p \nmid q(\Pi)$; cf. [FPRS19, Axiom 3(b) and (3.3)].

(1): Compare the degrees of $L_p(s)$, $L_p(s, \Pi)$ at a prime $p \nmid q(\Pi) \Delta(c)$.

(2): For some $v \in \mathbb{Z}$, the local Dirichlet polynomial $L_p(s - \frac{v}{2})$ has rational coefficients for all primes p . So by [ST14, (3.2)], there exists $u \in \mathbb{R}$ such that for all $p \nmid q(\Pi)$, the representation $\Pi_p \otimes (|\cdot|^u \circ \det)$ of $\operatorname{GL}_d(\mathbb{Q}_p)$ has field of rationality $\mathbb{Q}$, in the sense of [ST14, Definition 2.2]. So by strong multiplicity one, the field of rationality of $\Pi \otimes (|\cdot|^u \circ \det)$ is $\mathbb{Q}$. Now consider any $p \mid q(\Pi)$. Then $L_p(s, \Pi)$ has degree $< d$, so $L_p(s)$ has degree $< d$, whence $p \mid \Delta(c)$. By [ST14, Lemmas 3.11 and 3.13], then, $v_p(q(\Pi)) \ll_d 1$. So $q(\Pi) \mid \Delta(c)^{O_d(1)} \mid \Delta(c)^{O_m(1)}$.

(3): Let Π' be a cuspidal constituent of Π , so Π' is a cuspidal automorphic representation of $\operatorname{GL}_{d'}(\mathbb{A}_{\mathbb{Q}})$ for some $d' \geq 1$. Let $\omega' : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \rightarrow \mathbb{C}^\times$ be the central character of Π' . Then ω' corresponds to a classical character $n \mapsto |n|^z \chi(n)$ on $\mathbb{Z}$, where $z \in \mathbb{C}$ and χ is a Dirichlet character of conductor dividing $q(\Pi)$, such that $\omega'_p(p) = p^z \chi(p)$ for all primes $p \nmid q(\Pi)$. But at each prime $p \nmid q(\Pi)$, if we write $L_p(s, \Pi) = \prod_{1 \leq j \leq d'} (1 - \alpha_{\Pi, j}(p) p^{-s})^{-1}$, then $\omega'_p(p) = \prod_{1 \leq j \leq d'} \alpha_{\Pi, j}(p)$; cf. [FPRS19, (3.3) and its proof]. By (3.9) and the algebraicity of the eigenvalues $\tilde{\alpha}_j(p)$, it follows that for infinitely many primes p , we have $|p^z| = 1$ and $p^z \in \overline{\mathbb{Q}}$. So $\operatorname{Re}(z) = 0$, i.e. ω' is unitary; and then $\operatorname{Im}(z) = 0$ by the six exponentials theorem (cf. [FPRS19, proof of Lemma 4.9]), so ω' has finite order.

(4), (5), (6): Use (3) and results of Godement and Jacquet; cf. [FPRS19, Theorem 3.1].

(7): For some $v \in \mathbb{Z}$, the coefficients of $L(s - \frac{v}{2})$ are all rational. Hence $L(s)$ has real coefficients. So by (3) and strong multiplicity one, we have $L(s, \Pi) = L(s, \Pi^\vee)$ and thus Π is self-dual. The functional equation from (6) then implies $\varepsilon(\Pi) \in \mathbb{R}$, so $\varepsilon(\Pi) = \pm 1$.

(8): By (3.3), (5), (6), and Conjecture 1.3(2), we have $1/L(s, \psi) \ll_{d, \varepsilon} q(\psi)^\varepsilon (1 + |s|)^\varepsilon$ for $\operatorname{Re}(s) \geq \frac{1}{2} + \varepsilon$; see e.g. [IK04, Theorem 5.19 and the ensuing paragraph]. But $q(\psi) \mid q(\Pi) \mid \Delta(c)^{O_m(1)}$, so the desired bound on $1/L$ follows. By Perron's formula ([IK04, Proposition 5.54]) and contour integration, $\sum_{1 \leq n \leq N} \mu_\psi(n) n^{-it} \ll_{m, \varepsilon} \|c\|^\varepsilon (1 + |t|)^\varepsilon N^{1/2+\varepsilon}$. $\square$

4 Local control on polynomials and L -functions

We first recall some bounds on the local near-zero loci of a polynomial. Given $f \in \mathbb{Z}[x]$, let $N(f; q)$ be the number of solutions $x \bmod q$ to $f(x) \equiv 0 \bmod q$, and let $\mu_{\mathbb{R}}(f; \lambda)$ be the Lebesgue measure of

$\{x \in [-1, 1] : |f(x)| \leq \lambda\}$. Similarly, for $P \in \mathbb{Z}[y_1, \dots, y_n]$, define

$$\begin{aligned} N(P; q) &:= \#\{(y_1, \dots, y_n) \in (\mathbb{Z}/q\mathbb{Z})^n : P(y_1, \dots, y_n) \equiv 0 \pmod{q}\}, \\ \mu_{\mathbb{R}}(P; \lambda) &:= \text{vol}\{(y_1, \dots, y_n) \in [-1, 1]^n : |P(y_1, \dots, y_n)| \leq \lambda\}. \end{aligned}$$

Proposition 4.1. *Suppose $f \in \mathbb{Z}[x]$ has leading term ax^d with $a \neq 0$ and $d \geq 1$. Then*

$$N(f; q) \ll_d |a|^{1/d} q^{1-d^{-1}}, \quad (4.1)$$

uniformly over integers $q \geq 1$. Also, uniformly over reals $\lambda > 0$, we have

$$\mu_{\mathbb{R}}(f; \lambda) \ll_d |a|^{-1/d} \lambda^{1/d}. \quad (4.2)$$

Proof. The bound (4.2) goes back to Pólya; see e.g. [Gan01, Theorem 1.1]. Now let h denote the greatest common divisor of q and the coefficients of f . The bound (4.1) follows from [Kon79] if $h = 1$, and then in general from the inequality $N(f; q) \leq h \cdot N(f/h; q/h)$. $\square$

Corollary 4.2. *Fix a nonconstant polynomial $P \in \mathbb{Z}[y_1, \dots, y_n]$, where $n \geq 1$. Then*

$$N(P; q) \ll_P q^{n-(\deg P)^{-1}}, \quad (4.3)$$

uniformly over integers $q \geq 1$. Also, uniformly over reals $\lambda > 0$, we have

$$\mu_{\mathbb{R}}(P; \lambda) \ll_P \lambda^{1/\deg P}. \quad (4.4)$$

Proof. Let $d = \deg P$. Given a matrix $A \in \text{GL}_n(\mathbb{Q})$ with integral entries, let $Q = (\det A)^d \cdot P \circ A^{-1} \in \mathbb{Z}[y_1, \dots, y_n]$. Via the $\mathbb{Z}$ -linear map $\mathbf{y} \mapsto A\mathbf{y}$, we have $N(P; q) \ll_A N(Q; q)$ and $\mu_{\mathbb{R}}(P; \lambda) \ll_A \mu_{\mathbb{R}}(Q; \lambda)$. By [Eis95, p. 283, proof of Lemma 13.2.c], we may choose A so that Q has y_1 -leading term $a_1 y_1^d$, with $a_1 \neq 0$. Fix $y_2, \dots, y_n$; then $\#\{y_1 : Q(\mathbf{y}) \equiv 0\} \ll_{d, a_1} q^{1-d^{-1}}$ by (4.1). Summing over $y_2, \dots, y_n$ gives (4.3). Similarly, (4.2) implies (4.4).

(For nearly the same result and proof, see [PSW16, Lemmas 4.10 and 4.8].) $\square$

We now turn to local L -factors, defined in p -adic generality by Remark 3.1.

Proposition 4.3 ([Kis99]). *Fix a prime p and a vector $\mathbf{b} \in \mathbb{Z}_p^m$ with $\Delta(\mathbf{b}) \neq 0$. Then there exists an integer $l \geq 0$, depending only on p and $\mathbf{b}$, such that for all vectors $\mathbf{a} \in \mathbb{Z}_p^m$ with $\mathbf{a} \equiv \mathbf{b} \pmod{p^{1+l}}$, we have $\Delta(\mathbf{a}) \neq 0$ and $L_p(s, V_{\mathbf{a}}) = L_p(s, V_{\mathbf{b}})$.*

Proof. Let S be the open subscheme $\Delta(\mathbf{c}) \neq 0$ of $\mathbb{A}_{\mathbb{Q}_p}^m$. Let W be the closed subscheme $F(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} = 0$ of $\mathbb{P}_S^{m-1}$. Let $\ell \neq p$ be a prime. The maps $f_1: W \rightarrow S$ and $f_2: \mathbb{P}_S^{m-1} \rightarrow S$ induce local systems $\mathcal{L}_j := R^{m*}(f_j)_*(\mathbb{Z}_{\ell})$ on S . By [Kis99, Theorem 5.1, case (2), and its proof], there exists a p -adic neighborhood U of $\mathbf{b}$ in S such that the Galois representations $\rho_{j, \mathbf{a}}: G_{\mathbb{Q}_p} \rightarrow \text{Aut}(\mathcal{L}_{j, \mathbf{a}})$ for $\mathbf{a} \in U(\mathbb{Q}_p)$ all factor through $\pi_1(U)$ in an appropriate sense. So the isomorphism class of the representation $G_{\mathbb{Q}_p} \rightarrow \text{Aut}(\mathbb{Q}_{\ell} \otimes (\mathcal{L}_{1, \mathbf{a}}/\mathcal{L}_{2, \mathbf{a}}))$ is constant over $\mathbf{a} \in U(\mathbb{Q}_p)$, and thus $L_p(s, V_{\mathbf{a}})$ is too. $\square$

Lemma 4.4. Fix a prime p . For each $\mathbf{b} \in \mathbb{Z}_p^m$ with $\Delta(\mathbf{b}) \neq 0$, let $l(p, \mathbf{b}) \geq 0$ be the smallest integer for which the conclusion of Proposition 4.3 holds. Then for each integer $l \geq 0$, the set

$$\mathcal{L}(l) := \{\mathbf{b} \in \mathbb{Z}_p^m : \Delta(\mathbf{b}) \neq 0, l(p, \mathbf{b}) \leq l\}$$

is invariant under translation by any element of $p^{l+1}\mathbb{Z}_p^m$. Furthermore, the measure of $\mathcal{L}(l)$ tends to 1 as $l \rightarrow \infty$. (Here we use the usual Haar measure on $\mathbb{Z}_p^m$.)

Proof. Suppose $\mathbf{b}, \mathbf{c} \in \mathbb{Z}_p^m$ with $\Delta(\mathbf{b}) \neq 0$ and $\mathbf{c} \equiv \mathbf{b} \pmod{p^{1+l(p, \mathbf{b})}}$. Then $\Delta(\mathbf{c}) \neq 0$ and $L_p(s, V_{\mathbf{c}}) = L_p(s, V_{\mathbf{b}})$. For every $\mathbf{a} \equiv \mathbf{c} \pmod{p^{1+l(p, \mathbf{b})}}$, we have $\mathbf{a} \equiv \mathbf{b} \pmod{p^{1+l(p, \mathbf{b})}}$, whence $\Delta(\mathbf{a}) \neq 0$ and $L_p(s, V_{\mathbf{a}}) = L_p(s, V_{\mathbf{b}}) = L_p(s, V_{\mathbf{c}})$. Thus $l(p, \mathbf{c}) \leq l(p, \mathbf{b})$. But then $\mathbf{b} \equiv \mathbf{c} \pmod{p^{1+l(p, \mathbf{c})}}$, so a similar argument gives $l(p, \mathbf{b}) \leq l(p, \mathbf{c})$, whence

$$l(p, \mathbf{b}) = l(p, \mathbf{c}). \quad (4.5)$$

Therefore, if $\mathbf{b} \in \mathcal{L}(l)$ for some $l \geq 0$, then $\mathbf{c} \in \mathcal{L}(l)$, so indeed $\mathcal{L}(l) \supseteq \mathbf{b} + p^{l+1}\mathbb{Z}_p^m$.

Next, let $A \in \mathbb{Z}_{\geq 0}$. The set $S_A = \{\mathbf{c} \in \mathbb{Z}_p^m : v_p(\Delta(\mathbf{c})) \leq A\}$ is closed in $\mathbb{Z}_p^m$, and thus compact. The function $\mathbf{b} \mapsto l(p, \mathbf{b})$ on S_A is locally constant by (4.5), and thus has a (finite) maximum value. Therefore, for all l sufficiently large in terms of A , the set $\mathcal{L}(l)$ contains S_A . But by (4.3), the measure of S_A tends to 1 as $A \rightarrow \infty$. $\square$

5 General separation technique

At several points in the paper, we need to understand quantities that vary with $\mathbf{c}$ and n . A key tool for this is Lemma 5.2 below. It amounts to a smooth dyadic decomposition, followed by a separation of variables via Mellin inversion.

Let $d^\times r := dr/r$ and $\partial_{\log r} := r \cdot \partial_r$ for $r \in \mathbb{R}_{>0}$. For any $k \in \mathbb{Z}_{\geq 1}$ and $\mathbf{r} \in \mathbb{R}_{>0}^k$, let $d^\times \mathbf{r} = d^\times r_1 \cdots d^\times r_k$ and $\partial_{\log \mathbf{r}}^\alpha := \partial_{\log r_1}^{\alpha_1} \cdots \partial_{\log r_k}^{\alpha_k}$. Given $g \in C_c^\infty(\mathbb{R}_{>0}^k) \otimes \mathbb{C}$ and $\mathbf{s} \in \mathbb{C}^k$, let

$$g^\vee(\mathbf{s}) := \int_{\mathbf{r} > 0} d^\times \mathbf{r} g(\mathbf{r}) \mathbf{r}^\mathbf{s} = \int_{r_1, \dots, r_k > 0} d^\times r_1 \cdots d^\times r_k g(r_1, \dots, r_k) r_1^{s_1} \cdots r_k^{s_k}. \quad (5.1)$$

Then for all $\boldsymbol{\sigma} \in \mathbb{R}^k$, Mellin inversion [IK04, p. 90, (4.106)] gives

$$g(\mathbf{r}) = (2\pi)^{-k} \int_{\mathbf{t} \in \mathbb{R}^k} d\mathbf{t} g^\vee(\boldsymbol{\sigma} + i\mathbf{t}) \cdot \mathbf{r}^{-\boldsymbol{\sigma} - i\mathbf{t}}. \quad (5.2)$$

Let $\|\mathbf{s}\| := \max(|s_1|, \dots, |s_k|)$. We need the following standard Mellin bound.

Proposition 5.1. Let $A, B \in \mathbb{R}_{>0}$ with $A < B$. Let $k \in \mathbb{Z}_{\geq 1}$ and $(\mathbf{M}, \mathbf{s}) \in \mathbb{R}_{>0}^k \times \mathbb{C}^k$. Let $g \in C_c^\infty(\mathbb{R}_{>0}^k) \otimes \mathbb{C}$ with $\text{Supp } g \subseteq \prod_{j=1}^k [M_j A, M_j B]$. Then

$$g^\vee(\mathbf{s}) \ll_b \frac{\|g \circ \exp\|_{b, \infty}}{(1 + \|\mathbf{s}\|)^b} \prod_{1 \leq j \leq k} \left(M_j^{\text{Re}(s_j)} \max(A^{\text{Re}(s_j)}, B^{\text{Re}(s_j)}) \log(B/A) \right), \quad (5.3)$$

for all $b \in \mathbb{Z}_{\geq 0}$. Here $\|g \circ \exp\|_{b, \infty} := \sum_{|\alpha| \leq b} \sup_{\mathbf{r} \in \mathbb{R}_{>0}^k} |\partial_{\log \mathbf{r}}^\alpha g(\mathbf{r})|$.

Proof. If $\|\mathbf{s}\| \leq 1$, we may assume $b = 0$. Choose $i \in \{1, 2, \dots, k\}$ with $|s_i| = \|\mathbf{s}\|$. On the right-hand side of (5.1), integrate by parts b times in $\log r_i$ to get

$$g^\vee(\mathbf{s}) = (-1)^b \int_{\mathbf{r} > 0} d^\times \mathbf{r} \frac{\mathbf{r}^{\mathbf{s}}}{s_i^b} \partial_{\log r_i}^b g(\mathbf{r}) \ll \frac{\|g \circ \exp\|_{b, \infty}}{|s_i|^b} \prod_{1 \leq j \leq k} \left(\mathfrak{M}_j \int_{M_j A}^{M_j B} d^\times r \right),$$

where $\mathfrak{M}_j := \sup_{r \in [M_j A, M_j B]} (r^{\operatorname{Re}(s_j)})$. This suffices for (5.3). $\square$

Fix a function $v_2 \in C_c^\infty(\mathbb{R}_{>0})$, supported on $[1, 2]$, with

$$\int_{r > 0} d^\times r v_2(r)^2 = 1. \quad (5.4)$$

Lemma 5.2. Let $k \in \mathbb{Z}_{\geq 1}$ and $A \in \mathbb{R}_{\geq 1}$. Let $a: \mathbb{Z}_{\geq 1}^k \rightarrow \mathbb{C}$ be a function. Let $f: \mathbb{R}_{>0}^k \rightarrow \mathbb{C}$ be a smooth function supported on $\{\mathbf{r} \in \mathbb{R}_{>0}^k : \prod_{j=1}^k r_j \leq A\}$. Let $v = v_2$; let $v(\mathbf{r}/\mathbf{N}) := \prod_{j=1}^k v(r_j/N_j)$ and $g_{\mathbf{N}}(\mathbf{r}) := f(\mathbf{r})v(\mathbf{r}/\mathbf{N})$. Then

$$\sum_{\mathbf{n} \geq 1} a(\mathbf{n})f(\mathbf{n}) = (2\pi)^{-k} \int_{\mathbf{N} \in [1/2, \infty)^k} d^\times \mathbf{N} \int_{\mathbf{t} \in \mathbb{R}^k} dt g_{\mathbf{N}}^\vee(\mathbf{it}) \sum_{\mathbf{n} \geq 1} v(\mathbf{n}/\mathbf{N}) a(\mathbf{n}) \mathbf{n}^{i\mathbf{t}}. \quad (5.5)$$

Proof. For all $\mathbf{n} \geq 1$, we have $f(\mathbf{n}) = \int_{\mathbf{N} \geq 1/2} d^\times \mathbf{N} v(\mathbf{n}/\mathbf{N}) g_{\mathbf{N}}(\mathbf{n})$ by (5.4), since $v(\mathbf{n}/\mathbf{N}) = 0$ for $\mathbf{N} < 1/2$. Also, $g_{\mathbf{N}}(\mathbf{r}) \in C_c^\infty(\mathbb{R}_{>0}^k) \otimes \mathbb{C}$ for each $\mathbf{N} \geq 1/2$. So by (5.2), we get

$$\sum_{\mathbf{n} \geq 1} a(\mathbf{n})f(\mathbf{n}) = \sum_{\mathbf{n} \geq 1} a(\mathbf{n}) \int_{\mathbf{N} \geq 1/2} d^\times \mathbf{N} v(\mathbf{n}/\mathbf{N}) (2\pi)^{-k} \int_{\mathbf{t} \in \mathbb{R}^k} dt g_{\mathbf{N}}^\vee(\mathbf{it}) \mathbf{n}^{i\mathbf{t}}. \quad (5.6)$$

If $\mathbf{r} \in \operatorname{Supp} g_{\mathbf{N}}$, then $\prod_j r_j \leq A$ and $\mathbf{r} \in \prod_j [N_j, 2N_j]$, so $\prod_j N_j \leq A$. Therefore, there exist compact sets $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathbb{R}_{>0}^k$ such that if $(\mathbf{n}, \mathbf{N}) \notin \mathcal{K}_1 \times \mathcal{K}_2$ in (5.6), then $v(\mathbf{n}/\mathbf{N}) \cdot g_{\mathbf{N}}^\vee(\mathbf{it}) = 0$. Furthermore, there exists a compact set $\mathcal{K}_3 \subseteq \mathbb{R}_{>0}^k$ such that if $\mathbf{N} \in \mathcal{K}_2$, then $\operatorname{Supp} g_{\mathbf{N}} \subseteq \mathcal{K}_3$. So Proposition 5.1 gives $g_{\mathbf{N}}^\vee(\mathbf{it}) \ll_{f, v, b} (1 + \|\mathbf{t}\|)^{-b}$ for $\mathbf{N} \in \mathcal{K}_2$. Taking $b = k + 1$, we get

$$\sum_{\mathbf{n} \geq 1} \int_{\mathbf{N} \geq 1/2} d^\times \mathbf{N} \int_{\mathbf{t} \in \mathbb{R}^k} dt |a(\mathbf{n})v(\mathbf{n}/\mathbf{N})g_{\mathbf{N}}^\vee(\mathbf{it})| \ll_{a, f, v, k} \sum_{\mathbf{n} \in \mathcal{K}_1 \cap \mathbb{Z}^k} \int_{\mathbf{N} \in \mathcal{K}_2} d^\times \mathbf{N} < \infty.$$

Now (5.5) follows from (5.6) by Fubini's theorem. $\square$

For later use, we now do a general dyadic calculation of a unimodal nature.

Lemma 5.3. Let $q, a, b \in \mathbb{R}$ with $a \leq q < b$. Let $r_1, r_2, \tau \in \mathbb{R}_{>0}$. Then

$$\sum_{r_1 \leq r \leq r_2: \log_2(r) \in \mathbb{Z}} \frac{r^q}{r^a + \tau^{b-a} r^b}, \quad \int_{r_1 \leq r \leq r_2} d^\times r \frac{r^q}{r^a + \tau^{b-a} r^b},$$

are both $\ll_{q, a, b} \min(\tau^{-1}, r_2)^{q-a} \cdot (\log(1 + r_2/r_1))^{1_{q=a}}$.

Proof. We may assume $r_1 \leq r_2$, or else the sum and integral both vanish. By subtracting q, a, b by a , we may also assume $a = 0$. If $r_2 \leq \tau^{-1}$, then the result follows from the bound $1 + \tau^b r^b \geq 1$ and a geometric series (or corresponding integral). If $r_2 > \tau^{-1}$, then separately considering $r \leq \tau^{-1}$ and $r > \tau^{-1}$ leads to the result. $\square$

6 Statistics of families of L -functions

Throughout §6, assume $2 \mid m$. Recall the eigenvalue and coefficient notation from §3.

6.1 Computing local averages

Recall the definitions of $\mathcal{S}_0$ and $\mathcal{S}_1$ from (1.4). For each vector $\mathbf{c} \in \mathcal{S}_1$, let $\tilde{\alpha}_{\mathbf{c},j}(p) := \tilde{\alpha}_{V_{\mathbf{c}},j}(p)$ and $\lambda_{\mathbf{c}}^{\natural}(n) := \lambda_{V_{\mathbf{c}}}^{\natural}(n)$ for convenience, and let $\mu_{\mathbf{c}}(n)$ be the n th coefficient of the Dirichlet series $1/L(s, V_{\mathbf{c}})$. We have

$$1/L_p(s, V_{\mathbf{c}}) = \prod_j (1 - \tilde{\alpha}_{\mathbf{c},j}(p)p^{-s}) \quad (6.1)$$

by (3.2). So if $p \nmid \Delta(\mathbf{c})$, then (3.5), (3.7), (3.8), and $2 \nmid m_*$ imply

$$\mu_{\mathbf{c}}(p) = -\lambda_{\mathbf{c}}^{\natural}(p) = E_{\mathbf{c}}^{\natural}(p), \quad \mu_{\mathbf{c}}(p^2) = \lambda_{V_{\mathbf{c}}, \Lambda^2}^{\natural}(p) = \frac{1}{2}(E_{\mathbf{c}}^{\natural}(p)^2 + E_{\mathbf{c}}^{\natural}(p^2)). \quad (6.2)$$

Averages of $\lambda_{\mathbf{c}}^{\natural}(n)$, $\mu_{\mathbf{c}}(n)$ over $\mathbf{c}$ play a basic role in the statistics of $L(s, V_{\mathbf{c}})$ over $\mathbf{c}$. To prove that certain averages *exist*, we will use (3.3) and Lemma 4.4. But to *estimate* said averages, we will take a point-counting approach. The result is Proposition 6.1 below.

For any set $S \neq \emptyset$ with an obvious measure (e.g. the counting measure on a finite set, or the usual Haar measure on $\mathbb{Z}_p^m$), we let $\mathbb{E}_{b \in S}[f(b)]$ denote the average of $f(b)$ over $b \in S$. Moreover, we use the following notation for Proposition 6.1. Let $\mathcal{B} := \prod_{j=1}^m I_j \subseteq \mathbb{R}^m$, where $I_1, \dots, I_m \subseteq \mathbb{R}$ are compact intervals of positive length. Let $Z\mathcal{B} := \{Z\mathbf{x} : \mathbf{x} \in \mathcal{B}\}$. Let $\mathbf{a} \in \mathbb{Z}^m$ and $n_0, n, n_1, n_2 \in \mathbb{Z}_{\geq 1}$. Let $\mathbb{E}_{\mathbf{c} \in S}^{\mathbf{a}, n_0}[f]$ be the average of f over $\{\mathbf{c} \in S : \mathbf{c} \equiv \mathbf{a} \pmod{n_0}\}$, assuming this set is nonempty. Let $\mathbb{E}_{1 \leq \mathbf{c} \leq n}^{\mathbf{a}, n_0}[f] := \mathbb{E}_{\mathbf{c} \in \{1, 2, \dots, n\}^m}^{\mathbf{a}, n_0}[f]$. The reader may assume $(\mathbf{a}, n_0) = (\mathbf{0}, 1)$, but the general case is no harder, and is useful for future work.

Proposition 6.1. *The following limit exists, and is independent of $\mathcal{B}$:*

$$\bar{\mu}_{F,2}^{\mathbf{a}, n_0}(n_1, n_2) := \lim_{Z \rightarrow \infty} \mathbb{E}_{\mathbf{c} \in \mathcal{S}_1 \cap Z\mathcal{B}}^{\mathbf{a}, n_0}[\mu_{\mathbf{c}}(n_1)\mu_{\mathbf{c}}(n_2)].$$

Let $\bar{\mu}_{F,1}^{\mathbf{a}, n_0}(n) := \bar{\mu}_{F,2}^{\mathbf{a}, n_0}(n, 1) = \bar{\mu}_{F,2}^{\mathbf{a}, n_0}(1, n)$. If $\gcd(n_1 n_2, n'_1 n'_2) = 1$, then

$$\bar{\mu}_{F,2}^{\mathbf{a}, n_0}(n_1, n_2) \bar{\mu}_{F,2}^{\mathbf{a}, n_0}(n'_1, n'_2) = \bar{\mu}_{F,2}^{\mathbf{a}, n_0}(n_1 n'_1, n_2 n'_2). \quad (6.3)$$

Now let p be a prime, and let $l, l_1, l_2 \geq 0$ be integers. Then

$$\bar{\mu}_{F,1}^{\mathbf{a}, n_0}(p^l) \ll_{\varepsilon} p^{l\varepsilon}, \quad \bar{\mu}_{F,2}^{\mathbf{a}, n_0}(p^{l_1}, p^{l_2}) \ll_{\varepsilon} p^{(l_1+l_2)\varepsilon} \quad (6.4)$$

(uniformly over $p, l, l_1, l_2, \mathbf{a}, n_0$). Furthermore, if $p \nmid n_0$, then

$$\bar{\mu}_{F,1}^{\mathbf{a}, n_0}(p) = \lambda_V^{\natural}(p)p^{-1/2} + O(p^{-1}), \quad \bar{\mu}_{F,1}^{\mathbf{a}, n_0}(p^2) = 1 + O(p^{-1}), \quad (6.5)$$

$$\bar{\mu}_{F,2}^{\mathbf{a}, n_0}(p, p) = 1 + O(p^{-1}). \quad (6.6)$$

Proof. For vectors $\mathbf{c} \in \prod_{p \mid n} \mathbb{Z}_p^m$, define $\mu_{\mathbf{c}}(n) \mathbf{1}_{\Delta(\mathbf{c}) \neq 0}$ to be $\mu_{\mathbf{c}}(n)$ if $\Delta(\mathbf{c}) \neq 0$, and 0 if $\Delta(\mathbf{c}) = 0$. By (3.3) and (6.1), we have

$$\mu_{\mathbf{c}}(p^l) \mathbf{1}_{\Delta(\mathbf{c}) \neq 0} \ll_m 1, \quad \mu_{\mathbf{c}}(n) \mathbf{1}_{\Delta(\mathbf{c}) \neq 0} \ll_{\varepsilon} n^{\varepsilon}. \quad (6.7)$$

Since $\mu_{\mathbf{c}}(1)\mathbf{1}_{\Delta(\mathbf{c}) \neq 0} = \mathbf{1}_{\Delta(\mathbf{c}) \neq 0}$, we have $\bar{\mu}_{F,2}^{\mathbf{a},n_0}(1,1) = 1$, because $|\mathcal{S}_0 \cap Z\mathcal{B}| = o_{\mathcal{B};Z \rightarrow \infty}(Z^m)$ by (2.4). On the other hand, if $n \geq 2$ and $k \geq 1$, then by Lemma 4.4, the quantity $\mu_{\mathbf{c}}(n)\mathbf{1}_{\Delta(\mathbf{c}) \neq 0}$ depends only on $\mathbf{c} \bmod n^k$, unless $\mathbf{c}$ lies in one of $o_{n;k \rightarrow \infty}(n^{km})$ exceptional residue classes of $\mathbb{Z}^m$ (or $\prod_{p|n} \mathbb{Z}_p^m$) modulo n^k . This, together with (6.7) and the Chinese remainder theorem, implies that the following three quantities exist and equal one another:

$$\bar{\mu}_{F,2}^{\mathbf{a},n_0}(n_1, n_2), \quad \lim_{k \rightarrow \infty} \mathbb{E}_{1 \leq \mathbf{c} \leq (n_1 n_2)^k}^{\mathbf{a},n_0} [\mu_{\mathbf{c}}(n_1) \mu_{\mathbf{c}}(n_2) \mathbf{1}_{\Delta(\mathbf{c}) \neq 0}], \quad \prod_{p|n_1 n_2} \mathbb{E}_{\mathbf{c} \in \mathbb{Z}_p^m}^{\mathbf{a},n_0} [\mu_{\mathbf{c}}(p^{v_p(n_1)}) \mu_{\mathbf{c}}(p^{v_p(n_2)}) \mathbf{1}_{\Delta(\mathbf{c}) \neq 0}].$$

This establishes the required existence, independence, and multiplicativity of limits.

We now turn to the required estimates. First, (6.4) follows from (6.7). Now assume $p \nmid n_0$; we must prove (6.5) and (6.6). But $p \nmid n_0$ implies $\{\mathbf{c} \in \mathbb{Z}_p^m : \mathbf{c} \equiv \mathbf{a} \bmod n_0\} = \mathbb{Z}_p^m$, so $\mathbb{E}_{\mathbf{c} \in \mathbb{Z}_p^m}^{\mathbf{a},n_0}[f] = \mathbb{E}_{\mathbf{c} \in \mathbb{Z}_p^m}[f]$ for any quantity f . Therefore, it remains to prove the following:

1. $\mathbb{E}_{\mathbf{c} \in \mathbb{Z}_p^m} [\mu_{\mathbf{c}}(p) \mathbf{1}_{\Delta(\mathbf{c}) \neq 0}] = \lambda_V^{\natural}(p) p^{-1/2} + O(p^{-1})$.
2. $\mathbb{E}_{\mathbf{c} \in \mathbb{Z}_p^m} [\mu_{\mathbf{c}}(p^2) \mathbf{1}_{\Delta(\mathbf{c}) \neq 0}] = 1 + O(p^{-1})$.
3. $\mathbb{E}_{\mathbf{c} \in \mathbb{Z}_p^m} [\mu_{\mathbf{c}}(p)^2 \mathbf{1}_{\Delta(\mathbf{c}) \neq 0}] = 1 + O(p^{-1})$.

We now prove (1)–(3). By [Wan23a, Corollary 1.7], we have

$$\mathbb{E}_{\mathbf{c} \in \mathbb{F}_p^m} [E_{\mathbf{c}}^{\natural}(p) \mathbf{1}_{p \nmid \Delta(\mathbf{c})}] = \lambda_V^{\natural}(p) p^{-1/2} + O(p^{-1}), \quad (6.8)$$

$$\mathbb{E}_{\mathbf{c} \in \mathbb{F}_p^m} [E_{\mathbf{c}}^{\natural}(p^2) \mathbf{1}_{p \nmid \Delta(\mathbf{c})}] = 1 + O(p^{-1}), \quad (6.9)$$

$$\mathbb{E}_{\mathbf{c} \in \mathbb{F}_p^m} [E_{\mathbf{c}}^{\natural}(p)^2 \mathbf{1}_{p \nmid \Delta(\mathbf{c})}] = 1 + O(p^{-1}), \quad (6.10)$$

in view of our present definition of $E_{\mathbf{c}}^{\natural}$ in (2.12). But for each $f \in \{\mu_{\mathbf{c}}(p), \mu_{\mathbf{c}}(p^2), \mu_{\mathbf{c}}(p)^2\}$, we have $\mathbb{E}_{\mathbf{c} \in \mathbb{Z}_p^m} [f \mathbf{1}_{\Delta(\mathbf{c}) \neq 0}] = \mathbb{E}_{\mathbf{c} \in \mathbb{Z}_p^m} [f \mathbf{1}_{p \nmid \Delta(\mathbf{c})}] + O(p^{-1})$, by (6.7) and the Lang–Weil bound for the zero locus of $\Delta(\mathbf{c})$ over $\mathbb{F}_p$. After rewriting $f \mathbf{1}_{p \nmid \Delta(\mathbf{c})}$ using (6.2), we conclude that (6.8) implies (1), that (6.10) implies (3), and that (6.9), (6.10) imply (2). $\square$

For the rest of §6, assume that $2 \mid m$ and that Conjecture 1.3 holds.

6.2 The Sarnak–Shin–Templier framework

Using Conjecture 1.3 and (6.8), (6.9), (6.10), we can obtain useful statistical information on $L(s, V_{\mathbf{c}})$ over $\mathbf{c} \in \mathcal{S}_1$, where the set $\mathcal{S}_1$ is defined as in (1.4). Let $\Pi_{\mathbf{c}}$ be an isobaric automorphic representation over $\mathbb{Q}$ corresponding to $L(s, V_{\mathbf{c}})$ in Conjecture 1.3. Let $\mathcal{S}_2$ be the set of $\mathbf{c} \in \mathcal{S}_1$ for which $\Pi_{\mathbf{c}}$ is cuspidal, self-dual, and symplectic, in the sense of [SST16, p. 533]. By (3.1) and the fact that the gamma function $\Gamma(z)$ has no zeros, the gamma factor $L_{\infty}(s, V_{\mathbf{c}}, \wedge^2)$ is an invertible holomorphic function at $s = 1$. Therefore, for each $\mathbf{c} \in \mathcal{S}_2$, the L -function $L(s, V_{\mathbf{c}}, \wedge^2)$ has a pole at $s = 1$, whence there exists an isobaric automorphic representation $\phi_{\mathbf{c},2}$ over $\mathbb{Q}$ with

$$L(s, V_{\mathbf{c}}, \wedge^2) = \zeta(s) L(s, \phi_{\mathbf{c},2}). \quad (6.11)$$

Proposition 6.2. *Let $Z \in \mathbb{R}_{\geq 1}$. Then $|(\mathcal{S}_1 \setminus \mathcal{S}_2) \cap [-Z, Z]^m| \ll_{m,\varepsilon} Z^{m-1/2+\varepsilon}$.*

Proof. We want to show that the family $\mathbf{c} \mapsto \Pi_{\mathbf{c}}$ indexed by $\mathbf{c} \in \mathcal{S}_1$ is, in the sense of [SST16, p. 538, (i)–(iii)], *essentially* cuspidal, self-dual, and symplectic, with a power-saving exceptional set. We follow the GRH strategy suggested in [SST16].

Let v_0 be as in §1.2. Fix $\varepsilon \in (0, \frac{1}{2})$ and let $P = Z^{1-\varepsilon}$.

By Proposition 3.2(7), each $\Pi_{\mathbf{c}}$ is self-dual. Let $\mathcal{S}_{1,5}$ be the set of $\mathbf{c} \in \mathcal{S}_1$ for which $\Pi_{\mathbf{c}}$ is cuspidal. Then $L(s, V_{\mathbf{c}}, \otimes^2)$ has a pole at $s = 1$ of order exactly 1 if $\mathbf{c} \in \mathcal{S}_{1,5}$, and at least 2 if $\mathbf{c} \in \mathcal{S}_1 \setminus \mathcal{S}_{1,5}$. This follows from the theory of unramified Rankin–Selberg L -functions; cf. [FPRS19, proof of Lemma 2.3]. A calculation with $\frac{L'}{L}(s, V_{\mathbf{c}}, \otimes^2)$, using (3.3), GRH, and [IK04, §5.6’s Exercise 6 and §5.7’s Theorem 5.15], then yields

$$\sum_{\mathbf{c} \in \mathcal{S}_1} v_0(\mathbf{c}/Z) \sum_{p \leq P: p \nmid \Delta(\mathbf{c})} (\log p) \cdot (\lambda_{V_{\mathbf{c}}, \otimes^2}^{\natural}(p) - 1) \geq P \sum_{\mathbf{c} \in \mathcal{S}_1} v_0(\mathbf{c}/Z) (\mathbf{1}_{\mathbf{c} \notin \mathcal{S}_{1,5}} + O_{\varepsilon}(P^{-1/2+\varepsilon})). \quad (6.12)$$

On the other hand, $L(s, V_{\mathbf{c}}, \text{Sym}^2)$ has a pole at $s = 1$ if $\mathbf{c} \in \mathcal{S}_{1,5} \setminus \mathcal{S}_2$; this follows from [SST16, p. 533]. So a calculation with $\frac{L'}{L}(s, V_{\mathbf{c}}, \text{Sym}^2)$ gives

$$\sum_{\mathbf{c} \in \mathcal{S}_1} v_0(\mathbf{c}/Z) \sum_{p \leq P: p \nmid \Delta(\mathbf{c})} (\log p) \cdot \lambda_{V_{\mathbf{c}}, \text{Sym}^2}^{\natural}(p) \geq P \sum_{\mathbf{c} \in \mathcal{S}_1} v_0(\mathbf{c}/Z) (\mathbf{1}_{\mathbf{c} \in \mathcal{S}_{1,5} \setminus \mathcal{S}_2} + O_{\varepsilon}(P^{-1/2+\varepsilon})). \quad (6.13)$$

But for all primes p and vectors $\mathbf{c} \in \mathbb{Z}^m$ with $p \nmid \Delta(\mathbf{c})$, we have $\mathbf{c} \in \mathcal{S}_1$, and we can use (3.5), (3.6), (3.7) to write $\lambda_{V_{\mathbf{c}}, \otimes^2}^{\natural}(p) = \lambda_{\mathbf{c}}^{\natural}(p)^2 = E_{\mathbf{c}}^{\natural}(p)^2$ and $\lambda_{V_{\mathbf{c}}, \text{Sym}^2}^{\natural}(p) = \frac{1}{2}(E_{\mathbf{c}}^{\natural}(p)^2 - E_{\mathbf{c}}^{\natural}(p^2))$, since m_* is odd. So the left-hand side of (6.12) equals

$$\sum_{p \leq P} (\log p) \sum_{\mathbf{c} \in \mathbb{Z}^m} v_0(\mathbf{c}/Z) (E_{\mathbf{c}}^{\natural}(p)^2 - 1) \mathbf{1}_{p \nmid \Delta(\mathbf{c})} = \sum_{p \leq P} (\log p) Z^m (0 + O(p^{-1})) \ll_{\varepsilon} Z^m P^{\varepsilon},$$

by Poisson summation and (6.10). Similarly, by (6.10) and (6.9), the left-hand side of (6.13) is $\ll_{\varepsilon} Z^m P^{\varepsilon}$. Thus from (6.12) we get $|(\mathcal{S}_1 \setminus \mathcal{S}_{1,5}) \cap [-Z/2, Z/2]^m| \ll_{m, \varepsilon} Z^m P^{-1/2+\varepsilon}$, and from (6.13) we get $|(\mathcal{S}_{1,5} \setminus \mathcal{S}_2) \cap [-Z/2, Z/2]^m| \ll_{m, \varepsilon} Z^m P^{-1/2+\varepsilon}$. Now let $\varepsilon \rightarrow 0$. $\square$

6.3 The Ratios Recipe

Using Proposition 6.1 and the Ratios Recipe [CFZ08, §5.1], we will soon derive the Ratios Conjecture 1.5 (R2). It is easier to first explain a simpler instance, (R1), of the Ratios Conjecture. Recall $\sigma(Z) := \frac{1}{2} + \frac{1}{\log Z}$. We will soon define an Euler product $A_{F,1}(s)$, which converges absolutely for $\text{Re}(s) > \frac{1}{3}$.

Conjecture 6.3 (R1). *Write $s = \sigma(Z) + it$. Then there exists a real $\bar{h} > 0$ such that uniformly over reals $Z \geq 2$ and $t \in [-Z^{\bar{h}}, Z^{\bar{h}}]$, we have*

$$\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \Phi^{\mathbf{c}, 1}(s) = \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} (1 + o_{Z \rightarrow \infty}(1)) \cdot A_{F,1}(s).$$

6.3.1 Deriving (R1)

The Ratios Recipe [CFZ08, §5.1] produces the conjecture

$$\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \frac{1}{L(s, V_{\mathbf{c}})} = \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \mathcal{A}(s) (1 + O(Z^{-\bar{h}})) \quad (6.14)$$

for t, s as in Conjecture 6.3, for some $\hbar > 0$, where

$$\mathcal{A}(s) := \sum_{n \geq 1} \bar{\mu}_{F,1}^{0,1}(n) n^{-s}$$

is the *coefficient-wise average* of the series $L(s, V_c)^{-1} = \sum_{n \geq 1} \mu_c(n) n^{-s}$. The first step of the recipe is to define $\mathcal{L}_c(s; \gamma) := \frac{1}{L(s+\gamma, V_c)}$. For us, the second step expands $\frac{1}{L}$ as a series, and the third step does nothing. The fourth and fifth steps replace each coefficient $\mu_c(n)$ with its expected value $\bar{\mu}_{F,1}^{0,1}(n)$ as $Z \rightarrow \infty$, in the notation of Proposition 6.1, and call the total series $M_c(s; \gamma) := \mathcal{A}(s + \gamma)$. The sixth step plugs in $(\frac{1}{2}, s - \frac{1}{2})$ for (s, γ) , giving (6.14).

The series $\mathcal{A}(s)$ behaves like $\zeta(2s)L(s + \frac{1}{2}, V)$, as we now explain. Let

$$A_{F,1}(s) := \mathcal{A}(s) \zeta(2s)^{-1} L(s + \frac{1}{2}, V)^{-1} =: \sum_{n \geq 1} \bar{a}_{F,1}(n) n^{-s}.$$

Multiplying the definition of $A_{F,1}(s)$ out modulo $O(p^{-3s})$, we get

$$\bar{a}_{F,1}(p) = \bar{\mu}_{F,1}^{0,1}(p) - \lambda_V^{\natural}(p) p^{-1/2}, \quad \bar{a}_{F,1}(p^2) = \bar{\mu}_{F,1}^{0,1}(p^2) - \bar{\mu}_{F,1}^{0,1}(p) \lambda_V^{\natural}(p) p^{-1/2} - 1 + O(p^{-1}).$$

Also, $\bar{a}_{F,1}(n)$ is multiplicative in n by (6.3). Now by (6.4), (6.5) we have

$$\bar{a}_{F,1}(n) \ll_{\varepsilon} n^{\varepsilon}, \quad \bar{a}_{F,1}(p) \ll p^{-1}, \quad \bar{a}_{F,1}(p^2) \ll p^{-1/2}. \quad (6.15)$$

So $A_{F,1}(s)$ has an Euler product, and satisfies

$$|A_{F,1}(s)| \leq \sum_{n \geq 1} \frac{|\bar{a}_{F,1}(n)|}{n^{1/3+\varepsilon}} \leq \prod_p \left(1 + \frac{O(p^{-1})}{p^{1/3+\varepsilon}} + \frac{O(p^{-1/2})}{p^{2/3+2\varepsilon}} + \frac{O_{\varepsilon}(p^{\varepsilon})}{p^{1+3\varepsilon}} \right) \ll 1 \quad (6.16)$$

for $\operatorname{Re}(s) \geq \frac{1}{3} + \varepsilon$, for any $\varepsilon > 0$. On the other hand, $\zeta(2s)L(s + \frac{1}{2}, V)$ has a pole of order ≥ 1 at $s = \frac{1}{2}$, and thus is more dominant than $A_{F,1}(s)$ in $\mathcal{A}(s) = A_{F,1}(s) \zeta(2s)L(s + \frac{1}{2}, V)$.

Writing $\mathcal{A}(s) = A_{F,1}(s) \zeta(2s)L(s + \frac{1}{2}, V)$ in (6.14), then dividing by $\zeta(2s)L(s + \frac{1}{2}, V)$, gives Conjecture 6.3, in view of the definition (1.6) of $\Phi^{c,1}(s)$ in terms of $1/L(s, V_c)$. Here $\zeta(2s)$, $L(s + \frac{1}{2}, V)$ are called *polar factors*, or *polar terms*. As we will see shortly, an additional polar factor, $\zeta(s_1 + s_2)$, arises in (R2).

Before proceeding, the following clarifications may be helpful:

1. Since the ratio $1/L$ has no L 's in the numerator, the Ratios Recipe for $1/L(s, V_c)$ does not involve the approximate functional equation for $L(s, V_c)$.
2. Some $L(s, V_c)$ may not be primitive. But by Proposition 6.2 and GRH, there is no statistical difference between $\mathcal{S}_1$ and $\mathcal{S}_2$, on which each $L(s, V_c)$ is primitive.
3. We order our families not by conductor, but by the similar natural level function $\|c\|$; cf. [SST16, p. 535, Remark (i); and p. 560, second paragraph after (25)].
4. We restrict t to comfortably respect the constraint [CS07, (2.11c)] on vertical shifts. But it would be reasonable to allow $t \in [-Z^A, Z^A]$ for arbitrarily large $A > 0$; cf. [BC21, p. 4 of arXiv:2002.09466v2, the sentence before Conjecture 2].

6.3.2 Deriving (R2)

We follow §6.3.1 closely. Consider the coefficient-wise average

$$\mathcal{A}_{F,2}(s_1, s_2) := \sum_{n_1, n_2 \geq 1} \bar{\mu}_{F,2}^{\mathbf{0},1}(n_1, n_2) n_1^{-s_1} n_2^{-s_2}$$

of $L(s_1, V_{\mathbf{c}})^{-1} L(s_2, V_{\mathbf{c}})^{-1} = \sum_{n_1, n_2 \geq 1} \mu_{\mathbf{c}}(n_1) \mu_{\mathbf{c}}(n_2) n_1^{-s_1} n_2^{-s_2}$. The Ratios Recipe gives

$$\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \frac{1}{L(s_1, V_{\mathbf{c}}) L(s_2, V_{\mathbf{c}})} = \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \mathcal{A}_{F,2}(s_1, s_2) (1 + O(Z^{-\bar{h}})). \quad (6.17)$$

Let

$$\mathcal{A}_{F,2}(s_1, s_2) =: A_{F,2}(s_1, s_2) \zeta(s_1 + s_2) \prod_{1 \leq j \leq 2} (\zeta(2s_j) L(s_j + \tfrac{1}{2}, V)).$$

Then $A_{F,2}(s_1, s_2) = \sum_{n_1, n_2 \geq 1} \bar{a}_{F,2}(n_1, n_2) n_1^{-s_1} n_2^{-s_2}$, where

$$\begin{aligned} \bar{a}_{F,2}(p^l, 1) &= \bar{a}_{F,2}(1, p^l) = \bar{a}_{F,1}(p^l), \\ \bar{a}_{F,2}(p, p) &= \bar{\mu}_{F,2}^{\mathbf{0},1}(p, p) - 2\bar{\mu}_{F,1}^{\mathbf{0},1}(p) \lambda_V^{\natural}(p) p^{-1/2} - 1 + \lambda_V^{\natural}(p)^2 p^{-1}. \end{aligned}$$

Therefore, if $\operatorname{Re}(s_1), \operatorname{Re}(s_2) \geq \frac{1}{3} + \varepsilon$, then by (6.15) and Proposition 6.1, we have

$$|A_{F,2}(s_1, s_2)| \leq \sum_{n_1, n_2 \geq 1} |\bar{a}_{F,2}(n_1, n_2)| (n_1 n_2)^{-1/3-\varepsilon} \ll_{\varepsilon} 1; \quad (6.18)$$

cf. (6.16). Dividing (6.17) by $\prod_{1 \leq j \leq 2} (\zeta(2s_j) L(s_j + \frac{1}{2}, V))$ leads to Conjecture 1.5, after using (1.6) to write $\Phi^{\mathbf{c},1}(s_j)$ in terms of $1/L(s_j, V_{\mathbf{c}})$.

6.4 Applying (R2)

Write $A_{F,2}(\mathbf{s}) = A_{F,2}(s_1, s_2)$. Conjecture 1.5 implies, uniformly over $Z \geq 2$ and $\mathbf{t} \in \mathbb{R}^2$, that for $s_j = \sigma(Z) + it_j$, we have, for all $\varepsilon > 0$,

$$\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \Phi^{\mathbf{c},1}(s_1) \Phi^{\mathbf{c},1}(s_2) = O_{\varepsilon}(Z^m (1 + \|\mathbf{t}\|)^{\varepsilon}) + \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} A_{F,2}(\mathbf{s}) \zeta(s_1 + s_2). \quad (6.19)$$

This follows for $\|\mathbf{t}\| \leq Z^{\bar{h}}$ by Conjecture 1.5, and for $\|\mathbf{t}\| > Z^{\bar{h}}$ by GRH (see Proposition 3.2(8)). Using (6.19), we will derive the following conjecture:

Conjecture 6.4 (R2'). *Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be entire, with $f(s) \ll_{f,b} (1 + |\operatorname{Im}(s)|)^{-b}$ on the strip $0 \leq \operatorname{Re}(s) \leq 2$ for all $b \in \mathbb{Z}_{\geq 1}$. Let $Z, N \in \mathbb{R}_{\geq 1}$ with $N \leq Z^3$. If $\sigma_0 \in (1, 2)$, then*

$$\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \left| \int_{(\sigma_0)} \Phi^{\mathbf{c},1}(s) \cdot f(s) N^s ds \right|^2 \ll_F Z^m N \sup_{0 \leq \sigma \leq 2} \int_{\mathbb{R}} (1 + |t|)^2 |f(\sigma + it)|^2 dt. \quad (6.20)$$

Remark 6.5. There are several notable features of (R2') worth clarifying or discussing:

1. The contour (σ_0) in (6.20) runs from $s = \sigma_0 - i\infty$ to $s = \sigma_0 + i\infty$. The left-hand side of (6.20) is independent of $\sigma_0 \in (\frac{1}{2}, 2)$, under Conjecture 1.3 (HW2).
2. There are no log factors on the right-hand side of (6.20); the factor $\zeta(2s)^{-1}L(s + \frac{1}{2}, V)^{-1}$ in (1.6), and the integral over (σ_0) in (6.20), play a mollifying role.
3. In Theorem 1.7, one may assume Conjecture 6.4 (R2') in place of Conjecture 1.5 (R2).
4. The statement (R2') can be derived from (HW2) and (R2); see Proposition 6.6. Yet (R2') is a rough, order-of-magnitude prediction that could perhaps be explained in other ways, not just following the rather arithmetic Ratios Recipe.
5. (R2') essentially concerns cancellation in the coefficients of $\Phi^{c,1}(s)$ over moduli n in a dyadic range; see Proposition 6.8. It should be possible to obtain results towards (R2') using ideas of [Sou09, Har13, Sza24], perhaps over $\mathbb{F}_q[t]$ as in [BFK23, Flo24].
6. Since (6.20) is log-free, one may hope that “better than square-root cancellation” occurs in $\ell^{2-\varepsilon}$, by analogy with [Har23, (1.2)]. If true, this would provide additional cancellation in Proposition 7.5 (one of the key ingredients for Theorem 1.7), and thus provide an alternative approach to the second half of Theorem 1.2.

Proposition 6.6. *Assume Conjectures 1.3 and 1.5. Then Conjecture 6.4 holds.*

Proof. Let $\bar{f}(s) := \overline{f(\bar{s})}$. Let $Z, N \geq 1$ with $N \leq Z^3$. The left-hand side of (6.20) is independent of $\sigma_0 > \frac{1}{2}$. If $Z \ll 1$ then (6.20) is trivial, so suppose Z is large. Now shift contours to $\text{Re}(s) = \sigma(Z)$, and expand squares using self-duality of the L -functions in (1.6) (see Proposition 3.2(7)), to equate the left-hand side of (6.20) with the quantity

$$\Sigma_0 := \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \int_{\sigma(Z)-i\infty}^{\sigma(Z)+i\infty} ds_1 \int_{\sigma(Z)+i\infty}^{\sigma(Z)-i\infty} ds_2 \Phi^{c,1}(s_1) \Phi^{c,1}(s_2) \cdot f(s_1) \bar{f}(s_2) N^{s_1+s_2}.$$

After switching the order of $\mathbf{c}$ and $\mathbf{s}$ in Σ_0 using Fubini's theorem, and plugging in the estimate (6.19) and the bound $1 + \|\mathbf{t}\| \leq (1 + |t_1|)(1 + |t_2|)$ for each $\mathbf{s}$, we get the estimate

$$\Sigma_0 = O_\varepsilon(Z^m \Sigma_1^2) + \Sigma_2, \quad (6.21)$$

where $\Sigma_1 := \int_{t \in \mathbb{R}} dt (1 + |t|)^\varepsilon \cdot |f(\sigma(Z) + it)| N^{\sigma(Z)}$ and

$$\Sigma_2 := \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \int_{\sigma(Z)-i\infty}^{\sigma(Z)+i\infty} ds_1 \int_{\sigma(Z)+i\infty}^{\sigma(Z)-i\infty} ds_2 \zeta(s_1 + s_2) A_{F,2}(\mathbf{s}) \cdot f(s_1) \bar{f}(s_2) N^{s_1+s_2}.$$

Here $N \leq Z^{O(1)}$, so $\Sigma_1 \ll N^{1/2} \sup_{0 \leq \sigma \leq 2} \|(1 + |t|)^\varepsilon f(\sigma + it)\|_{L_t^1(\mathbb{R})}$. Also,

$$\|(1 + |t|)^\varepsilon f(\sigma + it)\|_{L_t^1(\mathbb{R})} \leq \|(1 + |t|)^{-1/2-\varepsilon}\|_{L_t^2(\mathbb{R})} \cdot \|(1 + |t|)^{1/2+2\varepsilon} f(\sigma + it)\|_{L_t^2(\mathbb{R})} \quad (6.22)$$

by the Cauchy–Schwarz inequality.

Now let $\delta = \frac{1}{20}$, and assume Z is large enough that $1 - \sigma(Z) - \delta \geq \frac{1}{3} + \delta$. Shifting s_2 in Σ_2 from $\operatorname{Re}(s_2) = \sigma(Z)$ to $\operatorname{Re}(s_2) = 1 - \sigma(Z) - \delta$ yields $\Sigma_2 = \Sigma_3 + O_{F,\varepsilon}(Z^m \Sigma_4)$, where

$$\Sigma_3 := \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \int_{\sigma(Z) - i\infty}^{\sigma(Z) + i\infty} ds_1 (-2\pi i) A_{F,2}(s_1, 1 - s_1) \cdot f(s_1) \bar{f}(1 - s_1) N$$

comes from the residue of $\zeta(s_1 + s_2)$ at $s_2 = 1 - s_1$, and where on $\operatorname{Re}(s_2) = 1 - \sigma(Z) - \delta$ we use (6.18) and the consequence $\zeta(s_1 + s_2) \ll_{\delta,\varepsilon} (1 + |t_1 + t_2|)^\varepsilon$ of RH to be able to take

$$\Sigma_4 := \int_{\mathbf{t} \in \mathbb{R}^2} dt_1 dt_2 (1 + |t_1 + t_2|)^\varepsilon |f(\sigma(Z) + it_1) f(1 - \sigma(Z) - \delta - it_2)| N^{1-\delta}.$$

In Σ_3 , shifting s_1 to $\operatorname{Re}(s_1) = \frac{1}{2}$ yields $\Sigma_3 \ll_F Z^m N \|f(\frac{1}{2} + it)\|_{L_t^2(\mathbb{R})}^2$, thanks to (6.18). Also, $\Sigma_4 \ll N^{1-\delta} \sup_{0 \leq \sigma \leq 2} \|(1 + |t|)^\varepsilon f(\sigma + it)\|_{L_t^1(\mathbb{R})}^2$, since $1 + |t_1 + t_2| \leq (1 + |t_1|)(1 + |t_2|)$. But $\Sigma_0 \leq O(Z^m \Sigma_1^2) + \Sigma_3 + O_{F,\varepsilon}(Z^m \Sigma_4)$, by (6.21). By (6.22) with $\varepsilon = \frac{1}{4}$, we get (6.20). $\square$

As a stepping stone from Conjecture 6.4 (R2') to Conjecture 7.4 (R2'E'), we now state (R2'E). Let $a_{\mathbf{c},1}(n)$ be the n th coefficient of the Dirichlet series $\Phi^{\mathbf{c},1}(s)$.

Conjecture 6.7 (R2'E). Fix a function $D \in C_c^\infty(\mathbb{R}_{>0})$. Let $t_0 \in \mathbb{R}$. Let $Z, N \in \mathbb{R}_{>0}$ with $N \leq Z^3$. Then for some real $A_3 > 0$ depending only on F , we have

$$\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \left| \sum_{n \geq 1} D(n/N) \cdot n^{-it_0} a_{\mathbf{c},1}(n) \right|^2 \ll_{F,D} (1 + |t_0|)^{A_3} Z^m N. \quad (6.23)$$

Proposition 6.8. Assume Conjecture 6.4. Then Conjecture 6.7 holds.

Proof. The case $N < 1$ is trivial, so assume $N \geq 1$. Let $\sigma_0 = 1.5$. By (5.2), $D(n/N) = (2\pi)^{-1} \int_{t \in \mathbb{R}} dt D^\vee(\sigma_0 + it)(N/n)^{\sigma_0 + it}$. Apply Conjecture 6.4 with $f(s) = D^\vee(s - it_0)$, and bound f using Proposition 5.1, to get (6.23) with $A_3 = 2$. $\square$

6.5 Bounding exterior squares

For each $\mathbf{c} \in \mathcal{S}_1$, let $\mu_{\mathbf{c},2}(n)$ denote the n th coefficient of the Dirichlet series $\zeta(s)/L(s, V_{\mathbf{c}}, \wedge^2)$. Recall $\mathcal{N}^{\mathbf{c}}$, $\mathcal{N}_{\mathbf{c}}$ from (2.5). Assume Conjecture 1.3.

Proposition 6.9. Let $A \geq 2$ be an even integer. Let $Z, N, \varepsilon \in \mathbb{R}_{>0}$ with $N \leq Z^{3/2}$. Then

$$\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \left| \sum_{N \leq n < 2N: n \in \mathcal{N}^{\mathbf{c}}} \mu_{\mathbf{c},2}(n) \right|^A \ll_{A,\varepsilon} Z^m N^{A-1/3+\varepsilon}. \quad (6.24)$$

Proof. For $N < 2$, the left-hand side of (6.24) is $\ll_A Z^m$ by (3.3). Now assume $N \geq 2$.

Let v_0 be as in §1.2. Let $f(Z, N)$ denote the left-hand side of (6.24). Let $g(Z, N)$ denote $f(Z, N)$ with $\sum_{\mathbf{c} \in \mathcal{S}_1} v_0(\mathbf{c}/Z) \cdots$ in place of $\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \cdots$. Then by positivity,

$$g(Z/2, N) \leq f(Z, N) \leq g(2Z, N) \quad (6.25)$$

for all $Z > 0$. But if $Z \geq (2N)^{A+1}$, then if we let $\mu_{\mathbf{c},2}(\mathbf{n}) := \mu_{\mathbf{c},2}(n_1) \cdots \mu_{\mathbf{c},2}(n_A)$ and $P := n_1 \cdots n_A$, and note that $\mathbf{1}_{\gcd(P, \Delta(\mathbf{c}))=1} \cdot \mu_{\mathbf{c},2}(\mathbf{n})$ is determined by P and $\mathbf{c} \bmod P$, we get

$$\begin{aligned} g(Z, N) &= \sum_{N \leq n_1, \dots, n_A < 2N} \sum_{\mathbf{c} \in \mathbb{Z}^m} v_0(\mathbf{c}/Z) \mathbf{1}_{\mathbf{c} \in \mathcal{S}_1} \mathbf{1}_{\gcd(P, \Delta(\mathbf{c}))=1} \cdot \mu_{\mathbf{c},2}(\mathbf{n}) \\ &= O_A(Z^m) + \sum_{N \leq n_1, \dots, n_A < 2N} (Z/P)^m \sum_{\mathbf{c} \in (\mathbb{Z}/P\mathbb{Z})^m} \mathbf{1}_{\gcd(P, \Delta(\mathbf{c}))=1} \cdot \mu_{\mathbf{c},2}(\mathbf{n}), \end{aligned}$$

by (3.3) and Poisson summation in residue classes modulo P ; note that $P \geq N^A > 1$, so the condition $\gcd(P, \Delta(\mathbf{c})) = 1$ automatically implies $\mathbf{c} \in \mathcal{S}_1$. Cf. the proof of Proposition 6.2. If $\tilde{g}(Z, N) := g(Z, N)/Z^m$, then for $Z \geq (2N)^{A+1}$ we conclude that

$$\tilde{g}(Z, N) = \tilde{g}((2N)^{A+1}, N) + O_A(1). \quad (6.26)$$

We now address $Z \leq (2N)^{A+1}$. Recall $\mathcal{S}_2$ from Proposition 6.2. If $\mathbf{c} \in \mathcal{S}_2$, then (6.11) and Proposition 3.2(8) for $L(s, \phi_{\mathbf{c},2})^{-1}$, when combined with (3.3) at primes $p \mid \Delta(\mathbf{c})$, yield $\sum_{N \leq n < 2N: n \in \mathcal{N}^c} \mu_{\mathbf{c},2}(n) \ll_\varepsilon \|\mathbf{c}\|^\varepsilon N^{1/2+\varepsilon}$. If $\mathbf{c} \in \mathcal{S}_1 \setminus \mathcal{S}_2$, we still have the bound $\sum_{N \leq n < 2N: n \in \mathcal{N}^c} \mu_{\mathbf{c},2}(n) \ll_\varepsilon N^{1+\varepsilon}$ due to (3.3). Therefore, Proposition 6.2 yields

$$g(Z, N) \ll_{A,\varepsilon} Z^{m+\varepsilon} N^{A/2+\varepsilon} + Z^{m-1/2+\varepsilon} N^{A+\varepsilon}$$

for all $Z > 0$. It follows that $\tilde{g}(Z, N) \ll_{A,\varepsilon} N^{A/2+\varepsilon} + N^{A-1/3+\varepsilon}$ for all $Z \in [N^{2/3}, (2N)^{A+1}]$, and thus for all $Z \geq N^{2/3}$ by (6.26). Now (6.24) follows from (6.25). $\square$

To handle primes $p \mid \Delta(\mathbf{c})$, we prove the following unconditional result:

Lemma 6.10. *If $Z, N, A, \varepsilon \in \mathbb{R}_{>0}$, then $\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} |\mathcal{N}_{\mathbf{c}} \cap [1, N]|^A \ll_{A,\varepsilon} Z^m N^\varepsilon$.*

Proof. Lemma 2.3 immediately suffices if $Z < N^A$. Now suppose $Z \geq N^A$. By Hölder's inequality, we may assume $A \in \mathbb{Z}_{\geq 1}$. The sum $\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} |\mathcal{N}_{\mathbf{c}} \cap [1, N]|^A$ then equals

$$\Sigma_5 := \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \sum_{u_1, \dots, u_A \leq N: u_i \mid \Delta(\mathbf{c})^\infty} 1 = \sum_{u_1, \dots, u_A \leq N} \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \mathbf{1}_{\text{rad}(u_1 \cdots u_A) \mid \Delta(\mathbf{c})}.$$

Here $u_1 \cdots u_A \leq N^A \leq Z$, so by the Lang–Weil bound and the Chinese remainder theorem,

$$\Sigma_5 \ll_\varepsilon \sum_{u_1, \dots, u_A \leq N} Z^m \text{rad}(u_1 \cdots u_A)^{\varepsilon-1} \leq \sum_{r \leq N^A} Z^m r^{\varepsilon-1} \sum_{u_1, \dots, u_A \leq N: u_i \mid r^\infty} 1.$$

By Lemma 2.3, we conclude that $\Sigma_5 \ll_{A,\varepsilon} \sum_{r \leq N^A} Z^m r^{\varepsilon-1} (Nr)^\varepsilon \ll_\varepsilon Z^m N^{(2A+1)\varepsilon}$. $\square$

In §7, we need the following technical complement to Proposition 6.8.

Proposition 6.11. *Assume Conjecture 1.3. Fix $A \in \mathbb{R}_{>0}$ and $f(r) \in \{\mathbf{1}_{r \leq 1}\} \cup C_c^\infty(\mathbb{R}_{>0})$. Let $Z, N \in \mathbb{R}_{>0}$ with $N \leq Z^{3/2}$. Let $t_0 \in \mathbb{R}$. For some $\eta_1 = \eta_1(A) > 0$, we have*

$$\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \left| \sum_{n \geq 1} f(n/N) n^{-it_0} \cdot \mu_{\mathbf{c},2}(n) \right|^A \ll_{A,f} (1 + |t_0|)^A Z^m N^{(1-\eta_1)A}. \quad (6.27)$$

Proof. First suppose $(f, t_0) = (\mathbf{1}_{r \leq 1}, 0)$ and $A \in \mathbb{Z}_{\geq 2}$. Let $g(M) = \sum_{r \leq M: r \in \mathcal{N}_{\mathbf{c}}} \mu_{\mathbf{c}, 2}(r)$. By multiplicativity and (3.3), we find that

$$\sum_{n \leq N} \mu_{\mathbf{c}, 2}(n) = \sum_{d \leq N: d \in \mathcal{N}_{\mathbf{c}}} \mu_{\mathbf{c}, 2}(d) g(N/d) \ll_{\varepsilon} N^{\varepsilon} \sum_{d \leq N: d \in \mathcal{N}_{\mathbf{c}}} |g(N/d)|.$$

So by Hölder's inequality over d , the left-hand side of (6.27) is at most $O_{A, \varepsilon}(N^{\varepsilon} \Sigma_6)$, where

$$\Sigma_6 := \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} |\mathcal{N}_{\mathbf{c}} \cap [1, N]|^{A-1} \sum_{d \leq N: d \in \mathcal{N}_{\mathbf{c}}} |g(N/d)|^A.$$

By positivity, $\Sigma_6 \leq \sum_{d \leq N} \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} |\mathcal{N}_{\mathbf{c}} \cap [1, N]|^{A-1} |g(N/d)|^A$. Then the Cauchy–Schwarz inequality over $\mathbf{c}$, followed by Lemma 6.10 and Proposition 6.9, gives

$$\Sigma_6 \ll_{A, \varepsilon} \sum_{d \leq N} (Z^m N^{\varepsilon})^{1/2} (Z^m (N/d)^{2A-1/3+\varepsilon})^{1/2} \ll_{A, \varepsilon} Z^m N^{A-1/6+\varepsilon},$$

where we use $A - 1/6 + \varepsilon/2 \geq 1 + \varepsilon/2$ to evaluate the sum over d . By Hölder's inequality, then, (6.27) holds with $\eta_1 = 1/7A$ if $A \geq 2$, and with $\eta_1 = 1/14$ if $0 < A \leq 2$.

The general case follows from partial summation and Hölder's inequality, in view of the derivative bounds $\frac{\partial}{\partial n} D(n/N) \ll_D 1/N$ and $\frac{\partial}{\partial n} n^{it_0} \ll |t_0|/n$. $\square$

7 Adapting L -function statistics to delta

7.1 Factorization

We need to mold the statistics from §6 into a form friendlier for the delta method. We first split the series $\Phi(\mathbf{c}, s)$ from (2.11) into more manageable pieces. Recall $\mathcal{N}^{\mathbf{c}}$, $\mathcal{N}_{\mathbf{c}}$ from (2.5). Given $\mathbf{c} \in \mathcal{S}_1$, consider the factorization $\Phi = \Phi^G \Phi^B$, where

$$\Phi^G(\mathbf{c}, s) := \prod_{p \nmid \Delta(\mathbf{c})} \Phi_p(\mathbf{c}, s) = \sum_{n \in \mathcal{N}^{\mathbf{c}}} S_{\mathbf{c}}^{\mathbf{d}}(n) n^{-s}, \quad (7.1)$$

$$\Phi^B(\mathbf{c}, s) := \prod_{p \mid \Delta(\mathbf{c})} \Phi_p(\mathbf{c}, s) = \sum_{n \in \mathcal{N}_{\mathbf{c}}} S_{\mathbf{c}}^{\mathbf{d}}(n) n^{-s}. \quad (7.2)$$

One can approximate Φ^G using Hasse–Weil L -functions. It would be interesting to also relate Φ_p for $p \mid \Delta(\mathbf{c})$ to L -functions, even in special cases like when $m = 4$ and $v_p(\Delta(\mathbf{c})) = 1$. For now, we study Φ^B very differently, in §9. In §7, we thus concentrate on Φ^G .

For the rest of §7, assume $2 \mid m$. We first factor Φ^G into three pieces: $\Phi^{\mathbf{c}, 1}$, $\Phi^{\mathbf{c}, 2}$, $\Phi^{\mathbf{c}, 3}$.

Definition 7.1. Let $\Phi^{\mathbf{c}, 1}(s) := L(s, V_{\mathbf{c}})^{-1} L(\frac{1}{2} + s, V)^{-1} \zeta(2s)^{-1}$ as in (1.6), and let

$$\Phi^{\mathbf{c}, 2}(s) := \zeta(2s) / L(2s, V_{\mathbf{c}}, \Lambda^2), \quad (7.3)$$

$$\Phi^{\mathbf{c}, 3}(s) := \Phi^G(\mathbf{c}, s) L(s, V_{\mathbf{c}}) L(\frac{1}{2} + s, V) L(2s, V_{\mathbf{c}}, \Lambda^2). \quad (7.4)$$

For each $j \in \{1, 2, 3\}$, let $a_{\mathbf{c}, j}(n)$ be the n th coefficient of the Dirichlet series $\Phi^{\mathbf{c}, j}(s)$.

The factors $\Phi^{\mathbf{c},2}, \Phi^{\mathbf{c},3}$ in $\Phi^{\mathbf{G}} = \Phi^{\mathbf{c},1}\Phi^{\mathbf{c},2}\Phi^{\mathbf{c},3}$ hinder any direct attempt to apply statistics on $\Phi^{\mathbf{c},1}$ to $\Phi^{\mathbf{G}}$. Fortunately, $\Phi^{\mathbf{c},2}, \Phi^{\mathbf{c},3}$ turn out to be suitably small on average. Proposition 6.11 lets us handle large moduli in $\Phi^{\mathbf{c},2}$. Indeed, $a_{\mathbf{c},2}(n) = 0$ if n is not a square, and

$$a_{\mathbf{c},2}(n) = \mu_{\mathbf{c},2}(n^{1/2}) \quad (7.5)$$

otherwise. Here $\mu_{\mathbf{c},2}$ is as in §6.5. We now prove results to handle large moduli in $\Phi^{\mathbf{c},3}$.

The factor $\Phi^{\mathbf{c},3}$ measures the quality of $\Phi^{\mathbf{c},1}\Phi^{\mathbf{c},2}$ as an approximation to $\Phi^{\mathbf{G}}$. Recall the approximation $\Phi(\mathbf{c}, s) = \Psi^{\mathbf{c},1}(s)\Psi^{\mathbf{c},2}(s)$ given by $\Psi^{\mathbf{c},1}, \Psi^{\mathbf{c},2}$ from (2.18); here $\Psi^{\mathbf{c},1}(s) = 1/L(s, V_{\mathbf{c}})$. The factor $\Psi^{\mathbf{c},2}(s)$ is only expected to converge absolutely for $\operatorname{Re}(s) > \frac{1}{2}$. As suggested in §2, this is a “source of ε ” in (2.17). On the other hand, the following result establishes absolute convergence for $\Phi^{\mathbf{c},3}(s)$ past the critical line $\operatorname{Re}(s) = \frac{1}{2}$.

Proposition 7.2. *Uniformly over $\mathbf{c} \in \mathcal{S}_1$, primes p , and integers $l \geq 1$, we have*

$$a_{\mathbf{c},3}(p) \cdot \mathbf{1}_{p|\Delta(\mathbf{c})} = 0, \quad a_{\mathbf{c},3}(p^2) \cdot \mathbf{1}_{p|\Delta(\mathbf{c})} \ll p^{-1/2}, \quad a_{\mathbf{c},3}(p^l) \ll_{\varepsilon} p^{l\varepsilon}. \quad (7.6)$$

In particular, if $\mathbf{c} \in \mathcal{S}_1$, then $\Phi^{\mathbf{c},3}(s)$ converges absolutely over $\operatorname{Re}(s) > \frac{1}{3}$.

Proof. Let $\mathbf{c} \in \mathcal{S}_1$. Let $\lambda_{\mathbf{c}}^{\natural} := \lambda_{V_{\mathbf{c}}}^{\natural}$, as in §6.1. Suppose first that $p \mid \Delta(\mathbf{c})$. Then $\Phi_p^{\mathbf{G}}(\mathbf{c}, s) = 1$ by (7.1). So by (7.4) and (3.3), we have $a_{\mathbf{c},3}(p^l) \ll_{\varepsilon} p^{l\varepsilon}$. So (7.6) holds.

Now suppose $p \nmid \Delta(\mathbf{c})$. Then $\Phi_p(\mathbf{c}, s) = 1 + S_{\mathbf{c}}^{\natural}(p)p^{-s}$ by (2.14), and

$$S_{\mathbf{c}}^{\natural}(p) = E_{\mathbf{c}}^{\natural}(p) - p^{-1/2}E_F^{\natural}(p) = -\lambda_{\mathbf{c}}^{\natural}(p) - p^{-1/2}\lambda_V^{\natural}(p)$$

by (2.13), (3.5), and (3.4). In particular, $a_{\mathbf{c},3}(p^l) \ll_{\varepsilon} p^{l\varepsilon}$ by (7.4) and (3.3). Furthermore,

$$\begin{aligned} \Phi_p(\mathbf{c}, s)L_p(s, V_{\mathbf{c}}) &= (1 - \lambda_{\mathbf{c}}^{\natural}(p)p^{-s} - \lambda_V^{\natural}(p)p^{-1/2-s})(1 + \lambda_{\mathbf{c}}^{\natural}(p)p^{-s} + \lambda_{\mathbf{c}}^{\natural}(p^2)p^{-2s} + O(p^{-3s})) \\ &= 1 - \lambda_V^{\natural}(p)p^{-1/2-s} + [\lambda_{\mathbf{c}}^{\natural}(p^2) - \lambda_{\mathbf{c}}^{\natural}(p)^2]p^{-2s} + O(p^{-1/2-2s}) + O(p^{-3s}). \end{aligned}$$

To get further cancellation, we multiply $\Phi_p(\mathbf{c}, s)L_p(s, V_{\mathbf{c}})$ by the two series

$$\begin{aligned} L_p(\tfrac{1}{2} + s, V) &= 1 + \lambda_V^{\natural}(p)p^{-1/2-s} + O(p^{-1-2s}), \\ L_p(2s, V_{\mathbf{c}}, \wedge^2) &= 1 + \lambda_{V_{\mathbf{c}}, \wedge^2}^{\natural}(p)p^{-2s} + O(p^{-4s}), \end{aligned}$$

and plug in the identity $\lambda_{V_{\mathbf{c}}, \wedge^2}^{\natural}(p) = \lambda_{\mathbf{c}}^{\natural}(p)^2 - \lambda_{\mathbf{c}}^{\natural}(p^2)$ from (3.7), to get

$$\Phi_p(\mathbf{c}, s)L_p(s, V_{\mathbf{c}})L_p(\tfrac{1}{2} + s, V)L_p(2s, V_{\mathbf{c}}, \wedge^2) = 1 + O(p^{-1/2-2s}) + O(p^{-3s}). \quad (7.7)$$

By (7.4), the left-hand side of (7.7) is precisely the local factor $\Phi_p^{\mathbf{c},3}(s)$ of $\Phi^{\mathbf{c},3}(s)$. Thus (7.7) completes the proof of (7.6). The convergence statement on $\Phi^{\mathbf{c},3}(s)$ follows from (7.6). $\square$

Corollary 7.3. *Fix $A \in \mathbb{R}_{>0}$. Then uniformly over $Z, N \in \mathbb{R}_{>0}$, we have*

$$\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \left(\sum_{n \leq N} |a_{\mathbf{c},3}(n)| \right)^A \ll_{A, \varepsilon} Z^m N^{(1/3+\varepsilon)A}. \quad (7.8)$$

Proof. Recall $\mathcal{N}^c, \mathcal{N}_c$ from (2.5). By the multiplicativity and positivity of $|a_{c,3}|$, we have

$$\sum_{n \leq N} |a_{c,3}(n)| \leq \sum_{d \leq N: d \in \mathcal{N}_c} |a_{c,3}(d)| \sum_{r \leq N: r \in \mathcal{N}^c} |a_{c,3}(r)|. \quad (7.9)$$

But for any $\varepsilon > 0$, the bounds in (7.6) imply $a_{c,3}(d) \ll_\varepsilon d^\varepsilon$, and that

$$\sum_{r \leq N: r \in \mathcal{N}^c} |a_{c,3}(r)| \leq N^{1/3+\varepsilon} \sum_{r \in \mathcal{N}^c} r^{-1/3-\varepsilon} |a_{c,3}(r)| \ll_\varepsilon N^{1/3+\varepsilon};$$

cf. (6.16). So by (7.9), the left-hand side of (7.8) is

$$\ll_{A,\varepsilon} N^{(1/3+2\varepsilon)A} \sum_{c \in \mathcal{S}_1 \cap [-Z, Z]^m} |\mathcal{N}_c \cap [1, N]|^A.$$

Now (7.8) follows from Lemma 6.10. $\square$

7.2 A near-second moment bound

We now build on Conjecture 6.7 (R2'E).

Conjecture 7.4 (R2'E'). *Fix a function $D \in C_c^\infty(\mathbb{R}_{>0})$. Let $t \in \mathbb{R}$. Let $Z, N, \varepsilon \in \mathbb{R}_{>0}$ with $N \leq Z^3$ and $\varepsilon \in (0, 1]$. Then for some real $A_4 = A_4(F, \varepsilon) > 0$, we have*

$$\sum_{c \in \mathcal{S}_1 \cap [-Z, Z]^m} \left| \sum_{n \in \mathcal{N}^c} D(n/N) \cdot n^{-it} S_c^\natural(n) \right|^{2-\varepsilon} \ll_{F,D,\varepsilon} (1 + |t|)^{A_4} Z^m N^{(2-\varepsilon)/2}. \quad (7.10)$$

Proposition 7.5. *Assume Conjectures 1.3 and 6.7. Then Conjecture 7.4 holds.*

Proof. Let $v_2 \in C_c^\infty(\mathbb{R}_{>0})$ be as in §5, so that $\text{Supp } v_2 \subseteq [1, 2]$ and we have (5.4). In view of (7.1) and the factorization $\Phi^G = \Phi^{c,1} \Phi^{c,2} \Phi^{c,3}$, we may use Lemma 5.2, with $k = 3$, $a(\mathbf{n}) = \prod_{1 \leq j \leq 3} (n_j^{-it} a_{c,j}(n_j))$, and $f(\mathbf{r}) = D(r_1 r_2 r_3 / N)$, to write

$$\sum_{n \in \mathcal{N}^c} D(n/N) \cdot n^{-it} S_c^\natural(n) = (2\pi)^{-3} \int_{\mathbf{N} \geq 1/2} d^\times \mathbf{N} \int_{\mathbf{t} \in \mathbb{R}^3} d\mathbf{t} g_{\mathbf{N}}^\vee(i\mathbf{t}) \prod_{1 \leq j \leq 3} \Sigma_{7,\mathbf{N}}^{c,j}(\mathbf{t}), \quad (7.11)$$

where $\mathbf{N} = (N_1, N_2, N_3)$, where $g_{\mathbf{N}}(\mathbf{r}) := D(r_1 r_2 r_3 / N) \prod_{1 \leq j \leq 3} v_2(r_j / N_j)$, and where

$$\Sigma_{7,\mathbf{N}}^{c,j}(\mathbf{t}) = \Sigma_{7,\mathbf{N}}^{c,j}(t_1, t_2, t_3) := \sum_{n_j \geq 1} v_2(n_j / N_j) n_j^{-i(t_j + t)} a_{c,j}(n_j). \quad (7.12)$$

Here N, t are independent of $\mathbf{N}, \mathbf{t}$. For all $\mathbf{N} \geq 1/2$ and $b \geq 0$, Proposition 5.1 gives

$$g_{\mathbf{N}}^\vee(i\mathbf{t}) \ll_{b,D,v_2} (1 + \|\mathbf{t}\|)^{-b}. \quad (7.13)$$

Fix an integer $A \geq 1$ for which $\text{Supp } D \subseteq [A^{-1}, A]$. If $\mathbf{r} \in \text{Supp } g_{\mathbf{N}}$, then $N/A \leq r_1 r_2 r_3 \leq AN$ and $N_j \leq r_j \leq 2N_j$ for all j , so $\mathbf{N}$ lies in the set

$$\mathcal{R}_7 := \{\mathbf{N} \geq 1/2 : N_1 N_2 N_3 \in [N/8A, AN]\}. \quad (7.14)$$

Thus the equality (7.11) remains true if we restrict the integral over $\mathbf{N} \geq 1/2$ to the region $\mathcal{R}_7$. Now set $W_1(\mathbf{N}, \mathbf{t}) := N_1^\eta \cdot |g_{\mathbf{N}}^\vee(\mathbf{it})|$ and $W_2(\mathbf{N}, \mathbf{t}) := N_1^{-(\beta-1)\eta} \cdot |g_{\mathbf{N}}^\vee(\mathbf{it})|$, for some constants $\eta, \beta > 0$ to be chosen later. Let $\mathcal{I}_1 := \int_{\mathcal{R}_7 \times \mathbb{R}^3} d^\times \mathbf{N} d\mathbf{t} W_1(\mathbf{N}, \mathbf{t})$. Letting $\beta := 2 - \varepsilon \in [1, 2)$, and using Hölder's inequality over $(\log \mathbf{N}, \mathbf{t}) \subseteq \mathbb{R}^3 \times \mathbb{R}^3$, restricted to $\mathbf{N} \in \mathcal{R}_7$, we obtain

$$\left| \sum_{n \in \mathbf{N}^c} D(n/N) n^{-it} S_{\mathbf{c}}^{\dagger}(n) \right|^\beta \leq \mathcal{I}_1^{\beta-1} \cdot \int_{\mathcal{R}_7 \times \mathbb{R}^3} d^\times \mathbf{N} d\mathbf{t} W_2(\mathbf{N}, \mathbf{t}) \prod_{1 \leq j \leq 3} |\Sigma_{7, \mathbf{N}}^{\mathbf{c}, j}(\mathbf{t})|^\beta, \quad (7.15)$$

since $\beta \geq 1$ and $W_1(\mathbf{N}, \mathbf{t})^{\beta-1} \cdot W_2(\mathbf{N}, \mathbf{t}) = |g_{\mathbf{N}}^\vee(\mathbf{it})|^\beta$.

By (7.13) and (7.14), we have $\mathcal{I}_1 \ll_{D, v_2} \int_{\mathcal{R}_7} d^\times \mathbf{N} N_1^\eta \ll_{A, \eta} N^\eta \int_{N_2, N_3 \geq 1/2} \frac{d^\times N_2 d^\times N_3}{(N_2 N_3)^\eta} \ll_\eta N^\eta$. Upon summing (7.15) over $\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m$, we thus find that the left-hand side of (7.10) is

$$\ll_{D, v_2, \eta} \int_{\mathcal{R}_7 \times \mathbb{R}^3} d^\times \mathbf{N} d\mathbf{t} (N/N_1)^{(\beta-1)\eta} \cdot |g_{\mathbf{N}}^\vee(\mathbf{it})| \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \prod_{1 \leq j \leq 3} |\Sigma_{7, \mathbf{N}}^{\mathbf{c}, j}(\mathbf{t})|^\beta. \quad (7.16)$$

Now let $(\gamma_1, \gamma_2, \gamma_3) := (2, 4\beta/\varepsilon, 4\beta/\varepsilon)$. Then $\sum_{1 \leq j \leq 3} \beta/\gamma_j = 1$, since $\beta = 2 - \varepsilon$. So

$$\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} |\heartsuit_7|^\beta = \|\heartsuit_7\|_{\ell_{\mathbf{c}}^\beta(\mathcal{S}_1 \cap [-Z, Z]^m)}^\beta \leq \prod_{1 \leq j \leq 3} \mathcal{M}_j^{\beta/\gamma_j} \quad (7.17)$$

by Hölder's inequality over $\mathbf{c}$, where

$$\heartsuit_7 := \prod_{1 \leq j \leq 3} \Sigma_{7, \mathbf{N}}^{\mathbf{c}, j}(\mathbf{t}), \quad \mathcal{M}_j := \|\Sigma_{7, \mathbf{N}}^{\mathbf{c}, j}(\mathbf{t})\|_{\ell_{\mathbf{c}}^{\gamma_j}(\mathcal{S}_1 \cap [-Z, Z]^m)}^{\gamma_j}.$$

We now bound the necessary ℓ^{γ_j} -norms. First,

$$\mathcal{M}_1 \ll_{v_2} (1 + |t_1 + t|)^{A_3} (Z + N_1^{1/3})^m N_1^{\gamma_1/2},$$

by (7.12) and Conjecture 6.7 with $Z + N_1^{1/3}$, N_1 in place of Z, N . Second, by (7.12), (7.5), and (6.27) with $A = \gamma_2$, with $f = v_2$, and with $Z + N_2^{1/3}$, $N_2^{1/2}$ in place of Z, N , we have

$$\mathcal{M}_2 \ll_{\gamma_2, v_2} (1 + |t_2 + t|)^{\gamma_2} (Z + N_2^{1/3})^m (N_2^{1/2})^{(1-\eta_1(\gamma_2))\gamma_2}.$$

Third, by (7.12), the bound $v_2(n_3/N_3) n_3^{-i(t_3+t)} \ll_{v_2} 1$, and Corollary 7.3, we have

$$\mathcal{M}_3 \ll_{\gamma_3, v_2} Z^m N_3^{11\gamma_3/30}.$$

Plugging into (7.17), and writing $1 + |t_j + t| \leq (1 + \|\mathbf{t}\|)(1 + |t|)$, we get that if $\mathbf{N} \in \mathcal{R}_7$, then the left-hand side of (7.17) is $\ll_{D, v_2, \beta} (1 + \|\mathbf{t}\|)^{A_4} (1 + |t|)^{A_4} \mathcal{P}$, where

$$\mathcal{P} := Z^{\sum_j m\beta/\gamma_j} N_1^{\beta/2} N_2^{(1-\eta_1(\gamma_2))\beta/2} N_3^{11\beta/30} \asymp_{D, \beta} Z^m (N^{1/2} N_2^{-\eta_1(\gamma_2)/2} N_3^{-4/30})^\beta,$$

and where $A_4 := (A_3\beta/\gamma_1 + \gamma_2\beta/\gamma_2) = (A_3/2 + 1)\beta$. Upon plugging this result into (7.16), we get that the left-hand side of (7.10) is at most $O_{D, v_2, \eta, \beta}(\mathcal{Q})$, where

$$\mathcal{Q} := \int_{\mathcal{R}_7 \times \mathbb{R}^3} d^\times \mathbf{N} d\mathbf{t} (N/N_1)^{(\beta-1)\eta} \cdot |g_{\mathbf{N}}^\vee(\mathbf{it})| \cdot (1 + \|\mathbf{t}\|)^{A_4} \frac{(1 + |t|)^{A_4} Z^m N^{\beta/2}}{(N_2^{\eta_1(\gamma_2)/2} N_3^{4/30})^\beta}.$$

Finally, let $\eta = \min(\eta_1(\gamma_2)/2, 4/30)$. Then for each $\mathbf{N} \in \mathcal{R}_7$, we have $(N/N_1)^\eta \ll_{D,\beta} N_2^{\eta_1(\gamma_2)/2} N_3^{4/30}$. Since $\beta - 1 \geq 0$, it follows that

$$\mathcal{Q} \ll_{D,\beta} \int_{\mathcal{R}_7 \times \mathbb{R}^3} d^\times \mathbf{N} d\mathbf{t} |g_{\mathbf{N}}^\vee(i\mathbf{t})| \cdot (1 + \|\mathbf{t}\|)^{A_4} \frac{(1 + |\mathbf{t}|)^{A_4} Z^m N^{\beta/2}}{N_2^{\eta_1(\gamma_2)/2} N_3^{4/30}} \ll_{D,\beta} (1 + |\mathbf{t}|)^{A_4} Z^m N^{\beta/2},$$

where we have used (7.13) to integrate over $\mathbf{t}$, and (7.14) to integrate first over N_1 , given N_2, N_3 , and then over $N_2, N_3 \geq 1/2$. Therefore, (7.10) holds with $A_4 = (A_3/2 + 1)(2 - \varepsilon)$. $\square$

8 New bounds on the integral factor

Recall $J_{\mathbf{c},X}(n)$ from (2.9). We need to go beyond the integral estimates from [HB96, HB98, Hoo14]. We will prove uniform bounds free of epsilons and logs, while also bringing discriminants into the picture via the following consequence of (2.3):

$$\Delta(\nabla F(\mathbf{x})) \ll_F |F(\mathbf{x})| \cdot \|\mathbf{x}\|^{2\deg(\Delta) - \deg(F)} \quad \text{for } \mathbf{x} \in \mathbb{R}^m. \quad (8.1)$$

To give clean, general bounds, we assume (2.6). We prove the following on $J_{\mathbf{c},X}(n)$:

Proposition 8.1. *Assume (2.6). Let $X, Z, n \in \mathbb{R}_{>0}$. Let $\mathbf{c} \in \mathbb{R}^m$ with $\|\mathbf{c}\| \leq Z$. Then*

$$\partial_{\log n}^j \partial_{\mathbf{c}}^\alpha J_{\mathbf{c},X}(n) \ll_{F,w,j,\alpha,b} \frac{(X/n)^{|\alpha|} (1 + X\|\mathbf{c}\|/n)^{1-m/2}}{(1 + \|\mathbf{c}\|/X^{1/2})^b (1 + X\|\Delta(\mathbf{c}/Z)\mathbf{c}\|/n)^b}$$

for all integers $j, b \geq 0$ and multi-indices $\alpha \geq 0$.

The factor $\Delta(\mathbf{c}/Z) \ll 1$ measures how close $V_{\mathbf{c}}(\mathbb{R})$ is to being singular, and it arises in our proof from cancellation over smoothed circular arcs.

To prove Proposition 8.1, we must first dig into some technical aspects of [DFI93, HB96]'s h -function. Let $\rho_0(x) := \mathbf{1}_{|x|<1} \cdot \exp(-(1-x^2)^{-1}) \in C_c^\infty(\mathbb{R})$ and

$$\rho(x) := 4(\int_{y \in \mathbb{R}} dy \rho_0(y))^{-1} \cdot \rho_0(4x-3) \in C_c^\infty(\mathbb{R}),$$

so that ρ is supported on $[1/2, 1]$. For $x > 0$ and $y \in \mathbb{R}$, let

$$h(x, y) := \sum_{j \geq 1} (xj)^{-1} [\rho(xj) - \rho(|y|/(xj))].$$

By (2.7) and (2.9), and a change of variables from $\mathbf{x}$ to $\tilde{\mathbf{x}} = \mathbf{x}/X$, we have

$$J_{\mathbf{c},X}(n) = \int_{\tilde{\mathbf{x}} \in \mathbb{R}^m} d\tilde{\mathbf{x}} w(\tilde{\mathbf{x}}) h(n/Y, F(\tilde{\mathbf{x}})) e(-X\mathbf{c} \cdot \tilde{\mathbf{x}}/n). \quad (8.2)$$

The following shows that we may take $A_0 = \sup_{\mathbf{x} \in \text{Supp } w} \max(1, 2|F(\mathbf{x})|)$ in Proposition 2.4.

Lemma 8.2 (See [HB96, Lemma 4]). *If $\xi \in \mathbb{R}$ and $r \geq \max(1, 2|\xi|)$, then $h(r, \xi) = 0$.*

Following [HB96], we now build a Fourier transform. Fix $v_1 \in C_c^\infty(\mathbb{R})$ such that $v_1(F(\mathbf{x})) \geq 1$ for all $\mathbf{x} \in \text{Supp } w$. Let $w_0(\mathbf{x}) := w(\mathbf{x})/v_1(F(\mathbf{x})) \in C_c^\infty(\mathbb{R})$. For any $r > 0$, let

$$p_r(u) := \int_{\xi \in \mathbb{R}} d\xi v_1(\xi) h(r, \xi) e(-u\xi)$$

be the Fourier transform of $v_1(\xi) h(r, \xi) \in C_c^\infty(\mathbb{R})$. Writing $w(\tilde{\mathbf{x}}) = w_0(\tilde{\mathbf{x}}) v_1(F(\tilde{\mathbf{x}}))$ and $v_1(\xi) h(r, \xi) = \int_{u \in \mathbb{R}} du p_r(u) e(u\xi)$ in (8.2), where $r = n/Y$ and $\xi = F(\tilde{\mathbf{x}})$, we get

$$J_{\mathbf{c}, X}(n) = \int_{u \in \mathbb{R}} du p_r(u) \int_{\tilde{\mathbf{x}} \in \mathbb{R}^m} d\tilde{\mathbf{x}} w_0(\tilde{\mathbf{x}}) e(uF(\tilde{\mathbf{x}}) - \mathbf{v} \cdot \tilde{\mathbf{x}}), \quad (8.3)$$

where $(r, \mathbf{v}) := (n/Y, X\mathbf{c}/n)$; cf. [HB96, Lemma 17].

Let $q_r(t) := p_r(t/r)$, so that $p_r(u) = q_r(ru)$. The following lemma shows that $q_r(t)$ is Schwartz in t and smoothly varying in $\log r$.

Lemma 8.3. *Let $r > 0$ and $t \in \mathbb{R}$. Then $\partial_{\log r}^j q_r^{(l)}(t) \ll_{l,j,k} (1 + |t|)^{-k}$ for all $j, k, l \in \mathbb{Z}_{\geq 0}$.*

Proof. We may assume $k = 0$ if $|t| \leq 1$; this lets us treat all t simultaneously.

Since $q_r(t) = p_r(t/r) = \int_{\xi \in \mathbb{R}} d\xi v_1(\xi) h(r, \xi) e(-t\xi/r)$, we have

$$q_r^{(l)}(t) = \int_{\xi \in \mathbb{R}} d\xi v_1(\xi) h(r, \xi) \cdot (-2\pi i \xi/r)^l e(-t\xi/r).$$

Since $\partial_{\log r}((\xi/r)^l) = -l(\xi/r)^l$ and $\partial_{\log r}(e(-t\xi/r)) = (2\pi i t \xi/r) e(-t\xi/r)$, we find by induction and the Leibniz product rule that for some constants $c_{a,b}^{l,j} \in \mathbb{C}$, we have

$$\partial_{\log r}^j q_r^{(l)}(t) = \sum_{a,b \geq 0: a+b \leq j} c_{a,b}^{l,j} \int_{\xi \in \mathbb{R}} d\xi v_1(\xi) \cdot (\partial_{\log r}^a h(r, \xi)) \cdot (\xi/r)^l (t\xi/r)^b e(-t\xi/r).$$

We now integrate by parts k times in ξ , repeatedly integrating $e(-t\xi/r)$ and differentiating the complementary factor. Taking absolute values, we then get

$$\partial_{\log r}^j q_r^{(l)}(t) \ll_{l,j,k,v_1} \sum_{\substack{a,b,\alpha,\beta \geq 0: \\ a+b \leq j, \alpha+\beta \leq k, \beta \leq l+b}} \int_{\xi \in \text{Supp } v_1} d\xi (\partial_{\xi}^\alpha \partial_{\log r}^a h(r, \xi)) \cdot \frac{|\xi/r|^l |t\xi/r|^b}{|\xi|^\beta} \frac{1}{|t/r|^k}.$$

By Lemma 8.2, we may assume $r \ll_{v_1} 1$. Then $\partial_r^c \partial_{\xi}^\alpha h(r, \xi) \ll_{c,\alpha,A} r^{-1-c-\alpha} \min(1, (r/|\xi|)^A)$ for $c, \alpha, A \geq 0$, by [HB96, Lemma 5]. Thus $\partial_{\log r}^a \partial_{\xi}^\alpha h(r, \xi) \ll_a \sum_{0 \leq c \leq a} r^c \cdot |\partial_r^c \partial_{\xi}^\alpha h(r, \xi)| \ll_{a,\alpha,A} r^{-1-\alpha} \min(1, (r/|\xi|)^A)$, which when inserted in the previous display gives

$$\partial_{\log r}^j q_r^{(l)}(t) \ll_{l,j,k,A} \sum_{\substack{a,b,\alpha,\beta \geq 0: \\ a+b \leq j, \alpha+\beta \leq k, \beta \leq l+b}} \int_{\xi \in \mathbb{R}} d\xi \frac{r^{-1-\alpha-l-b+k} |\xi|^{l+b-\beta} |t|^{b-k}}{(1 + |\xi/r|)^A}.$$

Let $A = l + j + 2$; then each integral here over ξ is $\ll_{l,j,k} r^{-1-\alpha-l-b+k} r^{1+l+b-\beta} |t|^{b-k}$ by Lemma 5.3, and thus $\ll_{l,j,k} r^{k-\alpha-\beta} |t|^{b-k} \ll |t|^{b-k}$, since $k - \alpha - \beta \geq 0$ and $r \ll 1$. If $|t| \leq 1$, we are done, since $k = 0$. If $|t| > 1$, we may replace k with $k + j$ to finish, since $b \leq j$. $\square$

We now put (8.3) in a broader framework. For any Schwartz functions $q: \mathbb{R} \rightarrow \mathbb{C}$ and $\phi \in C_c^\infty(\mathbb{R}^m) \otimes \mathbb{C}$ with $|q^{(l)}(t)| \leq C(l, k)(1 + |t|)^{-k}$ and $\text{Supp } \phi \subseteq \text{Supp } w$, let

$$\mathcal{J}_{r, \mathbf{v}}(q, \phi) := \int_{u \in \mathbb{R}} du q(ru) \int_{\mathbf{x} \in \mathbb{R}^m} d\mathbf{x} \phi(\mathbf{x}) e(uF(\mathbf{x}) - \mathbf{v} \cdot \mathbf{x}). \quad (8.4)$$

Then by (8.3), we have $J_{\mathbf{c}, X}(n) = \mathcal{J}_{r, \mathbf{v}}(q_r, w_0)$, where $(r, \mathbf{v}) = (n/Y, X\mathbf{c}/n)$.

Proposition 8.4. Assume (2.6). Let q , ϕ , and $C = C(l, k)$ be as above. Then for all $(k, \mathbf{v}) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^m$ and positive reals $r \leq A_0$ and $M \geq \|\mathbf{v}\|$, we have

$$\mathcal{J}_{r, \mathbf{v}}(q, \phi) \ll_{F, w, C, \phi, k} (1 + \|\mathbf{v}\|)^{1-m/2} \cdot (1 + \|r\mathbf{v}\|)^{-k} (1 + \|\Delta(\mathbf{v}/M)\mathbf{v}\|)^{-k}.$$

Before proving Proposition 8.4, we first explain why it implies the desired Proposition 8.1. To handle $\partial_{\log n}^j$ for $j \geq 1$, we need a recursion, (8.5), observed for $j = 1$ by [HB96].

Lemma 8.5. Let $(r, \mathbf{v}) = (n/Y, X\mathbf{c}/n)$. Let $q_1(t) := t \cdot q(t)$ and let $\phi_1(\mathbf{x}) := \text{div}(\phi_1(\mathbf{x})\mathbf{x}) = \mathbf{x} \cdot \nabla \phi_1(\mathbf{x}) = \sum_{j=1}^m x_j \cdot \partial_{x_j} \phi_1$. Let $\phi_{2,j}(\mathbf{x}) := x_j \cdot \phi(\mathbf{x})$. Then

$$\partial_{\log n} \mathcal{J}_{r, \mathbf{v}}(q, \phi) = (m-3) \mathcal{J}_{r, \mathbf{v}}(q, \phi) - 2 \mathcal{J}_{r, \mathbf{v}}(q_1, \phi) + \mathcal{J}_{r, \mathbf{v}}(q, \phi_1), \quad (8.5)$$

$$\partial_{c_j} \mathcal{J}_{r, \mathbf{v}}(q, \phi) = (-2\pi i X/n) \mathcal{J}_{r, \mathbf{v}}(q, \phi_{2,j}). \quad (8.6)$$

Proof. The formula (8.6) immediately follows upon differentiating (8.4) by c_j . It is possible to prove (8.5) by a clever integration by parts; cf. [HB96, p. 182, proof of Lemma 14]. We give a slightly shorter argument. Write $u = t/r^3$ and $\mathbf{x} = r\mathbf{y}$ in (8.4) to get

$$\mathcal{J}_{r, \mathbf{v}}(q, \phi) = r^{m-3} \int_{\mathbb{R}} dt q(t/r^2) \int_{\mathbb{R}^m} d\mathbf{y} \phi(r\mathbf{y}) e(tF(\mathbf{y}) - r\mathbf{v} \cdot \mathbf{y}). \quad (8.7)$$

Differentiating both sides of (8.7) by $\log n$, using $\partial_{\log n} r = r$ and the fact that $r\mathbf{v} = X\mathbf{c}/Y$ is independent of $\log n$, we get (8.5), by (8.7) applied to each of (q, ϕ) , (q_1, ϕ) , (q, ϕ_1) . $\square$

Proof of Proposition 8.1, assuming Proposition 8.4. We have $J_{\mathbf{c}, X}(n) = \mathcal{J}_{r, \mathbf{v}}(q_r, w_0)$ by (8.3), where $(r, \mathbf{v}) = (n/Y, X\mathbf{c}/n)$. By Lemmas 8.3 and 8.5, we may thus write $\partial_{\mathbf{c}}^\alpha \partial_{\log n}^j J_{\mathbf{c}, X}(n)$ as a finite linear combination of integrals $\mathcal{J}_{r, \mathbf{v}}(q, \phi)$, with coefficients $O_{m,j,\alpha}((X/n)^{|\alpha|})$, running over a set of at most 4^j pairs (q, ϕ) . Here we first use the chain rule, (8.5), and Lemma 8.3 when differentiating by $\log n$, and then use (8.6) when differentiating $\mathbf{c}$. For example, for $j = 1$ and $\alpha = \mathbf{0}$, we would use the chain rule and (8.5) to write $\partial_{\log n} \mathcal{J}_{r, \mathbf{v}}(q_r, w_0)$ as

$$\mathcal{J}_{r, \mathbf{v}}(\partial_{\log r} q_r, w_0) + (m-3) \mathcal{J}_{r, \mathbf{v}}(q_r, w_0) - 2 \mathcal{J}_{r, \mathbf{v}}(q_{r,1}, w_0) + \mathcal{J}_{r, \mathbf{v}}(q_r, w_{0,1}),$$

where $q_{r,1}(t) = t \cdot q_r(t)$ and $w_{0,1}(\mathbf{x}) := \text{div}(w_0(\mathbf{x})\mathbf{x})$. In any case, Proposition 8.1 now follows from an application of Proposition 8.4 to each individual $\mathcal{J}_{r, \mathbf{v}}(q, \phi)$. $\square$

To prove Proposition 8.4, we need the following lemma:

Lemma 8.6 (Non-stationary phase). *Let $\lambda, A \in \mathbb{R}_{>0}$ and $d, k \in \mathbb{Z}_{\geq 1}$. Suppose $a \in C_c^\infty(\mathbb{R}^d) \otimes \mathbb{C}$ is supported on $[-A, A]^d$, with $\sum_{|\alpha| \leq k} |\partial_{\mathbf{x}}^\alpha a(\mathbf{x})| \leq A$ for all $\mathbf{x} \in \mathbb{R}^d$. Suppose $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is smooth, with $\|\nabla f(\mathbf{x})\| \geq \lambda/A$ and $\sum_{2 \leq |\alpha| \leq k+1} |\partial_{\mathbf{x}}^\alpha f(\mathbf{x})| \leq A\lambda$ for all $\mathbf{x} \in \text{Supp } a$. Then*

$$\int_{\mathbf{x} \in \mathbb{R}^d} d\mathbf{x} a(\mathbf{x}) e(f(\mathbf{x})) \ll_{d,A,k} \lambda^{-k}.$$

Proof. See [HB96, Lemma 10], and its proof via repeated integration by parts; see [HB96, §2] for the definition of $\mathcal{C}(S)$, which allows for the required uniformity over weight functions. $\square$

Proof of Proposition 8.4. Certainly $\mathbf{0} \notin \text{Supp } w$ by (2.6). Since V is smooth, we thus have $\|\nabla F(\mathbf{x})\| > 0$ for all $\mathbf{x} \in \text{Supp } w$. Since $\text{Supp } w$ is compact, there thus exists $A_5 = A_5(F, w) \geq 2$ such that for all $\mathbf{x} \in \text{Supp } w$, we have $\|\mathbf{x}\|, \|\nabla F(\mathbf{x})\| \in [A_5^{-1}, A_5]$. Let $\mathcal{W} := \{a\mathbf{x} : (a, \mathbf{x}) \in [1/3A_5, 3A_5] \times \text{Supp } w\}$ be the union of all dilates of $\text{Supp } w$ by a scale factor $a \in [1/3A_5, 3A_5]$. The set $\mathcal{W}$ is compact, being the continuous image of a product of compact sets. Also, since the right-hand side of (2.6) is invariant under scaling, we have

$$\mathcal{W} \subseteq \{\mathbf{w} \in \mathbb{R}^m : \det(\text{Hess } F(\mathbf{w})) \neq 0\}. \quad (8.8)$$

Let $A_6 = A_6(F, w) \geq 2$ be a constant such that for all $\mathbf{w} \in \mathcal{W}$, we have

$$A_6^{-1} \leq \|\mathbf{w}\| \leq A_6, \quad A_6^{-1} \leq \|\nabla F(\mathbf{w})\| \leq A_6. \quad (8.9)$$

Our plan is to first consider the $\mathbf{x}$ -integral in $\mathcal{J}_{r,\mathbf{v}}(q, \phi)$ (see (8.4)) for a single value of u at a time, and then integrate over u . Given $u, \mathbf{v}$, let $\psi_0(\mathbf{x}) := uF(\mathbf{x}) - \mathbf{v} \cdot \mathbf{x}$ and $\mathcal{J}_{u,\mathbf{v}}(\phi) := \int_{\mathbf{x} \in \mathbb{R}^m} d\mathbf{x} \phi(\mathbf{x}) e(\psi_0(\mathbf{x}))$. Taking absolute values in $\mathcal{J}$ gives the trivial bound

$$\mathcal{J}_{u,\mathbf{v}}(\phi) \ll_{\phi} 1. \quad (8.10)$$

On the other hand, $\nabla \psi_0(\mathbf{x}) = u\nabla F(\mathbf{x}) - \mathbf{v}$. In particular, if $|u| \geq 2A_5\|\mathbf{v}\|$ or $|u| \leq \|\mathbf{v}\|/2A_5$, then we have $\|\nabla \psi_0(\mathbf{x})\| \geq \max(|u|/2A_5, \|\mathbf{v}\|/2)$ and $\partial_{\mathbf{x}}^\alpha \psi_0(\mathbf{x}) \ll_{F,w,\alpha} |u| + \|\mathbf{v}\|$ for all $\mathbf{x} \in \text{Supp } w$, and thus $\mathcal{J}_{u,\mathbf{v}}(\phi) \ll_{\phi,k} (|u| + \|\mathbf{v}\|)^{-k}$ by Lemma 8.6, provided $|u| + \|\mathbf{v}\| > 0$. This, together with (8.10) and the bound $q(ru) \ll_C 1$, gives

$$\int_{|u| \geq 2A_5\|\mathbf{v}\|} du |q(ru) \cdot \mathcal{J}_{u,\mathbf{v}}(\phi)| \ll_{C,\phi,b} \int_{|u| \geq 2A_5\|\mathbf{v}\|} \frac{du}{(1+|u|)^{b+1}} \ll_b (1+\|\mathbf{v}\|)^{-b}, \quad (8.11)$$

$$\int_{|u| \leq \|\mathbf{v}\|/2A_5} du |q(ru) \cdot \mathcal{J}_{u,\mathbf{v}}(\phi)| \ll_{C,\phi,b} \int_{|u| \leq \|\mathbf{v}\|/2A_5} \frac{du}{(1+\|\mathbf{v}\|)^{b+1}} \ll_b (1+\|\mathbf{v}\|)^{-b}, \quad (8.12)$$

for all integers $b \geq 1$. Fix $w_1 \in C_c^\infty(\mathbb{R})$ with

$$w_1|_{[-2A_5, 2A_5] \setminus [-1/2A_5, 1/2A_5]} = 1, \quad \text{Supp } w_1 \subseteq [-3A_5, 3A_5] \setminus [-1/3A_5, 1/3A_5].$$

Then by (8.11), (8.12), and the triangle inequality, we have, for all integers $b \geq 1$,

$$\mathcal{J}_{r,\mathbf{v}}(q, \phi) - \int_{u \in \mathbb{R}} du q(ru) w_1(u/\|\mathbf{v}\|) \mathcal{J}_{u,\mathbf{v}}(\phi) \ll_{C,\phi,b} (1+\|\mathbf{v}\|)^{-b}. \quad (8.13)$$

It remains to handle $|u|/\|\mathbf{v}\| \in [1/3A_5, 3A_5]$. Suppose first that $\|\mathbf{v}\| \leq 1$. Plugging (8.10) and the bound $q(ru) \ll_C 1$ into (8.13), we get $\mathcal{J}_{r,\mathbf{v}}(q, \phi) \ll_{C,\phi} (1 + \|\mathbf{v}\|)^{-1} + \int_{|u| \leq 3A_5} du \ll 1$. This bound on $\mathcal{J}_{r,\mathbf{v}}(q, \phi)$ fits in Proposition 8.4, since $\|r\mathbf{v}\| \leq A_0$ and $\|\Delta(\mathbf{v}/M)\mathbf{v}\| \ll_F 1$.

For the rest of the proof, suppose $\|\mathbf{v}\| \geq 1$, let $\tilde{u} := u/\|\mathbf{v}\|$ and $\tilde{\mathbf{v}} := \mathbf{v}/\|\mathbf{v}\|$, let $\mathbf{z} := |\tilde{u}|^{1/2}\mathbf{x}$, let $\text{sgn}(u) := u/|u| \in \{-1, 1\}$, and let

$$\psi_1(\mathbf{z}) := |\tilde{u}|^{1/2} \psi_0(\mathbf{x})/\|\mathbf{v}\| = |\tilde{u}|^{1/2} \psi_0(\mathbf{z}/|\tilde{u}|^{1/2})/\|\mathbf{v}\| = \text{sgn}(u)F(\mathbf{z}) - \tilde{\mathbf{v}} \cdot \mathbf{z}.$$

The normalization by $|\tilde{u}|^{1/2}$ makes ψ_1 nearly independent of u ; this is crucial later. Now, note that $\nabla \psi_1(\mathbf{z}) = \text{sgn}(u)\nabla F(\mathbf{z}) - \tilde{\mathbf{v}} = \tilde{u}\nabla F(\mathbf{x}) - \tilde{\mathbf{v}}$. Consider an individual $\tilde{u} \in \mathbb{R}$ with

$$1/3A_5 \leq |\tilde{u}| \leq 3A_5. \quad (8.14)$$

For all $\mathbf{x} \in \text{Supp } w$, we have $\mathbf{z} \in \mathcal{W}$, so $\|\mathbf{z}\| \in [1/A_6, A_6]$ and $\|\nabla \psi_1(\mathbf{z})\| \leq \|\nabla F(\mathbf{z})\| + 1 \leq 2A_6$. Furthermore, the condition (8.8) implies

$$\text{vol}\{\mathbf{z} \in \mathcal{W} : \|\nabla \psi_1(\mathbf{z})\| \leq m\lambda\} \ll_{F,w} \lambda^m, \quad (8.15)$$

uniformly over $\lambda \in \mathbb{R}_{>0}$, by calculus; see [HB96, Lemma 21 and its proof]. We will split $\mathcal{J}_{u,\mathbf{v}}(\phi)$ into pieces according to the size of $\|\nabla \psi_1(\mathbf{z})\|$; cf. [Hua20, $\mathcal{K}_i$ on p. 2061]. To avoid the need for explicit stationary phase expansions, which can be messy after the leading term, we will do the subdivision in a way inspired by [HB96, §8].

Recall the weights v_0, v_1 from §1.2. Let $\bar{v}_0 := 1 - v_0$. For reals $\lambda > 0$, let

$$\mathcal{J}_{0,u,\mathbf{v}} := \int_{\mathbb{R}^m} \frac{d\mathbf{z}}{|\tilde{u}|^{m/2}} v_0 \left(\frac{\|\nabla \psi_1(\mathbf{z})\|}{\|\mathbf{v}\|^{-1/2}} \right) \phi(\mathbf{x}) e \left(\frac{\|\mathbf{v}\| \psi_1(\mathbf{z})}{|\tilde{u}|^{1/2}} \right), \quad (8.16)$$

$$\mathcal{J}_{1,u,\mathbf{v},\lambda} := \int_{\mathbb{R}^m} \frac{d\mathbf{z}}{|\tilde{u}|^{m/2}} \bar{v}_0 \left(\frac{\|\nabla \psi_1(\mathbf{z})\|}{\|\mathbf{v}\|^{-1/2}} \right) v_1 \left(\frac{\|\nabla \psi_1(\mathbf{z})\|}{\lambda} \right) \phi(\mathbf{x}) e \left(\frac{\|\mathbf{v}\| \psi_1(\mathbf{z})}{|\tilde{u}|^{1/2}} \right), \quad (8.17)$$

where $\phi(\mathbf{x}) = \phi(\mathbf{z}/|\tilde{u}|^{1/2})$. By the definitions of v_0, v_1 , we have $\mathcal{J}_{1,u,\mathbf{v},\lambda} = 0$ unless there exists $\mathbf{z} \in \mathcal{W}$, corresponding to $\mathbf{x} \in \text{Supp } w$, satisfying $\|\nabla \psi_1(\mathbf{z})\| > \|\mathbf{v}\|^{-1/2}/2$ and $\lambda \leq \|\nabla \psi_1(\mathbf{z})\| \leq m\lambda$. Therefore, $\mathcal{J}_{1,u,\mathbf{v},\lambda} = 0$ unless $\|\mathbf{v}\|^{-1/2}/2m < \lambda \leq 2A_6$. Since $\int_{\lambda > 0} d^\times \lambda v_1(\mathbf{t}/\lambda) = 1$ for all $\mathbf{t} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$, we obtain the decomposition

$$\mathcal{J}_{u,\mathbf{v}}(\phi) - \mathcal{J}_{0,u,\mathbf{v}} = \int_{\|\mathbf{v}\|^{-1/2}/2m}^{2A_6} d^\times \lambda \mathcal{J}_{1,u,\mathbf{v},\lambda}, \quad (8.18)$$

under the assumption (8.14). Fix functions $\omega_1, \dots, \omega_m \in C_c^\infty(\mathbb{R}^m)$, all supported on $\text{Supp } v_1$, such that $\sum_{j=1}^m \omega_j = v_1$ and we have $1/2 \leq |t_j| \leq 2m$ for all $\mathbf{t} \in \text{Supp } \omega_j$. Let $\mathcal{J}_{1,j,u,\mathbf{v},\lambda}$ denote $\mathcal{J}_{1,u,\mathbf{v},\lambda}$ with ω_j in place of v_1 ; clearly $\mathcal{J}_{1,u,\mathbf{v},\lambda} = \sum_{j=1}^m \mathcal{J}_{1,j,u,\mathbf{v},\lambda}$.

We proceed based on the following rough idea: if $\|\nabla \psi_1(\mathbf{z})\| \gg \|\mathbf{v}\|^{-1/2}$ with a large constant, integration by parts over $\mathbf{z}$ is useful; and if $\|\nabla \psi_1(\mathbf{z})\| \ll |\Delta(\tilde{\mathbf{v}})|^{1/2}$ with a small constant, integration by parts over u is useful, if $\|\Delta(\tilde{\mathbf{v}})\mathbf{v}\| \gg 1$ with any constant.

Let $\lambda \in [4\|\mathbf{v}\|^{-1/2}, 2A_6]$. Since $\lambda \geq 4\|\mathbf{v}\|^{-1/2}$, the supports of $v_0(\nabla \psi_1(\mathbf{z})/\|\mathbf{v}\|^{-1/2})$ and $v_1(\nabla \psi_1(\mathbf{z})/\lambda)$ are disjoint, so $\bar{v}_0(\nabla \psi_1(\mathbf{z})/\|\mathbf{v}\|^{-1/2})\omega_j(\nabla \psi_1(\mathbf{z})/\lambda) = \omega_j(\nabla \psi_1(\mathbf{z})/\lambda)$. Thus

$$|\tilde{u}|^{m/2} \mathcal{J}_{1,j,u,\mathbf{v},\lambda} = \int_{\mathbb{R}^m} d\mathbf{z} w_{2,j,0}(\mathbf{z}) e \left(\frac{\|\mathbf{v}\| \psi_1(\mathbf{z})}{|\tilde{u}|^{1/2}} \right), \quad (8.19)$$

where $w_{2,j,0} = w_{2,j,0,u,v,\lambda}(\mathbf{z}) := \omega_j(\nabla \psi_1(\mathbf{z})/\lambda) \phi(\mathbf{z}/|\tilde{u}|^{1/2})$.

Since $\text{Supp } \omega_j \subseteq \{\mathbf{t} \in \mathbb{R}^m : 1/2 \leq |t_j| \leq 2m\} \cap \text{Supp } \mathbf{v}_1$, we have

$$|\partial_{z_j} \psi_1(\mathbf{z})| \geq \lambda/2, \quad \|\nabla \psi_1(\mathbf{z})\| \leq m\lambda, \quad (8.20)$$

for all $\mathbf{z} \in \text{Supp } w_{2,j,0}$. For $k \geq 1$, recursively define $w_{2,j,k} = w_{2,j,k,u,v,\lambda}(\mathbf{z})$, a sequence of smooth functions $\mathbb{R}^m \rightarrow \mathbb{C}$ supported on $\text{Supp } w_{2,j,0}$, by setting

$$w_{2,j,k} := -(\partial_{z_j} \psi_1)^k \cdot \partial_{z_j} (w_{2,j,k-1} / (\partial_{z_j} \psi_1)^k). \quad (8.21)$$

Integrating by parts k times in (8.19), we get

$$|\tilde{u}|^{m/2} \mathcal{J}_{1,j,u,v,\lambda} = \int_{\mathbb{R}^m} d\mathbf{z} \frac{w_{2,j,k,u,v,\lambda}(\mathbf{z})}{(\|\mathbf{v}\| \partial_{z_j} \psi_1(\mathbf{z}) / |\tilde{u}|^{1/2})^k} \cdot e(\|\mathbf{v}\| \psi_1(\mathbf{z}) / |\tilde{u}|^{1/2}); \quad (8.22)$$

cf. [HB96, (5.4), from the proof of Lemma 10].

We claim that for each $k \geq 0$, there exists a smooth function $w_{3,j,k} : \mathbb{R}^{m(k+2)+3} \rightarrow \mathbb{C}$ of the form $w_{3,j,k} = w_{3,j,k}(\mathbf{a}_{1,0}, \dots, \mathbf{a}_{1,k}, \mathbf{a}_{2,0}, a_{2,1}, a_3, a_4)$, defined in terms of F, j, k , independently of $u, \mathbf{v}, \lambda$, such that for all $u, \mathbf{v}, \lambda$ satisfying the present assumptions $\|\mathbf{v}\| \geq 1$, (8.14), and $\lambda \in [4\|\mathbf{v}\|^{-1/2}, 2A_6]$, and for all $\mathbf{z} \in \text{Supp } w_{2,j,0}$, we have

$$\lambda^k \cdot w_{2,j,k}(\mathbf{z}) = w_{3,j,k} \left(\frac{\nabla \psi_1(\mathbf{z})}{\lambda}, \nabla \partial_{z_j}^1 \psi_1(\mathbf{z}), \dots, \nabla \partial_{z_j}^k \psi_1(\mathbf{z}), \frac{\mathbf{z}}{|\tilde{u}|^{1/2}}, \frac{1}{|\tilde{u}|^{1/2}}, \frac{\lambda}{\partial_{z_j} \psi_1}, \lambda \right). \quad (8.23)$$

This claim can be easily proven by induction on $k \geq 0$, as follows. For $k = 0$, we take

$$w_{3,j,0}(\mathbf{a}_{1,0}, \mathbf{a}_{2,0}, a_{2,1}, a_3, a_4) := \omega_j(\mathbf{a}_{1,0}) \phi(\mathbf{a}_{2,0}).$$

For $k \geq 1$, we first use the product and chain rules in (8.21) to compute the quantity $\lambda^k w_{2,j,k} = \lambda \cdot \lambda^{k-1} w_{2,j,k}$ in terms of $\lambda^{k-1} w_{2,j,k-1}$ and its z_j -derivatives. We then observe that

$$\begin{aligned} \lambda \cdot \partial_{z_j} \left(\frac{\psi_1}{\lambda} \right) &= \partial_{z_j}^1 \psi_1, \quad \lambda \cdot \partial_{z_j} (\partial_{z_j}^l) = \lambda \cdot \partial_{z_j}^{l+1}, \quad \lambda \cdot \partial_{z_j} (z_i) = \lambda \cdot \mathbf{1}_{i=j}, \\ \lambda \cdot \partial_{z_j} \left(\frac{\lambda}{\partial_{z_j} \psi_1} \right) &= -(\partial_{z_j}^2 \psi_1) \cdot \left(\frac{\lambda}{\partial_{z_j} \psi_1} \right)^2, \quad (\partial_{z_j} \psi_1)^k \cdot \partial_{z_j} \left(\frac{\lambda}{(\partial_{z_j} \psi_1)^k} \right) = (\partial_{z_j}^2 \psi_1) \cdot \frac{-k\lambda}{\partial_{z_j} \psi_1} \end{aligned}$$

for all $l \geq 1$, that ψ_1 is smooth, and that $\partial_{z_j} \psi_1$ is nonvanishing on $\text{Supp } w_{2,j,0}$.

By (8.9), we have $\partial^\alpha \psi_1(\mathbf{z}) \ll_{\alpha} 1$ whenever $|\alpha| \geq 2$ and $\mathbf{z} \in \mathcal{W}$. Inserting this and the bounds (8.20), (8.14), (8.9), and $\lambda \ll 1$ into (8.23), we find that $\lambda^k w_{2,j,k}(\mathbf{z}) \ll_{w_{3,j,k}} 1$ for all $\mathbf{z} \in \mathbb{R}^m$. Thus (8.22) and (8.20) immediately give

$$|\tilde{u}|^{m/2} \mathcal{J}_{1,j,u,v,\lambda} \ll_k \int_{\mathbf{z} \in \mathcal{W}} d\mathbf{z} \frac{\mathbf{1}_{\|\nabla \psi_1(\mathbf{z})\| \leq m\lambda}}{(\lambda^2 \|\mathbf{v}\| / |\tilde{u}|^{1/2})^k}, \quad (8.24)$$

assuming $\lambda \in [4\|\mathbf{v}\|^{-1/2}, 2A_6]$ and (8.14). Our derivation of (8.24) has essentially followed the proof of Lemma 8.6, but in some key ranges we need to go further. We need to decide when to integrate by parts over $\tilde{u} \in \mathbb{R}$; this will be informed by the next two paragraphs.

By (8.8) and the inverse function theorem, we know that for some open neighborhood $U \subseteq \mathbb{R}^m$ of $\mathcal{W}$, the gradient map $\nabla F|_U: U \rightarrow \mathbb{R}^m$ is a local diffeomorphism. But $\mathcal{W}$ is compact, so after covering it by sufficiently small open sets, we find that there exists a constant $\eta_2 = \eta_2(F, w) > 0$ such that for any $\mathbf{z} \in \mathcal{W}$ and $\mathbf{b} \in [-1, 1]^m$ with $\|\nabla F(\mathbf{z}) - \mathbf{b}\| \leq \eta_2$, there exists $\mathbf{s} \in (\nabla F)^{-1}(\mathbf{b})$ with $\|\mathbf{z} - \mathbf{s}\| \leq \eta_2^{-1} \|\nabla F(\mathbf{z}) - \mathbf{b}\|$.

We apply this as follows. Let $\mathbf{b} = \text{sgn}(u)\tilde{\mathbf{v}}$ and $\mathbf{z} \in \mathcal{W}$, and suppose $\|\nabla \psi_1(\mathbf{z})\| \leq \eta_2$. Then there exists $\mathbf{s} \in \mathbb{R}^m$ with $\nabla \psi_1(\mathbf{s}) = \mathbf{0}$ and $\|\mathbf{z} - \mathbf{s}\| \leq \eta_2^{-1} \|\nabla \psi_1(\mathbf{z})\|$. Since $\nabla \psi_1(\mathbf{s}) = \mathbf{0}$, Taylor expansion of ψ_1 at $\mathbf{s}$ gives $\psi_1(\mathbf{z}) = \psi_1(\mathbf{s}) + O_{F,w}(\|\mathbf{z} - \mathbf{s}\|^2)$. Furthermore, $\nabla \psi_1(\mathbf{s}) = \mathbf{0}$ implies $\mathbf{b} \cdot \mathbf{s} = \nabla F(\mathbf{s}) \cdot \mathbf{s} = 3F(\mathbf{s})$, so $|\psi_1(\mathbf{s})| = |2F(\mathbf{s})|$. But $\Delta(\mathbf{b}) = \Delta(\nabla F(\mathbf{s})) \ll_{F,w} |F(\mathbf{s})|$ by (8.1). Combining the above, we get

$$|\psi_1(\mathbf{z})| \geq \eta_3 |\Delta(\tilde{\mathbf{v}})| - \eta_3^{-1} \|\nabla \psi_1(\mathbf{z})\|^2, \quad \text{for some } \eta_3 = \eta_3(F, w) > 0. \quad (8.25)$$

Let $A_7 := 1 + \sup_{\mathbf{a} \in [-1, 1]^m} |\Delta(\mathbf{a})|$, and let $\eta_4 := m^{-1} \min(\eta_2/A_7^{1/2}, \eta_3/2)$.

Our remaining analysis breaks into two cases: $\|\Delta(\tilde{\mathbf{v}})\mathbf{v}\| \geq (4/\eta_4)^2$ and $\|\Delta(\tilde{\mathbf{v}})\mathbf{v}\| < (4/\eta_4)^2$.

Suppose first that $\|\Delta(\tilde{\mathbf{v}})\mathbf{v}\| < (4/\eta_4)^2$. Consider a $\tilde{u} \in \mathbb{R}$ satisfying (8.14). By (8.15), the right-hand side of (8.24) is $\ll_k \lambda^m (\lambda^2 \|\mathbf{v}\|)^{-k} = \|\mathbf{v}\|^{-m/2} (\lambda \|\mathbf{v}\|^{1/2})^{m-2k}$. Choosing $k = \lceil m/2 + 1 \rceil$ and inserting (8.24) into (8.18), we then get

$$\mathcal{J}_{u,\mathbf{v}}(\phi) - \mathcal{J}_{0,u,\mathbf{v}} - \int_{\|\mathbf{v}\|^{-1/2}/2m}^{4\|\mathbf{v}\|^{-1/2}} d^\times \lambda \mathcal{J}_{1,u,\mathbf{v},\lambda} \ll_m \|\mathbf{v}\|^{-m/2},$$

since $\int_{4\|\mathbf{v}\|^{-1/2}}^\infty d^\times \lambda (\lambda \|\mathbf{v}\|^{1/2})^{m-2k} = \int_4^\infty d^\times a a^{m-2k} \ll 1$ via the substitution $a = \lambda \|\mathbf{v}\|^{1/2}$. But if we simply take absolute values in $\mathcal{J}_{0,u,\mathbf{v}}$ (see (8.16)) and in $\mathcal{J}_{1,u,\mathbf{v},\lambda}$ (see (8.17)), and then apply (8.15), we get $\mathcal{J}_{0,u,\mathbf{v}} \ll \|\mathbf{v}\|^{-m/2}$ and $\mathcal{J}_{1,u,\mathbf{v},\lambda} \ll \|\mathbf{v}\|^{-m/2}$ for $\lambda \in [\|\mathbf{v}\|^{-1/2}/2m, 4\|\mathbf{v}\|^{-1/2}]$. Integrating over λ in the previous display, we conclude that $\mathcal{J}_{u,\mathbf{v}}(\phi) \ll_m \|\mathbf{v}\|^{-m/2}$, under (8.14). Hence by (8.13), after writing $u = \|\mathbf{v}\|\tilde{u}$, we have

$$\mathcal{J}_{r,\mathbf{v}}(q, \phi) \ll_b (1 + \|\mathbf{v}\|)^{-b} + \|\mathbf{v}\| \int_{1/3A_5 \leq |\tilde{u}| \leq 3A_5} d\tilde{u} |q(r\|\mathbf{v}\|\tilde{u})| \cdot \|\mathbf{v}\|^{-m/2}.$$

But $q(r\|\mathbf{v}\|\tilde{u}) \ll_b (1 + r\|\mathbf{v}\|)^{-b}$, under (8.14). Thus $\mathcal{J}_{r,\mathbf{v}}(q, \phi) \ll_b \|\mathbf{v}\|^{1-m/2} (1 + r\|\mathbf{v}\|)^{-b}$. This suffices for Proposition 8.4, since $\|\mathbf{v}\| \geq 1$ and $\|\Delta(\mathbf{v}/M)\mathbf{v}\| \leq \|\Delta(\tilde{\mathbf{v}})\mathbf{v}\| \ll 1$.

For the rest of the proof, assume $\|\Delta(\tilde{\mathbf{v}})\mathbf{v}\| \geq (4/\eta_4)^2$, so that $4\|\mathbf{v}\|^{-1/2} \leq \eta_4 |\Delta(\tilde{\mathbf{v}})|^{1/2}$. We will integrate by parts over $\tilde{u}$ in the integrals

$$\mathcal{J}_0 := \int_{\mathbb{R}} d\tilde{u} q(r\|\mathbf{v}\|\tilde{u}) w_1(\tilde{u}) \mathcal{J}_{0,u,\mathbf{v}}, \quad \mathcal{J}_{1,\lambda} := \int_{\mathbb{R}} d\tilde{u} q(r\|\mathbf{v}\|\tilde{u}) w_1(\tilde{u}) \mathcal{J}_{1,u,\mathbf{v},\lambda}$$

for $\lambda \leq \eta_4 |\Delta(\tilde{\mathbf{v}})|^{1/2}$. It is crucial to work in terms of $\mathbf{z}$ rather than $\mathbf{x} = \mathbf{z}/|\tilde{u}|^{1/2}$. Before proceeding, note that inserting (8.18) into (8.13) and writing $u = \|\mathbf{v}\|\tilde{u}$ gives

$$\mathcal{J}_{r,\mathbf{v}} - \|\mathbf{v}\| \cdot \mathcal{J}_0 - \|\mathbf{v}\| \cdot \int_{\|\mathbf{v}\|^{-1/2}/2m}^{2A_6} d^\times \lambda \mathcal{J}_{1,\lambda} \ll_b (1 + \|\mathbf{v}\|)^{-b}. \quad (8.26)$$

For later reference, let $\mathbf{z} \in \mathcal{W}$ and suppose $\|\nabla \psi_1(\mathbf{z})\| \leq m\eta_4 |\Delta(\tilde{\mathbf{v}})|^{1/2}$. By the definition of η_4 , we then have $\|\nabla \psi_1(\mathbf{z})\| \leq \eta_2$ and $\|\nabla \psi_1(\mathbf{z})\| \leq \eta_3 |\Delta(\tilde{\mathbf{v}})|^{1/2}/2$, so by (8.25), we have

$$|\psi_1(\mathbf{z})| \geq 3\eta_3 |\Delta(\tilde{\mathbf{v}})|/4. \quad (8.27)$$

For convenience, let $\mathcal{W}(\lambda) := \{\mathbf{z} \in \mathcal{W} : \|\nabla \psi_1(\mathbf{z})\| \leq m\lambda\}$ for each $\lambda > 0$.

We now bound $\mathcal{J}_0$. For each $\tilde{u} \in \text{Supp } w_1$, the condition (8.14) holds, so if we plug in the definition (8.16) of $\mathcal{J}_{0,u,\mathbf{v}}$, and then switch the order of $u, \mathbf{z}$, we may rewrite $\mathcal{J}_0$ as

$$\int_{\mathcal{W}(\|\mathbf{v}\|^{-1/2})} d\mathbf{z} v_0 \left(\frac{\nabla \psi_1(\mathbf{z})}{\|\mathbf{v}\|^{-1/2}} \right) \int_{\mathbb{R}} \frac{d\tilde{u}}{|\tilde{u}|^{m/2}} q(r\|\mathbf{v}\|\tilde{u}) w_1(\tilde{u}) \phi \left(\frac{\mathbf{z}}{|\tilde{u}|^{1/2}} \right) e \left(\frac{\|\mathbf{v}\| \psi_1(\mathbf{z})}{|\tilde{u}|^{1/2}} \right).$$

Here ψ_1 is independent of $|\tilde{u}|$, when $\text{sgn}(u)$ is fixed. Therefore, for each $\mathbf{z} \in \mathcal{W}(\|\mathbf{v}\|^{-1/2})$, the inner integral over $\tilde{u}$ is $\ll_k (1+r\|\mathbf{v}\|)^{-k} \|\Delta(\tilde{\mathbf{v}})\mathbf{v}\|^{-k}$ by (8.27) and Lemma 8.6, because $|\tilde{u}| \asymp 1$ for $\tilde{u} \in \text{Supp } w_1$ and we have $\partial_{\tilde{u}}^l q(r\|\mathbf{v}\|\tilde{u}) = (r\|\mathbf{v}\|)^l q^{(l)}(r\|\mathbf{v}\|\tilde{u}) \ll_{C,l,k} (1+r\|\mathbf{v}\|)^{-k}$ for all $l, k \geq 0$. Applying this inner integral estimate for each $\mathbf{z} \in \mathcal{W}(\|\mathbf{v}\|^{-1/2})$, and then using (8.15), we get $\mathcal{J}_0 \ll_k \|\mathbf{v}\|^{-m/2} (1+r\|\mathbf{v}\|)^{-k} \|\Delta(\tilde{\mathbf{v}})\mathbf{v}\|^{-k}$.

Similarly, $\mathcal{J}_{1,\lambda} \ll_k \|\mathbf{v}\|^{-m/2} (1+r\|\mathbf{v}\|)^{-k} \|\Delta(\tilde{\mathbf{v}})\mathbf{v}\|^{-k}$ for $\lambda \in [\|\mathbf{v}\|^{-1/2}/2m, 4\|\mathbf{v}\|^{-1/2}]$.

Now suppose $4\|\mathbf{v}\|^{-1/2} \leq \lambda \leq \min(\eta_4|\Delta(\tilde{\mathbf{v}})|^{1/2}, 2A_6)$. Let $\mathcal{J}_{1,j,\lambda}$ denote $\mathcal{J}_{1,\lambda}$ with $\mathcal{J}_{1,j,u,\mathbf{v},\lambda}$ in place of $\mathcal{J}_{1,u,\mathbf{v},\lambda}$. Then $\mathcal{J}_{1,\lambda} = \sum_{j=1}^m \mathcal{J}_{1,j,\lambda}$. By (8.22), we have, for all integers $l \geq 0$,

$$\mathcal{J}_{1,j,\lambda} = \int_{\mathbf{z} \in \mathcal{W}(\lambda)} d\mathbf{z} \int_{\mathbb{R}} \frac{d\tilde{u}}{|\tilde{u}|^{m/2}} q(r\|\mathbf{v}\|\tilde{u}) w_1(\tilde{u}) \frac{w_{2,j,l,u,\mathbf{v},\lambda}(\mathbf{z}) e(\|\mathbf{v}\| \psi_1(\mathbf{z})/|\tilde{u}|^{1/2})}{(\|\mathbf{v}\| \partial_{z_j} \psi_1(\mathbf{z})/|\tilde{u}|^{1/2})^l}.$$

We now write $w_{2,j,l,u,\mathbf{v},\lambda}(\mathbf{z}) = \lambda^{-l} w_{3,j,l}(\cdots)$, using (8.23). Since ψ_1 is independent of $|\tilde{u}|$, and we have $\|\mathbf{z}\| \ll 1$ and $|\tilde{u}| \asymp 1$ for all $(\mathbf{z}, \tilde{u}) \in \mathcal{W} \times \text{Supp } w_1$, we find by (8.20), (8.27), and Lemma 8.6, applied to the inner integral over $\tilde{u}$ for each $\mathbf{z} \in \mathcal{W}(\lambda)$ for which there exists $u \in \mathbb{R}$ with $w_1(\tilde{u}) w_{2,j,l,u,\mathbf{v},\lambda}(\mathbf{z}) \neq 0$, that

$$\mathcal{J}_{1,j,\lambda} \ll_{k,l} \int_{\mathbf{z} \in \mathcal{W}(\lambda)} d\mathbf{z} \frac{(1+r\|\mathbf{v}\|)^{-k} \|\Delta(\tilde{\mathbf{v}})\mathbf{v}\|^{-k}}{(\lambda^2 \|\mathbf{v}\|)^l} \ll \frac{\lambda^m (1+r\|\mathbf{v}\|)^{-k} \|\Delta(\tilde{\mathbf{v}})\mathbf{v}\|^{-k}}{(\lambda^2 \|\mathbf{v}\|)^l}$$

for all integers $k, l \geq 0$, where in the final step we use (8.15).

Finally, suppose $\eta_4|\Delta(\tilde{\mathbf{v}})|^{1/2} \leq \lambda \leq 2A_6$. Then (8.24) and (8.15) directly give, for $k, l \geq 0$,

$$\mathcal{J}_{1,\lambda} \ll_l \int_{\mathbb{R}} d\tilde{u} \frac{|q(r\|\mathbf{v}\|\tilde{u}) w_1(\tilde{u})| \cdot \lambda^m}{(\lambda^2 \|\mathbf{v}\|)^l} \ll_k \frac{(1+r\|\mathbf{v}\|)^{-k} \|\mathbf{v}\|^{-m/2}}{(\lambda \|\mathbf{v}\|^{1/2})^{2l-m}}.$$

Finally, we apply (8.26) with $b = \lceil m/2 + 2k \rceil$. On the left-hand side of (8.26), we insert our work from the last four paragraphs, except the last one if $\eta_4|\Delta(\tilde{\mathbf{v}})|^{1/2} > 2A_6$. We get

$$\frac{\mathcal{J}_{r,\mathbf{v}}}{\|\mathbf{v}\|} \ll_{k,l} \frac{1 + \int_{\lambda \geq 4\|\mathbf{v}\|^{-1/2}} d^\times \lambda (\lambda \|\mathbf{v}\|^{1/2})^{m-2l}}{\|\mathbf{v}\|^{m/2} (1+r\|\mathbf{v}\|)^k \|\Delta(\tilde{\mathbf{v}})\mathbf{v}\|^k} + \frac{\int_{\lambda \geq \eta_4|\Delta(\tilde{\mathbf{v}})|^{1/2}} d^\times \lambda (\lambda \|\mathbf{v}\|^{1/2})^{m-2l+2k}}{\|\mathbf{v}\|^{m/2} (1+r\|\mathbf{v}\|)^k \|\Delta(\tilde{\mathbf{v}})\mathbf{v}\|^k}.$$

Take $l = \lceil m/2 + k + 1 \rceil$, and evaluate both integrals over λ , the first being $\ll 1$ since $4\|\mathbf{v}\|^{-1/2} \cdot \|\mathbf{v}\|^{1/2} = 4 \gg 1$, and the second being $\ll 1$ since $\eta_4|\Delta(\tilde{\mathbf{v}})\mathbf{v}|^{1/2} \geq 4 \gg 1$. We get $\mathcal{J}_{r,\mathbf{v}} \ll_k \|\mathbf{v}\|^{1-m/2} (1+r\|\mathbf{v}\|)^{-k} \|\Delta(\tilde{\mathbf{v}})\mathbf{v}\|^{-k}$, which suffices for Proposition 8.4. $\square$

9 New bounds on bad exponential sums

We first provide some general *vanishing and boundedness criteria* for $S_{\mathbf{c}}(p^l)$, which when combined with classical estimates from [Hoo86b, HB98] will, under Conjecture 1.6, allow us to break a critical ε -barrier behind (2.17). Recall $S_{\mathbf{c}}(n)$, $S_{\mathbf{c}}^\dagger(n)$ from (2.8), (2.9).

Lemma 9.1. *Let $\mathbf{c} \in \mathcal{S}_1$, and let p be a prime.*

1. *If $p \nmid \gcd(\Delta(\mathbf{c}), \partial_{c_1}\Delta(\mathbf{c}), \dots, \partial_{c_m}\Delta(\mathbf{c}))$, then $S_{\mathbf{c}}^{\natural}(p) \ll_m 1$.*
2. *If $v_p(\Delta(\mathbf{c})) \leq 1$, then $S_{\mathbf{c}}^{\natural}(p^2) \ll_F 1$.*
3. *We have $S_{\mathbf{c}}(p^l) = 0$ for all integers $l \geq 2 + v_p(\Delta(\mathbf{c}))$.*

Proof. (1): This follows from [Hoo94, Lemma 60] if $p \nmid \deg \Delta$, since $(\deg \Delta) \Delta(\mathbf{c}) = \mathbf{c} \cdot \nabla \Delta(\mathbf{c})$. On the other hand, if $p \mid \deg \Delta$, then $S_{\mathbf{c}}^{\natural}(p) \ll_m 1$ trivially.

We now turn to (2) and (3). For any vector $\mathbf{u} \in \mathbb{Z}^m$, let $v_p(\mathbf{u}) := v_p(\gcd(u_1, \dots, u_m))$. Write $\mathbf{c} = p^g \tilde{\mathbf{c}}$, where $g = v_p(\mathbf{c}) < \infty$ and $\tilde{\mathbf{c}} \in \mathcal{S}_1$. For integers $u, d \geq 0$, let

$$\mathcal{B}^{(d)}(\mathbf{c}; p, u) := \bigcup_{\lambda \in \mathbb{Z}: p \nmid \lambda} \{\mathbf{x} \in \mathbb{Z}^m : p \nmid \mathbf{x}, p^u \mid F(\mathbf{x}), p^u \mid \tilde{\mathbf{c}} \cdot \mathbf{x}, p^d \mid \nabla F(\mathbf{x}) - \lambda \mathbf{c}\}.$$

For any $u, d \geq 0$ and $\mathbf{x} \in \mathcal{B}^{(d)}(\mathbf{c}; p, u)$, Corollary 2.2 (with $\tilde{\mathbf{c}}$ in place of $\mathbf{c}$) implies

$$\gcd(p^u, p^{2d}) \mid \Delta(\tilde{\mathbf{c}}). \quad (9.1)$$

For any set $A \subseteq \mathbb{Z}^m$ that can be written as a finite union of residue classes $\mathcal{R} \subseteq \mathbb{Z}^m$, let $\mu(A)$ be the density of A in $\mathbb{Z}^m$. Now let $l \geq 2 + 2g$ be an integer, and let $d = \lfloor l/2 \rfloor \geq 1 + g$; then by [Wan24b, Proposition 7.4], we have

$$p^{-lm} \varphi(p^l) S'_{\mathbf{c}}(p^l) = p^{2l+g} \mu(\mathcal{B}^{(d)}(\mathbf{c}; p, l)) - p^{2l-2+g} \mu(\mathcal{B}^{(d)}(\mathbf{c}; p, l-1)), \quad (9.2)$$

where $S'_{\mathbf{c}}(p^l)$, the “restriction to $p \nmid \mathbf{x}$ ” of $S_{\mathbf{c}}(p^l)$, is defined as in [Wan24b, (7.5)].

(2): Suppose $v_p(\Delta(\mathbf{c})) \leq 1$. Then $p \nmid \mathbf{c}$, since $\deg \Delta \geq 2$. So $g = 0$ and $\tilde{\mathbf{c}} = \mathbf{c}$. In particular, $S_{\mathbf{c}}(p^2) = S'_{\mathbf{c}}(p^2)$ by [Wan24b, Lemma 7.2]. By (9.2) with $l = 2$ and $d = \lfloor l/2 \rfloor = 1$, it remains to analyze $\mathcal{B}^{(1)}(\mathbf{c}; p, u)$ for $u \in \{1, 2\}$. By (9.1), we have $\mathcal{B}^{(1)}(\mathbf{c}; p, 2) = \emptyset$, since $p^2 \nmid \Delta(\tilde{\mathbf{c}})$. We now analyze $\mathcal{B}^{(1)}(\mathbf{c}; p, 1)$.

For each $\mathbf{a} \in \mathcal{B}^{(1)}(\mathbf{c}; p, 1)$, the variety $F(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x} = 0$ in $\mathbb{P}_{\mathbb{F}_p}^{m-1}$ is singular at the point $[a_1 : \dots : a_m] \in \mathbb{P}^{m-1}(\mathbb{F}_p)$. Since $p \nmid \mathbf{c}$, this variety is isomorphic to a cubic hypersurface $f(y_1, \dots, y_{m-1}) = 0$ in $\mathbb{P}_{\mathbb{F}_p}^{m-2}$. Moreover, its singular locus has dimension ≤ 0 if $p \nmid \text{disc}(F)$ [Wan23a, Theorem 2.3, due to Zak]. So if $p \nmid 3 \text{disc}(F)$, then $p^m \mu(\mathcal{B}^{(1)}(\mathbf{c}; p, 1)) \leq (p-1) \cdot 2^{m-1}$ by Bézout’s theorem applied to the system $\nabla f = 0$ in $\mathbb{P}_{\mathbb{F}_p}^{m-2}$; note that $3f = \mathbf{y} \cdot \nabla f$.

But if $p \mid 3 \text{disc}(F)$, then $p^m \mu(\mathcal{B}^{(1)}(\mathbf{c}; p, 1)) \leq p^m - 1$ trivially. So in every case, $p^m \mu(\mathcal{B}^{(1)}(\mathbf{c}; p, 1)) \ll_F p - 1$. Now by (2.9) and (9.2), we have

$$S_{\mathbf{c}}^{\natural}(p^2) = \frac{S_{\mathbf{c}}(p^2)}{p^{1+m}} = \frac{S'_{\mathbf{c}}(p^2)}{p^{1+m}} = \frac{p^{m-1}}{\varphi(p^2)} \cdot \left(p^4 \cdot 0 - p^2 \cdot \mu(\mathcal{B}^{(1)}(\mathbf{c}; p, 1)) \right) \ll_F \frac{p(p-1)}{\varphi(p^2)} = 1.$$

(3): Take a counterexample $(\mathbf{c}, l) \in \mathcal{S}_1 \times \mathbb{Z}_{\geq 0}$ with l minimal. Then $l \geq 2 + v_p(\Delta(\mathbf{c}))$ and $S_{\mathbf{c}}(p^l) \neq 0$. So $l \geq 2 + 2g$, because $\deg \Delta \geq 2$. Also, $d = \lfloor l/2 \rfloor \geq (l-1)/2$. By (9.1), we have $\mathcal{B}^{(d)}(\mathbf{c}; p, u) = \emptyset$ for all $u \geq l-1$, since $2d \geq l-1$ and $p^{l-1} \nmid \Delta(\mathbf{c})$. So (9.2) gives $S'_{\mathbf{c}}(p^l) = 0$, whence $S_{\mathbf{c}}(p^l) \neq S'_{\mathbf{c}}(p^l)$. If

$g = 0$, this contradicts [Wan24b, Lemma 7.2]. If $g = 1$, then $l \geq 4$, so $S_{\mathbf{c}}(p^l) - S'_{\mathbf{c}}(p^l) = 0$ by [Wan24b, Lemma 7.2(2)]; again, a contradiction. Now suppose $g \geq 2$. Then $p^2 \mid \mathbf{c}$ and $l - 3 \geq 2 + v_p(\Delta(\mathbf{c}/p^2))$, since $\deg \Delta \geq 3/2$. So $S_{\mathbf{c}/p^2}(p^{l-3}) = 0$ by the minimality hypothesis. But [Wan24b, Lemma 7.2(2)] then gives $S_{\mathbf{c}}(p^l) - S'_{\mathbf{c}}(p^l) = 0$; another contradiction. Therefore, no counterexample $(\mathbf{c}, l) \in \mathcal{S}_1 \times \mathbb{Z}_{\geq 0}$ to (3) in fact exists. $\square$

Remark 9.2. If F were quadratic, rather than cubic, then Lemma 9.1(1) would be false whenever $2 \mid m$. See [Wan22, Remark 7.2.4] or [Wan23a, sentence after Theorem 1.1].

Let $\mathcal{N}_{\leq}(t) := \{n \geq 1 : p \mid n \Rightarrow v_p(n) \leq t\}$ and $\mathcal{N}_{\geq}(t) := \{n \geq 1 : p \mid n \Rightarrow v_p(n) \geq t\}$ for each integer $t \geq 1$. For any integers $N, t \geq 1$, we have (see e.g. [BG58])

$$|\{N \leq n < 2N : n \in \mathcal{N}_{\geq}(t)\}| \ll_t N^{1/t}. \quad (9.3)$$

Recall $\mathcal{N}_{\mathbf{c}}$ from (2.5). Lemma 9.1(1) has a useful unconditional consequence:

Proposition 9.3. *Let $A \in \mathbb{R}_{\geq 1}$. Uniformly over reals $Z, N > 0$ with $N \leq Z^3$, we have*

$$\sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \left(\sum_{n \in \mathcal{N}_{\mathbf{c}} \cap \mathcal{N}_{\leq}(1) \cap [N, 2N]} n^{-1/2} |S_{\mathbf{c}}^{\natural}(n)| \right)^{2A} \ll_{F, A, \varepsilon} Z^m \min(N, Z)^{-1+\varepsilon}. \quad (9.4)$$

Proof. The case $N < 1$ is trivial, so assume $N \geq 1$. Let $\mathcal{D} = \{1, 2, 4, 8, \dots\}$. Let

$$\mathcal{N}_{\mathbf{c}, 0} := \{n \in \mathcal{N}_{\leq}(1) : p \mid n \Rightarrow p \mid \gcd(\Delta(\mathbf{c}), \partial_{c_1} \Delta(\mathbf{c}), \dots, \partial_{c_m} \Delta(\mathbf{c}))\} \quad (9.5)$$

for each $\mathbf{c} \in \mathbb{Z}^m$. For each $Q \in \mathcal{D}$, let $\mathcal{S}_3(Q)$ be the set of vectors $\mathbf{c} \in \mathcal{S}_1$ for which $\mathcal{N}_{\mathbf{c}, 0} \cap [Q, 2Q] \neq \emptyset$. Note that $1 \in \mathcal{N}_{\mathbf{c}, 0}$, so $\mathcal{S}_3(1) = \mathcal{S}_1$.

By e.g. [HB83, Lemma 11], it is known that $S_{\mathbf{c}}^{\natural}(p) \ll_m p^{1/2}$ for all $\mathbf{c} \in \mathbb{Z}^m$ and primes p . So for all $\mathbf{c} \in \mathcal{S}_1$ and $n \in \mathcal{N}_{\leq}(1)$, Lemma 9.1(1) implies

$$S_{\mathbf{c}}^{\natural}(n) \ll_{m, \varepsilon} n^{\varepsilon} \cdot \max_{q \mid n : q \in \mathcal{N}_{\mathbf{c}, 0}} q^{1/2} \ll n^{\varepsilon} \cdot (\max \{Q \in \mathcal{D} \cap [1, n] : \mathbf{c} \in \mathcal{S}_3(Q)\})^{1/2}.$$

Therefore, the left-hand side of (9.4) is at most $O_{m, A, \varepsilon}(\mathcal{T})$, where

$$\mathcal{T} := \sum_{\substack{\mathbf{c} \in \mathcal{S}_1 : \\ \|\mathbf{c}\| \leq Z}} |\mathcal{N}_{\mathbf{c}} \cap [N, 2N]|^{2A} \sum_{\substack{Q \in \mathcal{D} \cap [1, 2N] : \\ \mathbf{c} \in \mathcal{S}_3(Q)}} \frac{N^{\varepsilon} Q^A}{N^A} = \sum_{Q \in \mathcal{D} \cap [1, 2N]} \frac{N^{\varepsilon} Q^A}{N^A} \sum_{\substack{\mathbf{c} \in \mathcal{S}_3(Q) : \\ \|\mathbf{c}\| \leq Z}} |\mathcal{N}_{\mathbf{c}} \cap [N, 2N]|^{2A}.$$

Let $\varepsilon \in (0, 1)$, and consider an individual $Q \in \mathcal{D}$. By Lemma 6.10 and Hölder's inequality,

$$\sum_{\mathbf{c} \in \mathcal{S}_3(Q) \cap [-Z, Z]^m} |\mathcal{N}_{\mathbf{c}} \cap [N, 2N]|^{2A} \ll_{A, \varepsilon} |\mathcal{S}_3(Q) \cap [-Z, Z]^m|^{1-\varepsilon} \cdot (Z^m N^{\varepsilon})^{\varepsilon}. \quad (9.6)$$

On the other hand, the scheme $\Delta = \partial_{c_1} \Delta = \dots = \partial_{c_m} \Delta = 0$ in $\mathbb{A}_{\mathbb{Q}}^m$ has dimension $\leq m - 2$, since the absolute irreducibility of Δ implies that $\Delta = 0$ is generically smooth. So by a quantitative form of

[Eke91]'s geometric sieve (see [Bha14, Theorem 3.3] for the case of prime moduli, which extends to square-free moduli as in [Bha25, §5, Case III]), we have

$$|\mathcal{S}_3(Q) \cap [-Z, Z]^m| \ll_{\Delta, \varepsilon} Q^\varepsilon Z^m (Q^{-1} + Z^{-1}). \quad (9.7)$$

Upon inserting (9.7) into (9.6), we find that

$$\mathcal{T} \ll_{A, \varepsilon} \sum_{Q \in \mathcal{D} \cap [1, 2N]} \frac{N^{2\varepsilon} Q^A}{N^A} \cdot Z^m (Q^{\varepsilon-1} + Z^{\varepsilon-1}) \ll_{A, \varepsilon} N^{3\varepsilon} \cdot Z^m (N^{\varepsilon-1} + Z^{\varepsilon-1}),$$

since $A \geq 1$. Since $N^\varepsilon \leq \min(N, Z)^{3\varepsilon}$, the bound (9.4) follows immediately. $\square$

We now build on Conjecture 1.6.

Conjecture 9.4 (SF'). *There exists a real $\eta_5 = \eta_5(\Delta) > 0$ such that*

$$\#\{c \in [-Z, Z]^m : \exists q \in \mathcal{N}_{\geq}(2) \cap [Q, 2Q] \text{ with } q \mid \Delta(c)\} \ll_{\Delta} Z^m Q^{-\eta_5} \quad (9.8)$$

holds uniformly over reals $Z, Q \geq 1$ with $Q \leq Z^3$.

Proposition 9.5. *Assume Conjecture 1.6. Then Conjecture 9.4 holds.*

Proof. Suppose $Z, Q \geq 1$ are reals with $Q \leq Z^3$.

Case 1: $Q \leq Z$. Recall the notation $N(P; q)$ from §4. Let $\delta = (\deg \Delta)^{-1}$ and $\varepsilon \in (0, \delta)$. The left-hand side of (9.8) is at most

$$\sum_{q \in \mathcal{N}_{\geq}(2) \cap [Q, 2Q]} N(\Delta; q) \cdot O\left(\frac{Z^m}{q^m}\right) \ll_{\varepsilon} \frac{Z^m}{Q^{\delta/2-\varepsilon}} \sum_{q \in \mathcal{N}_{\geq}(2)} q^{\delta/2-\varepsilon} \cdot \frac{N(\Delta; q)}{q^m}. \quad (9.9)$$

Let p be a prime and $l \geq 2$ an integer. By (4.3), we have $N(\Delta; p^l) \ll_{\Delta} p^{l(m-\delta)}$. On the other hand, by Hensel's lemma (resp. the Lang–Weil bound) over the smooth (resp. singular) locus of $\Delta = 0$, we have $N(\Delta; p^2) \ll_{\Delta} p^{2m-2}$, and thus $N(\Delta; p^l) \leq p^{(l-2)m} N(\Delta; p^2) \ll_{\Delta} p^{lm-2}$. So by the Chinese remainder theorem, the right-hand side of (9.9) is

$$\leq \frac{Z^m}{Q^{\delta/2-\varepsilon}} \prod_p \left(1 + \sum_{l \geq 2} p^{(\delta/2-\varepsilon)l} \cdot O_{\Delta}(p^{-\max(\delta l, 2)}) \right) \leq \frac{Z^m}{Q^{\delta/2-\varepsilon}} \prod_p \left(1 + \sum_{l \geq 2} O_{\Delta}(p^{-1-\varepsilon l}) \right),$$

since $(\delta/2 - \varepsilon)l - \max(\delta l, 2) \leq -1 - \varepsilon l$. Since $\varepsilon > 0$, it follows that the right-hand side of (9.9) is $\ll_{\Delta, \varepsilon} Z^m Q^{-\delta/2+\varepsilon}$. Hence the left-hand side of (9.8) is $\ll_{\Delta, \varepsilon} Z^m Q^{-\delta/2+\varepsilon}$.

Case 2: $Q > Z$. Suppose $q \in \mathcal{N}_{\geq}(2) \cap [Q, 2Q]$. Let s be the largest square divisor of q ; then $s \geq q^{2/3} \geq Q^{2/3} > Z^{2/3}$. The integer $s^{1/2}$ thus lies in $(Z^{1/4}, 2Q^{1/2}]$, and hence either has a prime factor $p > Z^{1/4}$, in which case $p \in [P, 2P]$ for some $P \in [Z^{1/4}, Q^{1/2}]$, or an integer factor $d \in (Z^{1/4}, Z^{1/2}]$, in which case $d^2 \in [D, 2D]$ for some $D \in [Z^{1/2}, Z]$. So by Conjecture 1.6 with $P \in [Z^{1/4}, Q^{1/2}]$, and by Case 1 with $D \in [Z^{1/2}, Z]$ in place of Q , we find on summing over P and D that the left-hand side of (9.8) is

$$\ll_{\Delta, \varepsilon} Z^m (Z^{1/4})^{-\eta_0} + Z^m (Z^{1/2})^{-\delta/2+\varepsilon}.$$

Since $Z \geq Q^{1/3}$, it follows that (9.8) holds with $\eta_5 = \frac{1}{12} \min(\eta_0, \frac{9}{10}(\deg \Delta)^{-1})$. $\square$

For all $W \in \mathbb{Z}[\mathbf{c}] = \mathbb{Z}[c_1, \dots, c_m]$ and $Z, N, A \in \mathbb{R}_{>0}$, let

$$\Sigma_8^{W,A}(Z, N) := \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m : W(\mathbf{c}) \neq 0} \left(\sum_{n \in \mathcal{N}_{\mathbf{c}} \cap [N, 2N]} n^{-1/2} |S_{\mathbf{c}}^{\natural}(n)| \right)^A. \quad (9.10)$$

Close analogs of the moment $\Sigma_8^{1,A}(Z, N)$ have been considered for $A = 1$ classically (e.g. in [HB83, Hoo86b, HB98]), and for $A = 2$ in [Wan22, §4.1].

Conjecture 9.6 (B2). *Let $Z, N \in \mathbb{R}_{\geq 1}$ with $N \leq Z^3$. Then $\Sigma_8^{1,2}(Z, N) \ll_{F, \varepsilon} Z^{m+\varepsilon}$.*

Proposition 9.7. *Suppose F is diagonal. Then Conjecture 9.6 holds.*

Proposition 9.7 is useful, but we also need to go one step further.

Conjecture 9.8 (B3G). *Let $1 \leq A \leq 2$. There exists a nonzero polynomial $W = W_{F,A} \in \mathbb{Z}[\mathbf{c}]$, and a real $\eta_6 = \eta_6(F, A) > 0$, such that if $1 \leq N \leq Z^3$, then $\Sigma_8^{W,A}(Z, N) \ll_{F,A} Z^m N^{-\eta_6}$.*

One can extend Conjecture 9.8 to $A = 2 + \delta$, almost for free (see Proposition 10.3), but it is not clear to us what the limit is. Also, for $1 \leq A \leq 2$, one might hope for $\eta_6(F, A) \approx A/2$ to be admissible, by comparing with Proposition 9.3 or [SX91, Conjecture 1].

Proposition 9.9. *Suppose F is diagonal. Assume Conjecture 1.6. Then Conjecture 9.8 holds with $W = \prod_{i=1}^m c_i$, for all $A \in [1, 2]$.*

We need a classical pointwise bound on $S_{\mathbf{c}}^{\natural}(n)$. For integers $c \neq 0$, let $\text{sq}(c) := \prod_{p^2|c} p^{v_p(c)}$ and $\text{cub}(c) := \prod_{p^3|c} p^{v_p(c)}$. Also let $\text{sq}(0) := 0$.

Proposition 9.10 ([Hoo86b, HB98]; see e.g. [Wan24b, (5.2)]). *Assume F is diagonal. Let $\varepsilon > 0$. There exists $A_8(F, \varepsilon) \in \mathbb{R}_{\geq 1}$ such that if $\mathbf{c} \in \mathbb{Z}^m$ and $n \in \mathbb{Z}_{\geq 1}$, then*

$$n^{-1/2} |S_{\mathbf{c}}^{\natural}(n)| \leq A_8(F, \varepsilon) n^{\varepsilon} \prod_{1 \leq j \leq m} \gcd(\text{cub}(n)^2, \gcd(\text{cub}(n), \text{sq}(c_j))^3)^{1/12}.$$

We now prove Propositions 9.7 and 9.9. For the rest of §9, let $W = W(\mathbf{c}) := \prod_{i=1}^m c_i$ and let $\mathcal{S}_4 := \{\mathbf{c} \in \mathcal{S}_1 : W(\mathbf{c}) \neq 0\}$.

Proof of Proposition 9.7. Suppose $1 \leq N \leq Z^3$. For each $\mathbf{c} \in \mathcal{S}_4$, Proposition 9.10 implies $n^{-1/2} S_{\mathbf{c}}^{\natural}(n) \ll_{F, \varepsilon} n^{\varepsilon} \prod_{1 \leq j \leq m} \text{sq}(c_j)^{1/4}$. Plugging this into (9.10), and using Lemma 2.3 to bound $|\mathcal{N}_{\mathbf{c}} \cap [N, 2N]|$, we get $\Sigma_8^{W,2}(Z, N) \ll_{F, \varepsilon} Z^{\varepsilon} \prod_{1 \leq j \leq m} \sum_{1 \leq |c_j| \leq Z} \text{sq}(c_j)^{1/2}$. However,

$$\sum_{1 \leq |c| \leq Z} \text{sq}(c)^{1/2} \ll \sum_{e \in \mathcal{N}_{\geq}(2)} e^{1/2} \cdot (Z/e) \ll Z \log(1+Z) \quad (9.11)$$

by (9.3). Thus $\Sigma_8^{W,2}(Z, N) \ll_{F, \varepsilon} Z^{m+\varepsilon}$. Similarly, $\Sigma_8^{1,2}(Z, N) - \Sigma_8^{W,2}(Z, N) \ll_{F, \varepsilon} Z^{m-1+\varepsilon} N^{1/3}$ by Proposition 9.10, Lemma 2.3, and the inequality

$$\text{cub}(n)^{(m-t)/3} \prod_{1 \leq i \leq t} \sum_{1 \leq |c_i| \leq Z} \text{sq}(c_i)^{1/2} \ll_{t, \varepsilon} N^{(m-t)/3} Z^{t+\varepsilon} \leq N^{1/3} Z^{m-1+\varepsilon}$$

for $n \in [N, 2N]$, $t \in [1, m-1]$. Since $N \leq Z^3$, we obtain $\Sigma_8^{1,2}(Z, N) \ll_{F, \varepsilon} Z^{m+\varepsilon}$. $\square$

Proof of Proposition 9.9. Suppose $1 \leq N \leq Z^3$. For each $\mathbf{c} \in \mathbb{Z}^m$ and $n \in \mathbb{Z}_{\geq 1}$, let

$$S_{\mathbf{c}}^{\sharp}(n) := \prod_{p|n} \max \left(1, \frac{|S_{\mathbf{c}}^{\sharp}(p^{v_p(n)})|}{p^{v_p(n)/2}} \right).$$

Let $P \geq 1$ be a real parameter to be specified later. Let

$$\Sigma_9 := \sum_{\substack{\mathbf{c} \in \mathcal{S}_1: \\ \|\mathbf{c}\| \leq Z}} \left(\sum_{\substack{n \in \mathcal{N}_{\mathbf{c}} \cap [N, 2N): \\ S_{\mathbf{c}}^{\sharp}(n) < P}} \frac{|S_{\mathbf{c}}^{\sharp}(n)|}{n^{1/2}} \right)^2, \quad \Sigma_{10} := \sum_{\substack{\mathbf{c} \in \mathcal{S}_4: \\ \|\mathbf{c}\| \leq Z}} \left(\sum_{\substack{n \in \mathcal{N}_{\mathbf{c}} \cap [N, 2N): \\ S_{\mathbf{c}}^{\sharp}(n) \geq P}} \frac{|S_{\mathbf{c}}^{\sharp}(n)|}{n^{1/2}} \right)^2.$$

Then $\Sigma_8^{W,2}(Z, N) \leq 2(\Sigma_9 + \Sigma_{10})$, since $(a+b)^2 \leq 2(a^2 + b^2)$ for $a, b \in \mathbb{R}$. We will bound Σ_9 conditionally, using Conjecture 1.6, and Σ_{10} unconditionally, using the diagonality of F . We will lose factors of Z^{ε} at first, and then remove Z^{ε} later.

We first handle Σ_9 . Given $\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m$, let

$$\begin{aligned} \mathcal{N}_{\mathbf{c},1} &:= \{n \in \mathcal{N}_{\geq}(2) : p \mid n \Rightarrow \text{lcm}(p^2, p^{v_p(n)-1}) \nmid \Delta(\mathbf{c})\}, \\ \mathcal{N}_{\mathbf{c},2} &:= \{n \in \mathcal{N}_{\geq}(2) : p \mid n \Rightarrow \text{lcm}(p^2, p^{v_p(n)-1}) \mid \Delta(\mathbf{c})\}. \end{aligned}$$

Any integer $n \geq 1$ can be written uniquely as qn_1n_2 , where q, n_1, n_2 are pairwise coprime integers satisfying $q \in \mathcal{N}_{\leq}(1)$ and $n_j \in \mathcal{N}_{\mathbf{c},j}$ for $j \in \{1, 2\}$. We then have

$$n^{-1/2} |S_{\mathbf{c}}^{\sharp}(n)| \leq \min_{d|n} (S_{\mathbf{c}}^{\sharp}(n/d) \cdot d^{-1/2} |S_{\mathbf{c}}^{\sharp}(d)|) \leq S_{\mathbf{c}}^{\sharp}(n) \min(1, n_1^{-1/2} |S_{\mathbf{c}}^{\sharp}(n_1)|, q^{-1/2} |S_{\mathbf{c}}^{\sharp}(q)|)$$

by the definition of $S^{\sharp}$. Here $S_{\mathbf{c}}^{\sharp}(n_1) \ll_{\varepsilon} n_1^{\varepsilon}$ by parts (2) and (3) of Lemma 9.1. Also, the integer $\prod_{p|n_2} \text{lcm}(p^2, p^{v_p(n_2)-1}) \in \mathcal{N}_{\geq 2} \cap [n_2^{2/3}, n_2]$ divides $\Delta(\mathbf{c})$. Since $|\mathcal{N}_{\mathbf{c}} \cap [1, 2N]| \ll_{\varepsilon} (ZN)^{\varepsilon}$ by Lemma 2.3, and $\max(n_2, n_1, q) \geq n^{1/3}$ by the identity $n = qn_1n_2$, we conclude that

$$\Sigma_9 \ll_{F,\varepsilon} P^2 Z^{\varepsilon} \sum_{\mathbf{c} \in \mathcal{S}_1 \cap [-Z, Z]^m} \left(\mathbf{1}_{\mathbf{c} \in \mathcal{S}_5} + \frac{N^{\varepsilon}}{(N^{1/3})^{1/2}} + \sum_{q \in \mathcal{N}_{\mathbf{c}} \cap \mathcal{N}_{\leq}(1) \cap [N^{1/3}, 2N]} \frac{|S_{\mathbf{c}}^{\sharp}(q)|}{q^{1/2}} \right)^2,$$

where $\mathcal{S}_5$ denotes the set of vectors $\mathbf{c} \in \mathcal{S}_1$ for which there exists $d \in \mathcal{N}_{\geq 2} \cap [(N^{1/3})^{2/3}, 2N]$ with $d \mid \Delta(\mathbf{c})$. Since $(r+s+t)^2 \leq 3(r^2 + s^2 + t^2)$ for $r, s, t \in \mathbb{R}$, the sum over $\mathbf{c}$ on the right above is $\ll_{F,\varepsilon} Z^m (N^{2/9})^{-\eta_5} + Z^m N^{2\varepsilon-1/3} + Z^m N^{\varepsilon} \min(N^{1/3}, Z)^{-1+\varepsilon}$, by Propositions 9.5 and 9.3. Since $N \leq Z^3$, it follows that

$$\Sigma_9 \ll_{F,\varepsilon} P^2 Z^{m+\varepsilon} (N^{-2\eta_5/9} + N^{-1/3}). \quad (9.12)$$

We next handle Σ_{10} , via an Ekedahl-type phenomenon. For each $\mathbf{c} \in \mathcal{S}_1$, let

$$\mathcal{N}_{\mathbf{c},3}(P) := \{n \in \mathcal{N}_{\mathbf{c}} : S_{\mathbf{c}}^{\sharp}(n) \neq 0, S_{\mathbf{c}}^{\sharp}(n) \geq P\}.$$

Let $\varepsilon \in (0, 1)$, to be specified. Suppose $P^{1/10} \geq A_8(F, \varepsilon) \cdot (2N)^{\varepsilon}$. Now consider an individual $\mathbf{c} \in \mathcal{S}_4 \cap [-Z, Z]^m$ and $n \in \mathcal{N}_{\mathbf{c},3}(P) \cap [N, 2N]$. Here $S_{\mathbf{c}}^{\sharp}(n) \neq 0$, so $\prod_{p|\text{cub}(n)} p^{v_p(n)-1} \mid \Delta(\mathbf{c})$ by Lemma 9.1(3). Also, $S_{\mathbf{c}}^{\sharp}(n) \geq P$, so

$$P^{9/10} \leq \frac{S_{\mathbf{c}}^{\sharp}(n)}{P^{1/10}} \leq \frac{S_{\mathbf{c}}^{\sharp}(n)}{A_8(F, \varepsilon) n^{\varepsilon}} \leq \prod_{1 \leq j \leq m} \text{gcd}(\text{cub}(n), \text{sq}(c_j))^{1/4},$$

by Proposition 9.10. Thus there exists j with $d := \gcd(\text{cub}(n), \text{sq}(c_j)) \geq P^{3.6/m}$. Clearly $d \in \mathcal{N}_{\geq}(2)$. Also, $d \mid c_j \neq 0$, so $d \leq Z$; and $d \mid \text{cub}(n) \mid \prod_{p \mid \text{cub}(n)} p^{2(v_p(n)-1)} \mid \Delta(\mathbf{c})^2$. By taking $n \in \mathcal{N}_{\mathbf{c},3}(P) \cap [N, 2N)$ with $n^{-1/2} |\mathcal{S}_{\mathbf{c}}^{\dagger}(n)|$ maximal (if such an n exists), it follows that

$$\Sigma_{10} \leq \sum_{1 \leq j \leq m} \sum_{\substack{d \in \mathcal{N}_{\geq}(2): \\ P^{3.6/m} \leq d \leq Z}} \sum_{\substack{\mathbf{c} \in \mathcal{S}_6(j,d): \\ \|\mathbf{c}\| \leq Z}} |\mathcal{N}_{\mathbf{c}} \cap [N, 2N)|^2 \cdot A_8(F, \varepsilon)^2 (2N)^{2\varepsilon} d^{1/2} \prod_{\substack{1 \leq i \leq m: \\ i \neq j}} \text{sq}(c_i)^{1/2},$$

where $\mathcal{S}_6(j, d) := \{\mathbf{c} \in \mathcal{S}_4 : d \mid \gcd(c_j, \Delta(\mathbf{c})^2)\}$. Using Lemma 2.3 to bound $|\mathcal{N}_{\mathbf{c}} \cap [N, 2N)|$, and recalling that $N \leq Z^3$, we get

$$\Sigma_{10} \ll_{F, \varepsilon} Z^{\varepsilon} \sum_{1 \leq j \leq m} \sum_{\substack{d \in \mathcal{N}_{\geq}(2): \\ P^{3.6/m} \leq d \leq Z}} \sum_{\substack{\mathbf{c} \in \mathcal{S}_6(j,d): \\ \|\mathbf{c}\| \leq Z}} d^{1/2} \prod_{\substack{1 \leq i \leq m: \\ i \neq j}} \text{sq}(c_i)^{1/2}. \quad (9.13)$$

For notational simplicity, suppose $j = m$. For $k \in \{0, 1\}$, let

$$\mathcal{S}_{6,k}(m) := \{(c_1, \dots, c_{m-1}) \in (\mathbb{Z} \setminus \{0\})^{m-1} : \mathbf{1}_{\Delta(c_1, \dots, c_{m-1}, 0) \neq 0} = k\}.$$

For each $(c_1, \dots, c_{m-1}) \in \mathcal{S}_{6,1}(m) \cap [-Z, Z]^{m-1}$, we have

$$\sum_{\substack{d \in \mathcal{N}_{\geq}(2): \\ P^{3.6/m} \leq d \leq Z}} \sum_{\substack{1 \leq |c_m| \leq Z: \\ (c_1, \dots, c_m) \in \mathcal{S}_6(m, d)}} d^{1/2} \leq \sum_{\substack{d \mid \Delta(c_1, \dots, c_{m-1}, 0)^2: \\ P^{3.6/m} \leq d \leq Z}} \sum_{\substack{1 \leq |c_m| \leq Z: \\ d \mid c_m}} d^{1/2} \ll_{F, \varepsilon} Z^{\varepsilon} \cdot Z / (P^{3.6/m})^{1/2},$$

by the divisor bound for $\Delta(c_1, \dots, c_{m-1}, 0)^2 \neq 0$. Therefore, the contribution to the right-hand side of (9.13) from $\mathbf{c}$ with $(c_1, \dots, c_{m-1}) \in \mathcal{S}_{6,1}(m)$ (or the analogous condition if $j \neq m$) is

$$\ll_{F, \varepsilon} \frac{Z^{1+2\varepsilon}}{P^{1.8/m}} \prod_{1 \leq i \leq m-1} \sum_{1 \leq |c_i| \leq Z} \text{sq}(c_i)^{1/2} \ll_{m, \varepsilon} \frac{Z^{m+3\varepsilon}}{P^{1.8/m}},$$

where we use (9.11) in the final step.

On the other hand, for any $c_1, \dots, c_{m-2} \in \mathbb{Z}$, there are at most $\deg \Delta$ integers c_{m-1} for which $(c_1, \dots, c_{m-1}) \in \mathcal{S}_{6,0}(m)$. This is because for diagonal F , the $c_i^{\deg \Delta}$ coefficient of Δ is nonzero for all i ; see e.g. [HB98, (4.2)]. Therefore, the contribution to the right-hand side of (9.13) from $\mathbf{c}$ with $(c_1, \dots, c_{m-1}) \in \mathcal{S}_{6,0}(m)$ (or the analogous condition if $j \neq m$) is

$$\ll_{F, \varepsilon} Z^{\varepsilon} \sum_{\substack{d \in \mathcal{N}_{\geq}(2): \\ P^{3.6/m} \leq d \leq Z}} \sum_{\substack{1 \leq |c_m| \leq Z: \\ d \mid c_m}} d^{1/2} \max_{1 \leq |c_{m-1}| \leq Z} \text{sq}(c_{m-1})^{1/2} \prod_{1 \leq i \leq m-2} \sum_{1 \leq |c_i| \leq Z} \text{sq}(c_i)^{1/2},$$

which by (9.11) is $\ll_{F, \varepsilon} Z^{2\varepsilon} \cdot Z \cdot Z^{1/2} \cdot Z^{m-2} = Z^{m-1/2+2\varepsilon}$.

The previous two paragraphs together imply that the right-hand side of (9.13) is, in total, $\ll_{F, \varepsilon} Z^{m+3\varepsilon} (P^{-1.8/m} + Z^{-1/2})$. This, combined with (9.12), gives

$$\Sigma_8^{W,2}(Z, N) \leq 2(\Sigma_9 + \Sigma_{10}) \ll_{F, \varepsilon} P^2 Z^{m+\varepsilon} (N^{-2\eta_5/9} + N^{-1/3}) + Z^{m+3\varepsilon} (P^{-1.8/m} + Z^{-1/2}).$$

Suppose $\varepsilon \leq 2 \min(1, \eta_5)/270$, and let $P = A_8(F, \varepsilon)^{10} 2^{10\varepsilon} N^{2 \min(1, \eta_5)/27} \geq 1$; then we get

$$\Sigma_8^{W,2}(Z, N) \ll_{F, \varepsilon} Z^{m+3\varepsilon} (N^{-3.6 \min(1, \eta_5)/27m} + Z^{-1/2}) \ll Z^{m+3\varepsilon} N^{-\min(1, \eta_5)/9m},$$

since $m \geq 4$ and $Z \geq N^{1/3}$. Upon redefining ε , now that we can forget about P , it follows that $\Sigma_8^{W,2}(Z, N) \ll_{F, \varepsilon} Z^{m+\varepsilon} N^{-\min(1, \eta_5)/9m}$ for all $\varepsilon > 0$.

It remains to replace Z^ε with N^ε . This is trivial if $Z \leq N^{O(1)}$. In general, expanding the square in (9.10), and noting that $S_{\mathbf{c}}(n)$ depends only on $\mathbf{c} \bmod n$, gives

$$\Sigma_8^{W,2}(Z, N) \leq \sum_{n_1, n_2 \in [N, 2N]} \left(1 + \frac{2Z}{n_1 n_2}\right)^m \sum_{\substack{1 \leq \mathbf{a} \leq n_1 n_2: \\ \text{rad}(n_1 n_2) | \Delta(\mathbf{a})}} (n_1 n_2)^{-1/2} |S_{\mathbf{a}}^\natural(n_1) S_{\mathbf{a}}^\natural(n_2)|$$

for all $Z, N \geq 1$. Yet for any $n_1, n_2 \geq 1$ and $\mathbf{a} \in [1, n_1 n_2]^m$, there exists $\mathbf{c} \in \mathcal{S}_4 \cap [1, B n_1 n_2]^m$ with $\mathbf{c} \equiv \mathbf{a} \bmod n_1 n_2$, where $B = 1 + \deg(\Delta(\mathbf{c})W(\mathbf{c}))$. So

$$\Sigma_8^{W,2}(4BN^2, N) \geq \sum_{n_1, n_2 \in [N, 2N]} \sum_{\substack{1 \leq \mathbf{a} \leq n_1 n_2: \\ \text{rad}(n_1 n_2) | \Delta(\mathbf{a})}} (n_1 n_2)^{-1/2} |S_{\mathbf{a}}^\natural(n_1) S_{\mathbf{a}}^\natural(n_2)|.$$

Hence for all $Z \geq N^2$ we have

$$\Sigma_8^{W,2}(Z, N) \leq (3Z/N^2)^m \cdot \Sigma_8^{W,2}(4BN^2, N) \ll_{F, \varepsilon} (Z/N^2)^m (4BN^2)^{m+\varepsilon} N^{-\min(1, \eta_5)/9m},$$

since $1 \leq N \leq (4BN^2)^3$. Thus for all $Z \geq N^{1/3}$, including $Z \in [N^{1/3}, N^2]$, we have

$$\Sigma_8^{W,2}(Z, N) \ll_{F, \varepsilon} Z^m N^{\varepsilon - \min(1, \eta_5)/9m}.$$

Therefore, Conjecture 9.8 holds with $W = \prod_{i=1}^m c_i$ and $\eta_6(F, 2r) = r \min(1, \eta_5)/10m$, first if $r = 1$, and then for $r \in [1/2, 1]$ by Hölder's inequality over $\mathbf{c}$. $\square$

To end §9, we prove a result clarifying the nature of Conjecture 1.6 for diagonal F :

Proposition 9.11. *Assume Conjecture 1.6 holds when $(m, F) = (4, x_1^3 + x_2^3 + x_3^3 + x_4^3)$. Then Conjecture 1.6 holds whenever F is diagonal and $m \geq 4$.*

Proof. (This result is not used in the rest of the paper, so we confine ourselves to a sketch.)

Assume $1 \leq P \leq Z^{3/2}$. Recall $\mathcal{N}_{\mathbf{c},0}$ from (9.5). By [Bha14, Theorem 3.3], we have

$$\#\{\mathbf{c} \in [-Z, Z]^m : \exists p \in [P, 2P) \cap \mathcal{N}_{\mathbf{c},0}\} \ll_{\Delta} Z^m P^{-\delta_1}. \quad (9.14)$$

By (9.14), we see that for any given F , Conjecture 1.6 is equivalent to the statement

$$\#\{\mathbf{c} \in [-Z, Z]^m : \exists p \in [P, 2P) \setminus \mathcal{N}_{\mathbf{c},0} \text{ with } p^2 \mid \Delta(\mathbf{c})\} \ll Z^m P^{-\delta_2}. \quad (9.15)$$

Let $\mathcal{U}_1, \mathcal{U}_2$ be the respective smooth loci of the hypersurfaces $F = 0, \Delta = 0$ in $\mathbb{P}_{\mathbb{Z}[1/6]}^{m-1}$. By (2.3) we have a Gauss map $[\nabla F]: \mathcal{U}_1 \rightarrow \{\Delta = 0\}$; let $\mathcal{U}_3 := \mathcal{U}_1 \times_{[\nabla F]} \mathcal{U}_2 \subseteq \mathcal{U}_1$. The map $[\nabla F]: \mathcal{U}_3 \rightarrow \mathcal{U}_2$ is an isomorphism over $\mathbb{Q}$ (with inverse $[\nabla \Delta]: \mathcal{U}_2 \rightarrow \{F = 0\}$), and thus too over $\mathbb{Z}$, since $\mathcal{U}_2, \mathcal{U}_3$ are flat over

$\mathbb{Z}$. In particular, $\mathcal{U}_2(\mathbb{Z}/p^2\mathbb{Z}) = [\nabla F](\mathcal{U}_3(\mathbb{Z}/p^2\mathbb{Z}))$ for $p > 3$. Furthermore, it is known that $\mathcal{U}_3$ lies in the open subscheme $\det(\text{Hess } F(\mathbf{x})) \neq 0$ of $\mathbb{P}_{\mathbb{Z}}^{m-1}$. These two facts, plus (9.14), imply that (9.15) is equivalent to

$$\#\{\mathbf{c} \in [-Z, Z]^m : \exists p \in [P, 2P], \mathbf{x} \in \mathbb{Z}^m, \lambda \in [1, p-1] \text{ with} \\ p \nmid \det(\text{Hess } F(\mathbf{x})), p^2 \mid F(\mathbf{x}), \nabla F(\mathbf{x}) - \lambda \mathbf{c}\} \ll Z^m P^{-\delta_3}. \quad (9.16)$$

We would like to convert each “exists” into a sum. Each p on the left-hand side of (9.16) satisfies $p \mid \Delta(\mathbf{c})$, so there are $\ll_{\varepsilon} Z^{\varepsilon}$ possibilities for p if $\mathbf{c} \in \mathcal{S}_1$. Also, $|\mathcal{S}_0 \cap [-Z, Z]^m| \cdot P \ll_m Z^{m-1/2}$ by (2.4), since $m \geq 4$. Therefore, the statement (9.16) is equivalent to

$$\sum_{\mathbf{c} \in [-Z, Z]^m} \sum_{p \in [P, 2P]} \sum_{1 \leq \lambda \leq p^2} \mathbf{1}_{p \nmid \det(\text{Hess } F(\mathbf{x}))} \mathbf{1}_{p^2 \mid F(\mathbf{x})} \mathbb{E}_{1 \leq \lambda \leq p-1} [\mathbf{1}_{p^2 \mid \nabla F(\mathbf{x}) - \lambda \mathbf{c}}] \ll Z^m P^{-\delta_4}. \quad (9.17)$$

Now suppose $F = \sum_{j=1}^m F_j x_j^3$, where $F_j \in \mathbb{Z} \setminus \{0\}$. For each $G, \lambda \in \mathbb{Z}$, let

$$T_{3/2, G, \lambda}(a, p^2) := \sum_{\substack{c \in [-Z, Z], 1 \leq x \leq p^2: \\ p \nmid x, p^2 \mid 3Gx^2 - \lambda c}} e_{p^2}(aGx^3) \in \mathbb{R}.$$

Then the left-hand side of (9.17) equals $\mathcal{Q}/(p-1)$, where

$$\mathcal{Q} := \sum_{\substack{p \in [P, 2P], \\ 1 \leq a \leq p^2, 1 \leq \lambda \leq p-1}} \prod_{1 \leq j \leq m} T_{3/2, F_j, \lambda}(a, p^2) \ll_F Z^{m-4} \sum_{1 \leq j \leq 4} \sum_{\substack{p \in [P, 2P], \\ 1 \leq a \leq p^2, 1 \leq \lambda \leq p-1}} T_{3/2, F_j, \lambda}(a, p^2)^4$$

by the trivial bounds $T_{3/2, G, \lambda} \ll \gcd(p, 3G)^2 Z \ll_G Z$ and $\prod_{j=1}^4 |T_{3/2, F_j, \lambda}| \leq \sum_{j=1}^4 T_{3/2, F_j, \lambda}^4$, where we note that $T_{3/2, G, \lambda} \in \mathbb{R}$.

Since Conjecture 1.6 holds by assumption when $F = x_1^3 + x_2^3 + x_3^3 + x_4^3$, it also holds when $F = F_j \cdot (x_1^3 + x_2^3 + x_3^3 + x_4^3)$ for any fixed j . So by the last display, the statement (9.17) holds for all diagonal F when $m \geq 4$. Hence (9.16), (9.15), and Conjecture 1.6 hold too. $\square$

10 Delta endgame

Throughout §10, assume $2 \mid m$. Recall $\Sigma(X, \mathcal{S}), \Sigma^{\natural}(X, \mathcal{S})$ from (2.15). Explicitly, we have

$$\Sigma^{\natural}(X, \mathcal{S}) = \sum_{n \geq 1} \sum_{\mathbf{c} \in \mathcal{S}} n^{(1-m)/2} S_{\mathbf{c}}^{\natural}(n) J_{\mathbf{c}, X}(n) = \sum_{\mathbf{c} \in \mathcal{S}} \sum_{n \geq 1} n^{(1-m)/2} S_{\mathbf{c}}^{\natural}(n) J_{\mathbf{c}, X}(n), \quad (10.1)$$

by Proposition 2.4 and Fubini’s theorem. We are finally prepared to analyze these sums for various sets $\mathcal{S} \subseteq \mathcal{S}_1$, and eventually for $\mathcal{S} = \mathcal{S}_1$.

10.1 Delta decomposition

Consider an individual $\mathbf{c} \in \mathcal{S}_1$. In view of (7.1), (7.2) and the factorization $\Phi = \Phi^G \Phi^B$, we may use Lemma 5.2 with $k = 2$,

$$a(\mathbf{n}) = a(n_0, n_1) = (n_0^{-it_0} S_{\mathbf{c}}^{\natural}(n_0) \mathbf{1}_{n_0 \in \mathcal{N}_{\mathbf{c}}}) \cdot (n_1^{-it_1} S_{\mathbf{c}}^{\natural}(n_1) \mathbf{1}_{n_1 \in \mathcal{N}_{\mathbf{c}}}),$$

and $f(\mathbf{r}) = f(r_0, r_1) = (r_0 r_1)^{(1-m)/2} J_{\mathbf{c},X}(r_0 r_1)$, to write

$$\sum_{n \geq 1} n^{(1-m)/2} S_{\mathbf{c}}^{\natural}(n) J_{\mathbf{c},X}(n) = (2\pi)^{-2} \int_{\mathbf{N} \geq 1/2} d^{\times} \mathbf{N} \int_{\mathbf{t} \in \mathbb{R}^2} d\mathbf{t} g_{\mathbf{c},X,\mathbf{N}}^{\vee}(i\mathbf{t}) \prod_{0 \leq j \leq 1} \Sigma_{11,\mathbf{N}}^{\mathbf{c},j}(\mathbf{t}), \quad (10.2)$$

where $\mathbf{N} = (N_0, N_1)$ runs over $[1/2, \infty)^2$, where $\mathbf{t} = (t_0, t_1) \in \mathbb{R}^2$, and where

$$g_{\mathbf{c},X,\mathbf{N}}(\mathbf{r}) := (r_0 r_1)^{(1-m)/2} J_{\mathbf{c},X}(r_0 r_1) \cdot v_2(r_0/N_0) v_2(r_1/N_1), \quad (10.3)$$

$$\Sigma_{11,\mathbf{N}}^{\mathbf{c},0}(\mathbf{t}) := \sum_{n_0 \in \mathbb{N}_{\mathbf{c}}} v_2(n_0/N_0) n_0^{-it_0} S_{\mathbf{c}}^{\natural}(n_0), \quad \Sigma_{11,\mathbf{N}}^{\mathbf{c},1}(\mathbf{t}) := \sum_{n_1 \in \mathbb{N}_{\mathbf{c}}} v_2(n_1/N_1) n_1^{-it_1} S_{\mathbf{c}}^{\natural}(n_1). \quad (10.4)$$

Since $J_{\mathbf{c},X}(r)$ is supported on $r \leq A_0 Y$ by Proposition 2.4, and v_2 is supported on $[1, 2]$, we have $g_{\mathbf{c},X,\mathbf{N}}(\mathbf{r}) = 0$ unless $r_0 r_1 \leq A_0 Y$ and $N_j \leq r_j \leq 2N_j$ for all j . Thus $g_{\mathbf{c},X,\mathbf{N}} = 0$ identically unless $N_0 N_1 \leq A_0 Y$. So $g_{\mathbf{c},X,\mathbf{N}}^{\vee}(i\mathbf{t}) = 0$ unless $\mathbf{N}$ lies in the set

$$\mathcal{R}_{11}(X) := \{\mathbf{N} \geq 1/2 : N_0 N_1 \leq A_0 Y\}; \quad (10.5)$$

cf. the region $\mathcal{R}_7$ from (7.14). So (10.2) holds even if we restrict $\mathbf{N}$ to $\mathcal{R}_{11}(X)$.

Given $\mathbf{N} = (N_0, N_1)$, let $N := N_0 N_1$. For integers $C \geq 1$ and reals $\lambda > 0$, let

$$\mathcal{S}_1(C, \lambda) := \{\mathbf{c} \in \mathcal{S}_1 : \|\mathbf{c}\| \in [C, 2C), |\Delta(\mathbf{c}/C)| \in (\lambda/2, \lambda]\}.$$

Assume (2.6). The definition (10.3) and Propositions 5.1 and 8.1 imply, for $\mathbf{c} \in \mathcal{S}_1(C, \lambda)$ and $b \in \mathbb{Z}_{\geq 0}$, that for all multi-indices $\alpha \geq 0$,

$$N^{1/2} \partial_{\mathbf{c}}^{\alpha} g_{\mathbf{c},X,\mathbf{N}}^{\vee}(i\mathbf{t}) \ll_{F,w,v_2,\alpha,b} \frac{(X/N)^{|\alpha|} (N + XC)^{1-m/2}}{(1 + \|\mathbf{t}\|)^b (1 + C/X^{1/2})^b (1 + XC\lambda/N)^b}. \quad (10.6)$$

For each $\mathbf{c} \in \mathcal{S}_1$, we have $\mathbf{c} \in \mathbb{Z}^m$ and $\Delta(\mathbf{c}) \neq 0$, so $\|\mathbf{c}\| \geq 1$ and $1 \leq |\Delta(\mathbf{c})| \ll \|\mathbf{c}\|^{\deg \Delta}$. Let $\mathcal{D} = \{1, 2, 4, 8, \dots\}$ and $\mathcal{D}_2 = \mathcal{D} \cup \{1/d : d \in \mathcal{D}\}$; then we deduce that

$$\mathcal{S}_1 \subseteq \bigcup_{C \in \mathcal{D}, \lambda \in \mathcal{D}_2 : 1/C^{\deg \Delta} \leq \lambda \leq A_9} \mathcal{S}_1(C, \lambda) \quad (10.7)$$

for some $A_9 = A_9(F) \geq 1$. Also, for all $C \geq 1$ and $\lambda \leq A_9$, the volume bound (4.4) implies

$$|\mathcal{S}_1(C, \lambda)| \ll_F C^m (\lambda + C^{-1})^{1/\deg \Delta}, \quad (10.8)$$

because $|\Delta(\mathbf{z})| \ll_F |\Delta(\mathbf{y})| + C^{-1}$ for all $\mathbf{y} \in [-2, 2]^m$ and $\mathbf{z} \in \mathbf{y} + [-C^{-1}, C^{-1}]^m$.

10.2 Sharp delta bounds

In [Wan22, §4.1], we used the Cauchy–Schwarz inequality on $1/L$, ΦL over $\mathbf{c}$ to prove (2.17), for diagonal F with $2 \mid m$, under a large-sieve hypothesis, based on the first-order approximation $\Phi \approx 1/L$. Now, in §10.2, we will use Hölder’s inequality in a similar spirit to prove Theorem 1.7. This relies crucially on several new features, including the more precise approximation of Φ captured in $\Sigma_{11,\mathbf{N}}^{\mathbf{c},1}(\mathbf{t})$.

For the next four results, let $C \in \mathcal{D}$, let $\mathbf{N} \in \mathcal{R}_{11}(X)$, let $\mathbf{t} \in \mathbb{R}^2$, let $\mathcal{S} \subseteq \mathcal{S}_1 \cap [-C, C]^m$, and let $\kappa > 0$. Roughly speaking, (10.11) will be useful when $C \leq X^{1/2-\delta}$; and otherwise, (10.9) will be useful for small N_0 relative to XC/N , and (10.13) useful for larger N_0 .

Lemma 10.1. Assume Conjecture 7.4. Then for some $A_{10} = A_{10}(F, \kappa) > 0$, we have

$$\|\Sigma_{11, \mathbf{N}}^{\mathbf{c}, 0}(\mathbf{t}) \cdot \Sigma_{11, \mathbf{N}}^{\mathbf{c}, 1}(\mathbf{t})\|_{\ell_{\mathbf{c}}^1(\mathcal{S})} \ll_{F, \kappa} (1 + |t_1|)^{A_{10}} \cdot |\mathcal{S}|^{1/2 - \kappa} (C^m + N_1^{m/3})^{1/2 + \kappa} N_0^{m/2 + \kappa} N^{1/2}. \quad (10.9)$$

Proof. Plugging the trivial bound $|\mathcal{S}_{\mathbf{c}}^{\mathbf{h}}(n_0)| \leq n_0^{(1+m)/2}$ into (10.4) gives

$$\Sigma_{11, \mathbf{N}}^{\mathbf{c}, 0}(\mathbf{t}) \ll_{v_2} N_0^{(1+m)/2} \cdot |\mathcal{N}_{\mathbf{c}} \cap [N_0, 2N_0]| \quad (10.10)$$

for all $\mathbf{c} \in \mathcal{S}$. So by Lemma 6.10 with $A = (2 - \varepsilon)/\varepsilon$, Conjecture 7.4 with $C + N_1^{1/3}$, N_1 in place of Z , N , and Hölder's inequality over $\mathbf{c} \in \mathcal{S}$, the left-hand side of (10.9) is

$$\ll_{F, \varepsilon} N_0^{(1+m)/2} \cdot (C^m N_0^\varepsilon)^{\varepsilon/(2-\varepsilon)} \cdot |\mathcal{S}|^{(1-2\varepsilon)/(2-\varepsilon)} \cdot [(1 + |t_1|)^{A_4(F, \varepsilon)} (C^m + N_1^{m/3}) N_1^{(2-\varepsilon)/2}]^{1/(2-\varepsilon)}$$

for all $\varepsilon \in (0, \frac{1}{2})$, since $1 = \frac{\varepsilon}{2-\varepsilon} + \frac{1-2\varepsilon}{2-\varepsilon} + \frac{1}{2-\varepsilon}$. Writing $N_0^{(1+m)/2} N_1^{1/2} = N_0^{m/2} N^{1/2}$ gives (10.9) for some $\kappa \ll \varepsilon$, since $\frac{1-2\varepsilon}{2-\varepsilon} \geq \frac{1}{2} - O(\varepsilon)$ and $|\mathcal{S}| \leq C^m \leq C^m + N_1^{m/3}$. $\square$

Lemma 10.2. Assume Conjectures 1.3 and 9.6. Then

$$\|\Sigma_{11, \mathbf{N}}^{\mathbf{c}, 0}(\mathbf{t}) \cdot \Sigma_{11, \mathbf{N}}^{\mathbf{c}, 1}(\mathbf{t})\|_{\ell_{\mathbf{c}}^1(\mathcal{S})} \ll_{F, \varepsilon} (1 + |t_1|)^\varepsilon \cdot |\mathcal{S}|^{1/2} (C^m + N_0^{m/3})^{1/2 + \varepsilon} N^{1/2 + \varepsilon}. \quad (10.11)$$

Proof. By Proposition 3.2(8) and partial summation, $|\Sigma_{11, \mathbf{N}}^{\mathbf{c}, 1}(\mathbf{t})| \ll_{v_2, \varepsilon} C^\varepsilon (1 + |t_1|)^\varepsilon N_1^{1/2 + \varepsilon}$. By the Cauchy-Schwarz inequality and (9.10), the left-hand side of (10.11) is

$$\ll_{F, \varepsilon} [N_0 \cdot \Sigma_8^{1, 2}(C, N_0)]^{1/2} \cdot |\mathcal{S}|^{1/2} \cdot C^\varepsilon (1 + |t_1|)^\varepsilon N_1^{1/2 + \varepsilon}.$$

Now use Conjecture 9.6 with $C + N_0^{1/3}$, N_0 in place of Z , N . $\square$

Proposition 10.3. Assume Conjecture 9.8 with some W . Then for any real $\delta \geq 0$, we have

$$\sum_{\mathbf{c} \in \mathcal{S}: W(\mathbf{c}) \neq 0} |N_0^{-1/2} \cdot \Sigma_{11, \mathbf{N}}^{\mathbf{c}, 0}(\mathbf{t})|^{2+\delta} \ll_{F, \delta} (C^m + N_0^{m/3}) N_0^{m\delta/2 - 9\eta_6(F, 2)/10}. \quad (10.12)$$

Proof. By Hölder's inequality over $\mathcal{S} \cap \{W \neq 0\}$, the left-hand side of (10.12) is

$$\leq \|(N_0^{-1/2} \cdot \Sigma_{11, \mathbf{N}}^{\mathbf{c}, 0}(\mathbf{t}))^{\delta + \varepsilon}\|_{\ell_{\mathbf{c}}^{2/\varepsilon}(\mathcal{S} \cap \{W \neq 0\})} \cdot \|(N_0^{-1/2} \cdot \Sigma_{11, \mathbf{N}}^{\mathbf{c}, 0}(\mathbf{t}))^{2-\varepsilon}\|_{\ell_{\mathbf{c}}^{2/(2-\varepsilon)}(\mathcal{S} \cap \{W \neq 0\})},$$

for any $\varepsilon \in (0, \delta]$, since $\frac{\varepsilon}{2} + \frac{2-\varepsilon}{2} = 1$. The first factor here is $\ll_{\delta, \varepsilon} (N_0^{m/2})^{\delta + \varepsilon} (C^m N_0^\varepsilon)^{\varepsilon/2}$ by (10.10) and Lemma 6.10; the second factor is $\ll_\varepsilon [(C + N_0^{1/3})^m N_0^{-\eta_6(F, 2)}]^{(2-\varepsilon)/2}$ by Conjecture 9.8. Taking ε sufficiently small in terms of m , $\eta_6(F, 2)$ gives (10.12). $\square$

Lemma 10.4. Assume Conjectures 1.3, 7.4, 9.6, and 9.8. Assume $N \ll C^6$. Then for some $A_{11} = A_{11}(F, \kappa) > 0$, we have

$$\|\Sigma_{11, \mathbf{N}}^{\mathbf{c}, 0}(\mathbf{t}) \cdot \Sigma_{11, \mathbf{N}}^{\mathbf{c}, 1}(\mathbf{t})\|_{\ell_{\mathbf{c}}^1(\mathcal{S})} \ll_{F, \kappa} (1 + |t_1|)^{A_{11}} \cdot \frac{(C^m)^{1/2 - \kappa} (C^m + N^{m/3})^{1/2 + \kappa} N^{1/2}}{\min(C^{9/20}, N_0^{2\eta_6(F, 2)/5})}. \quad (10.13)$$

Proof. *Case 1:* $\mathcal{S} \subseteq \{W = 0\}$. Then $|\mathcal{S}| \ll_W C^{m-1}$ by [Bha14, Lemma 3.1]. Inserting this into (10.11), we get (10.13) for some $\kappa \ll \varepsilon$, upon writing $C^{(m-1)/2} N^\varepsilon \ll_\varepsilon C^{(m-1)/2+6\varepsilon}$.

Case 2: $\mathcal{S} \subseteq \{W \neq 0\}$. Since $1 = \frac{1-\varepsilon}{2-\varepsilon} + \frac{1}{2-\varepsilon}$, we may use (10.12) with $\delta = \frac{\varepsilon}{1-\varepsilon}$, together with Conjecture 7.4, and Hölder's inequality over $\mathbf{c} \in \mathcal{S}$, to bound the left-hand side of (10.13) by $O_{F,\varepsilon}(N^{1/2}\mathcal{T})$, where

$$\mathcal{T} := [(C^m + N_0^{m/3})N_0^{m\delta/2-9\eta_6(F,2)/10}]^{(1-\varepsilon)/(2-\varepsilon)} \cdot [(1 + |t_1|)^{A_4(F,\varepsilon)}(C^m + N_1^{m/3})]^{1/(2-\varepsilon)}.$$

Since $C^m + N_j^{m/3} \ll C^m$ for some $j \in \{0, 1\}$, we get (10.13) for some $\kappa \ll \varepsilon$.

Case 3: The general case. Decompose $\mathcal{S}$ as $(\mathcal{S} \cap \{W = 0\}) \cup (\mathcal{S} \setminus \{W = 0\})$. □

Theorem 10.5. Assume (2.6) and $2 \mid m$. Assume Conjectures 1.3, 6.4, 9.6, and 9.8. Then

$$|\Sigma^1(X, \mathcal{S}_1)| \leq \sum_{\mathbf{c} \in \mathcal{S}_1} \left| \sum_{n \geq 1} n^{(1-m)/2} S_{\mathbf{c}}^1(n) J_{\mathbf{c},X}(n) \right| \ll_{F,w} X^{(6-m)/4}. \quad (10.14)$$

Moreover, for any $X \in \mathbb{R}_{\geq 1}$ and $P_0, P \in \mathbb{R}_{>0}$, we have, for some constant $\eta_7 = \eta_7(F) > 0$,

$$\int_{\mathbf{N} \in \mathcal{R}_{11,P_0,P}(X)} d^\times \mathbf{N} \int_{\mathbf{t} \in \mathbb{R}^2} dt \sum_{\mathbf{c} \in \mathcal{S}_1} |g_{\mathbf{c},X,\mathbf{N}}^\vee(it)| \prod_{0 \leq j \leq 1} |\Sigma_{11,\mathbf{N}}^{\mathbf{c},j}(\mathbf{t})| \ll_{F,w} \frac{X^{(6-m)/4}}{\min(P_0 P, X)^{\eta_7}}, \quad (10.15)$$

where $\mathcal{R}_{11,P_0,P}(X) := \{\mathbf{N} \in \mathcal{R}_{11}(X) : N_0 \geq P_0/2, N \leq A_0 Y/P\}$.

Proof. By (10.2) and (10.5), the $P_0 = P = 1$ case of the bound (10.15) implies the second inequality in (10.14). The first inequality in (10.14) follows from (10.1). It remains to prove (10.15). For this, we may assume $P_0 \geq 1$ and $P \geq 1$, since if $P_0 < 1$ or $P < 1$, then we may increase P_0 or P while keeping the set $\mathcal{R}_{11,P_0,P}(X)$ the same. Now, we decompose $\mathcal{S}_1$ using (10.7), and then plug in (10.6) with $\boldsymbol{\alpha} = \mathbf{0}$. This bounds the left-hand side of (10.15) by

$$\ll_{F,w,b} \int_{\mathcal{R}_{11,P_0,P}(X)} d^\times \mathbf{N} \int_{\mathbb{R}^2} dt \sum_{C \in \mathcal{D}, \lambda \in \mathcal{D}_2(C)} f(\mathbf{N}, \mathbf{t}, C, \lambda),$$

where $\mathcal{D}_2(C) := \{\lambda \in \mathcal{D}_2 : 1/C^{\deg \Delta} \leq \lambda \leq A_9\}$ and

$$f(\mathbf{N}, \mathbf{t}, C, \lambda) := \frac{\|\Sigma_{11,\mathbf{N}}^{\mathbf{c},0}(\mathbf{t}) \cdot \Sigma_{11,\mathbf{N}}^{\mathbf{c},1}(\mathbf{t})\|_{\ell^1(\mathcal{S}_1(C,\lambda))} \cdot N^{-1/2} (N + XC)^{1-m/2}}{(1 + \|\mathbf{t}\|)^b (1 + C/X^{1/2})^b (1 + XC\lambda/N)^b}. \quad (10.16)$$

Next, note that Conjecture 7.4 holds by Propositions 6.8 and 7.5. So (10.9), (10.11), (10.13) are all at our disposal. Let $\theta = \frac{1}{10m}$ and $\varepsilon = \frac{\theta}{100m}$, say. Suppose $\mathbf{N} \in \mathcal{R}_{11}(X)$ and $b \geq 3 + \max(A_{10}(F, \varepsilon), \varepsilon, A_{11}(F, \varepsilon))$. Let $C \in \mathcal{D}$ and $\lambda \in \mathcal{D}_2(C)$.

Plugging (10.8) into (10.9) for $\mathcal{S} = \mathcal{S}_1(C, \lambda)$, and integrating (10.16) over $\mathbf{t}$, gives

$$\int_{\mathbb{R}^2} dt f(\mathbf{N}, \mathbf{t}, C, \lambda) \ll \frac{[C^m(\lambda + C^{-1})^{1/\deg \Delta}]^{1/2-\varepsilon} (C^m + N_1^{m/3})^{1/2+\varepsilon} N_0^{m/2+\varepsilon}}{(N + XC)^{m/2-1} (1 + C/X^{1/2})^b (1 + XC\lambda/N)^b}.$$

Let $\alpha = (1/2 - \varepsilon)/\deg \Delta$. Summing over $\lambda \in \mathcal{D}_2(C)$ using Lemma 5.3 with $r = \lambda$, $\tau = \frac{XC}{N}$, $a = 0$, and $q \in \{\alpha, 0\}$, noting that $0 \leq \alpha < b$ and $(\lambda + C^{-1})^\alpha \ll \lambda^\alpha + C^{-\alpha}$, we get

$$\sum_{\lambda \in \mathcal{D}_2(C)} \int_{\mathbb{R}^2} dt f \ll \frac{C^{(1/2-\varepsilon)m} [\min(\frac{N}{XC}, 1)^\alpha + C^{-(1-\varepsilon)\alpha}] (C^m + N_1^{m/3})^{1/2+\varepsilon} N_0^{m/2+\varepsilon}}{(N + XC)^{m/2-1} (1 + C/X^{1/2})^b}.$$

Writing $\min(\frac{N}{XC}, 1) \leq \frac{N}{XC}$ and $(N + XC)^{m/2-1} \geq (XC)^{m/2-1}$, and then summing over $C \in \mathcal{D}$ using Lemma 5.3 with $r = C$ and $\tau = X^{-1/2}$, noting that $(\frac{1}{2} - \varepsilon)m > \alpha + (\frac{m}{2} - 1)$, we get

$$\sum_{C \in \mathcal{D}, \lambda \in \mathcal{D}_2(C)} \int_{\mathbb{R}^2} dt f \ll \frac{[(\frac{N}{Y})^\alpha + X^{-(1-\varepsilon)\alpha/2}] (X^{m/2} + X^{(1/2-\varepsilon)m/2} N_1^{(1/2+\varepsilon)m/3}) N_0^{m/2+\varepsilon}}{Y^{m/2-1}},$$

since $X^{3/2} = Y$. The right-hand side is $\ll [(\frac{N}{Y})^\alpha + X^{-(1-\varepsilon)\alpha/2}] N_0^{m/2+\varepsilon} X^{(6-m)/4}$, since $N_1 \ll X^{3/2}$ by (10.5). Letting $\mathcal{R}_{12} := \{\mathbf{N} \in \mathcal{R}_{11, P_0, P}(X) : N_0^{m+\varepsilon} \leq \min(\frac{Y}{N}, X^{(1-\varepsilon)/2})^\alpha\}$, and writing $N_0^{m/2+\varepsilon} \leq N_0^{-m/2} \cdot \min(\frac{Y}{N}, X^{(1-\varepsilon)/2})^\alpha$ for each $\mathbf{N} \in \mathcal{R}_{12}$, we get

$$\int_{\mathcal{R}_{12}} d^\times \mathbf{N} \sum_{C \in \mathcal{D}, \lambda \in \mathcal{D}_2(C)} \int_{\mathbb{R}^2} dt f \ll \int_{1/4}^{A_0 Y/P} d^\times N [(\frac{N}{Y})^\alpha + X^{-(1-\varepsilon)\alpha/4}] P_0^{-m/2} X^{(6-m)/4},$$

by integrating over $N_0 \geq P_0/2$ for each fixed valued of $N_0 N_1 = N$. Thus

$$\int_{\mathcal{R}_{12}} d^\times \mathbf{N} \sum_{C \in \mathcal{D}, \lambda \in \mathcal{D}_2(C)} \int_{\mathbb{R}^2} dt f \ll [P^{-\alpha/2} + X^{-(1-2\varepsilon)\alpha/4}] P_0^{-m/2} X^{(6-m)/4}. \quad (10.17)$$

On the other hand, discarding the factor $(1 + XC\lambda/N)^b$ in (10.16) using $1 + XC\lambda/N \geq 1$, plugging $|\mathcal{S}| \ll C^m$ into (10.11) for $\mathcal{S} = \bigcup_{\lambda \in \mathcal{D}_2(C)} \mathcal{S}_1(C, \lambda)$, and integrating over $\mathbf{t}$ gives

$$\sum_{\lambda \in \mathcal{D}_2(C)} \int_{\mathbb{R}^2} dt f(\mathbf{N}, \mathbf{t}, C, \lambda) \ll \frac{C^{m/2} (C^m + N_0^{m/3})^{1/2+\varepsilon} N^\varepsilon}{(N + XC)^{m/2-1} (1 + C/X^{1/2})^b}.$$

Writing $(N + XC)^{m/2-1} \geq (XC)^{m/2-1}$, and then summing over $1 \leq C \leq X^{1/2-\theta}$, we get

$$\sum_{C \in \mathcal{D}: 1 \leq C \leq X^{1/2-\theta}} \sum_{\lambda \in \mathcal{D}_2(C)} \int_{\mathbb{R}^2} dt f \ll \frac{(X^{1/2-\theta})^{m/2} ((X^{1/2-\theta})^m + N_0^{m/3})^{1/2+\varepsilon} N^\varepsilon}{(X^{3/2-\theta})^{m/2-1}}.$$

But $N_0^{m/3} \ll X^{m/2}$ by (10.5); integrating the previous display over $\mathcal{R}_{11}(X)$ thus gives

$$\int_{\mathcal{R}_{11}(X)} d^\times \mathbf{N} \sum_{C \in \mathcal{D}: 1 \leq C \leq X^{1/2-\theta}} \sum_{\lambda \in \mathcal{D}_2(C)} \int_{\mathbb{R}^2} dt f \ll \frac{X^{(6-m)/4} X^{(m/2+2)\varepsilon}}{X^\theta} \leq \frac{X^{(6-m)/4}}{X^{9\theta/10}}. \quad (10.18)$$

Now suppose $C > X^{1/2-\theta}$; then $C > X^{1/2-1/10} = X^{2/5} \gg N^{1/6}$. Discarding $(1 + XC\lambda/N)^b$ in (10.16), taking $\mathcal{S} = \bigcup_{\lambda \in \mathcal{D}_2(C)} \mathcal{S}_1(C, \lambda)$ in (10.13), and integrating over $\mathbf{t}$ gives

$$\sum_{\lambda \in \mathcal{D}_2(C)} \int_{\mathbb{R}^2} dt f(\mathbf{N}, \mathbf{t}, C, \lambda) \ll \frac{C^{(1/2-\varepsilon)m} (C^m + N^{m/3})^{1/2+\varepsilon} \cdot (N + XC)^{1-m/2}}{\min(C^{9/20}, N_0^{2\eta_6(F,2)/5}) (1 + C/X^{1/2})^b}.$$

Writing $(N + XC)^{1-m/2} \leq (XC)^{1-m/2}$, and summing over C using Lemma 5.3, gives

$$\sum_{C \in \mathcal{D}: C > X^{1/2-\theta}} \sum_{\lambda \in \mathcal{D}_2(C)} \int_{\mathbb{R}^2} dt f \ll \frac{X^{(1/2-\varepsilon)m/2} (X^{m/2} + N^{m/3})^{1/2+\varepsilon} \cdot Y^{1-m/2}}{\min(X^9/50, N_0^{2\eta_6(F,2)/5})}.$$

Here $N_1, N \ll X^{3/2}$, so the right-hand side is $\ll X^{(6-m)/4} \cdot (X^{-9/50} + N_0^{-2\eta_6(F,2)/5})$. Let $\mathcal{R}_{13} := \mathcal{R}_{11, P_0, P}(X) \setminus \mathcal{R}_{12}$. Each $N \in \mathcal{R}_{13}$ satisfies $N_0^{m+\varepsilon} > \min(\frac{Y}{N}, X^{(1-\varepsilon)/2})^{\alpha/2}$, and thus $N_0^{-\eta_6(F,2)} < \min(\frac{Y}{N}, X^{(1-\varepsilon)/2})^{-\beta} \leq (\frac{N}{Y})^\beta + X^{-(1-\varepsilon)\beta/2}$, where $\beta = \frac{\eta_6(F,2)\alpha}{2(m+\varepsilon)}$. Thus

$$\begin{aligned} & \int_{\mathcal{R}_{13}} d^\times N \sum_{C \in \mathcal{D}: C > X^{1/2-\theta}} \sum_{\lambda \in \mathcal{D}_2(C)} \int_{\mathbb{R}^2} dt f \\ & \ll \int_{\mathcal{R}_{11, P_0, P}(X)} d^\times N X^{(6-m)/4} \cdot (X^{-9/50} + N_0^{-\eta_6(F,2)/5} [(\frac{N}{Y})^\beta + X^{-(1-\varepsilon)\beta/10}]) \\ & \ll (X^{\varepsilon-9/50} + P_0^{-\eta_6(F,2)/5} [P^{-\beta/5} + X^{-(1-2\varepsilon)\beta/10}]) \cdot X^{(6-m)/4}, \end{aligned}$$

where to prove the last inequality we integrate first over $N_0 \geq P_0/2$ when N is fixed, and then integrate over $N \leq A_0 Y/P$. This, when combined with (10.17) and (10.18), establishes (10.15) with $\eta_7 = \min(\frac{\alpha}{2}, \frac{0.9\alpha}{4}, \frac{9\theta}{10}, \frac{17}{100}, \frac{\eta_6(F,2)}{5}, \frac{\beta}{5}, \frac{0.9\beta}{10})$, where $\theta = \frac{1}{10m}$, $\alpha \geq \frac{0.4}{\deg \Delta}$, and $\beta \geq \frac{\eta_6(F,2)}{10m \deg \Delta}$. $\square$

Remark 10.6. Each of (10.9), (10.11), (10.13) is based on Hölder's inequality in the form $1 = \frac{1}{p} + \frac{1}{q}$ with $p \approx 2$. The input required for Theorem 10.5 could maybe be slightly relaxed by applying Hölder's inequality with more varied exponents, in different ranges of C, N_0, N_1 .

Proof of Theorem 1.7. Let $\mathcal{D} = \{1, 2, 4, 8, \dots\}$. Let $T_{0,X}(\theta) := \sum_{|x| \leq X} e(\theta x^3)$ for $\theta \in \mathbb{R}$; then

$$N_F(X) = \int_{[0,1]} d\theta |T_{0,X}(\theta)|^6 = \|T_{0,X}(\theta)\|_{L^1_\theta([0,1])}^6. \quad (10.19)$$

Now let $T_{1,X}(\theta) := \sum_{|x| \in (X/2, X]} e(\theta x^3)$. Let $\|f(\lambda)\|_{\ell^q_\lambda(\mathcal{D})} := (\sum_{\lambda \in \mathcal{D}} |f(\lambda)|^q)^{1/q}$ for $q \geq 1$; then

$$T_{0,X}(\theta) = 1 + \sum_{\lambda \in \mathcal{D}} T_{1,X/\lambda}(\theta) \ll 1 + \|(X/\lambda)^\rho\|_{\ell^{6/5}_\lambda(\mathcal{D})} \cdot \|(X/\lambda)^{-\rho} T_{1,X/\lambda}(\theta)\|_{\ell^6_\lambda(\mathcal{D})} \quad (10.20)$$

by Hölder's inequality, where $\rho := \frac{1}{10}$. Here $\|(X/\lambda)^\rho\|_{\ell^{6/5}_\lambda(\mathcal{D})} \ll X^\rho$. Upon inserting (10.20) into (10.19), we deduce that if $\|T_{1,X/\lambda}(\theta)\|_{L^1_\theta([0,1])}^6 \ll (X/\lambda)^3$ holds, then

$$N_F(X) \ll 1 + X^{6\rho} \cdot \|(X/\lambda)^{-6\rho} T_{1,X/\lambda}(\theta)\|_{L^1_{(\lambda, \theta)}(\mathcal{D} \times [0,1])}^6 \ll X^3, \quad (10.21)$$

since $6\rho < 3$.

Now let $w_\star(\mathbf{x}) := \prod_{1 \leq j \leq 6} \mathbf{1}_{|x_j| \in (1/2, 1]}$, and choose $w \in C_c^\infty((\mathbb{R} \setminus \{0\})^6)$ with $w \geq w_\star$. Then

$$\|T_{1,X}(\theta)\|_{L^1_\theta([0,1])}^6 = N_{F, w_\star}(X) \leq N_{F, w}(X). \quad (10.22)$$

But w satisfies (2.6). Also, for Theorem 1.7, we are assuming Conjectures 1.3, 1.5, and 1.6; in particular, Conjectures 9.6 and 9.8 hold by Propositions 9.7 and 9.9, respectively, and Conjecture 6.4 holds by Proposition 6.6. So Theorem 10.5 gives $\Sigma^{\natural}(X, \mathcal{S}_1) \ll 1$. But $\Sigma^{\natural}(X, \mathcal{S}_0) \ll 1$ by Theorem 2.5. So by (2.10) and the definitions (2.15), (1.4), we have

$$N_{F,w}(X)/X^3 = O_A(Y^{-A}) + \Sigma^{\natural}(X, \mathcal{S}_0) + \Sigma^{\natural}(X, \mathcal{S}_1) \ll 1.$$

Hence $N_F(X) \ll X^3$ by (10.22), (10.21). Given (1.8), a standard Cauchy–Schwarz argument then leads to the desired application to $F_0(S^3)$. $\square$

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References

- [BC21] S. Bettin and J. B. Conrey, *Averages of long Dirichlet polynomials*, Riv. Math. Univ. Parma (N.S.) **12** (2021), no. 1, 1–27. MR4284770 [↑202](#)
- [BDPW23] J. Bergström, A. Diaconu, D. Petersen, and C. Westerland, *Hyperelliptic curves, the scanning map, and moments of families of quadratic L-functions*, arXiv:2302.07664 (2023). [↑184](#)
- [Beu99] F. Beukers, *Integral points on cubic surfaces*, Number theory (Ottawa, ON, 1996), 1999, pp. 25–33. MR1684588 [↑183](#)
- [BFK23] H. M. Bui, A. Florea, and J. P. Keating, *The Ratios Conjecture and upper bounds for negative moments of L-functions over function fields*, Trans. Amer. Math. Soc. **376** (2023), no. 6, 4453–4510. MR4586817 [↑204](#)
- [BG58] P. T. Bateman and E. Grosswald, *On a theorem of Erdős and Szekeres*, Illinois J. Math. **2** (1958), 88–98. MR95804 [↑220](#)
- [BGW24] T. Browning, J. Glas, and V. Y. Wang, *Sums of three cubes over a function field*, arXiv:2402.07146 (2024). [↑184](#)
- [BGW25] ———, *Optimal sums of three cubes in $\mathbb{F}_q[t]$* , Math. Z. **310** (2025), no. 4, Paper No. 65, 25. MR4910970 [↑184](#)
- [Bha14] M. Bhargava, *The geometric sieve and the density of squarefree values of invariant polynomials*, arXiv:1402.0031 (2014). [↑186](#), [188](#), [221](#), [225](#), [229](#)
- [Bha25] ———, *Galois groups of random integer polynomials and van der Waerden’s conjecture*, Ann. of Math. (2) **201** (2025), no. 2, 339–377. MR4878222 [↑221](#)
- [BJ14] L. Busé and J.-P. Jouanolou, *On the discriminant scheme of homogeneous polynomials*, Math. Comput. Sci. **8** (2014), no. 2, 175–234. MR3224627 [↑188](#), [189](#)

- [BK16] A. R. Booker and M. Krishnamurthy, *A converse theorem for $GL(n)$* , Adv. Math. **296** (2016), 154–180. MR3490766 [↑185](#)
- [BS21] A. R. Booker and A. V. Sutherland, *On a question of Mordell*, Proc. Natl. Acad. Sci. USA **118** (2021), no. 11, Paper No. 2022377118, 11 pp. MR4279690 [↑183](#)
- [BV15] T. D. Browning and P. Vishe, *Rational points on cubic hypersurfaces over $\mathbb{F}_q(t)$* , Geom. Funct. Anal. **25** (2015), no. 3, 671–732. MR3361770 [↑184](#)
- [Čec24] M. Čech, *The ratios conjecture for real Dirichlet characters and multiple Dirichlet series*, Trans. Amer. Math. Soc. **377** (2024), no. 5, 3487–3528. MR4744786 [↑186](#)
- [CCDN20] W. Castryck, R. Cluckers, P. Dittmann, and K. H. Nguyen, *The dimension growth conjecture, polynomial in the degree and without logarithmic factors*, Algebra Number Theory **14** (2020), no. 8, 2261–2294. MR4172708 [↑190](#)
- [CFK⁺05] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, *Integral moments of L -functions*, Proc. London Math. Soc. (3) **91** (2005), no. 1, 33–104. MR2149530 [↑186](#)
- [CFZ08] J. B. Conrey, D. W. Farmer, and M. R. Zirnbauer, *Autocorrelation of ratios of L -functions*, Commun. Number Theory Phys. **2** (2008), no. 3, 593–636. MR2482944 [↑186](#), [201](#)
- [CS07] J. B. Conrey and N. C. Snaith, *Applications of the L -functions ratios conjectures*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 3, 594–646. MR2325314 [↑186](#), [202](#)
- [Dav39] H. Davenport, *On Waring’s problem for cubes*, Acta Math. **71** (1939), 123–143. MR0000026 [↑183](#)
- [DFI93] W. Duke, J. B. Friedlander, and H. Iwaniec, *Bounds for automorphic L -functions*, Invent. Math. **112** (1993), no. 1, 1–8. MR1207474 [↑187](#), [211](#)
- [DGH03] A. Diaconu, D. Goldfeld, and J. Hoffstein, *Multiple Dirichlet series and moments of zeta and L -functions*, Compositio Math. **139** (2003), no. 3, 297–360. MR2041614 [↑186](#)
- [DHL06] J.-M. Deshouillers, F. Hennecart, and B. Landreau, *On the density of sums of three cubes*, Algorithmic number theory, 2006, pp. 141–155. MR2282921 [↑184](#)
- [Eis95] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry. MR1322960 [↑196](#)
- [Eke91] T. Ekedahl, *An infinite version of the Chinese remainder theorem*, Comment. Math. Univ. St. Paul. **40** (1991), no. 1, 53–59. MR1104780 [↑188](#), [221](#)
- [Flo24] A. Florea, *Negative moments of L -functions with small shifts over function fields*, Int. Math. Res. Not. IMRN **3** (2024), 2298–2337. MR4702277 [↑204](#)
- [FMT89] J. Franke, Y. I. Manin, and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. **95** (1989), no. 2, 421–435. MR974910 [↑185](#)
- [FPRS19] D. W. Farmer, A. Pitale, N. C. Ryan, and R. Schmidt, *Analytic L -functions: definitions, theorems, and connections*, Bull. Amer. Math. Soc. (N.S.) **56** (2019), no. 2, 261–280. MR3923345 [↑185](#), [194](#), [195](#), [201](#)
- [Gan01] M. I. Ganzburg, *Polynomial inequalities on measurable sets and their applications*, Constr. Approx. **17** (2001), no. 2, 275–306. MR1814358 [↑196](#)
- [GH25] J. Glas and L. Hochfilzer, *On a question of Davenport and diagonal cubic forms over $\mathbb{F}_q(t)$* , Math. Ann. **391** (2025), no. 4, 5485–5533. MR4884553 [↑184](#)
- [Gra98] A. Granville, *ABC allows us to count squarefrees*, Internat. Math. Res. Notices **19** (1998), 991–1009. MR1654759 [↑186](#)
- [GS22] A. Ghosh and P. Sarnak, *Integral points on Markoff type cubic surfaces*, Invent. Math. **229** (2022), no. 2, 689–749. MR4448994 [↑184](#)
- [Har13] A. J. Harper, *Sharp conditional bounds for moments of the Riemann zeta function*, arXiv:1305.4618 (2013). [↑204](#)
- [Har23] ———, *The typical size of character and zeta sums is $o(\sqrt{x})$* , arXiv:2301.04390 (2023). [↑204](#)

- [HB83] D. R. Heath-Brown, *Cubic forms in ten variables*, Proc. London Math. Soc. (3) **47** (1983), no. 2, 225–257. MR703978 ↑220, 222
- [HB92] ———, *The density of zeros of forms for which weak approximation fails*, Math. Comp. **59** (1992), no. 200, 613–623. MR1146835 ↑183
- [HB96] ———, *A new form of the circle method, and its application to quadratic forms*, J. Reine Angew. Math. **481** (1996), 149–206. MR1421949 ↑187, 211, 212, 213, 214, 215, 216
- [HB98] ———, *The circle method and diagonal cubic forms*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. **356** (1998), no. 1738, 673–699. MR1620820 ↑184, 185, 187, 190, 191, 192, 211, 218, 222, 224
- [Hoo14] C. Hooley, *On octonary cubic forms*, Proc. Lond. Math. Soc. (3) **109** (2014), no. 1, 241–281. MR3237742 ↑191, 211
- [Hoo86a] ———, *On some topics connected with Waring’s problem*, J. Reine Angew. Math. **369** (1986), 110–153. MR850631 ↑184, 185
- [Hoo86b] ———, *On Waring’s problem*, Acta Math. **157** (1986), no. 1-2, 49–97. MR857679 ↑184, 185, 191, 192, 218, 222
- [Hoo94] ———, *On nonary cubic forms. III*, J. Reine Angew. Math. **456** (1994), 53–63. MR1301451 ↑219
- [Hoo97] ———, *On Hypothesis K^* in Waring’s problem*, Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995), 1997, pp. 175–185. MR1635754 ↑184, 185, 187, 191, 192
- [HT01] B. Hassett and Y. Tschinkel, *Density of integral points on algebraic varieties*, Rational points on algebraic varieties, 2001, pp. 169–197. MR1875174 ↑183
- [Hua20] J.-J. Huang, *The density of rational points near hypersurfaces*, Duke Math. J. **169** (2020), no. 11, 2045–2077. MR4132580 ↑188, 215
- [Hua38] L.-K. Hua, *On Waring’s problem*, Quart. J. Math. Oxford Ser. **9** (1938), no. 1, 199–202. ↑185
- [IK04] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004. MR2061214 ↑195, 197, 201
- [Kah20] B. Kahn, *Zeta and L-functions of varieties and motives*, London Mathematical Society Lecture Note Series, vol. 462, Cambridge University Press, Cambridge, 2020. Translated from the 2018 French original [3839285]. MR4382436 ↑192
- [Kis99] M. Kisin, *Local constancy in p -adic families of Galois representations*, Math. Z. **230** (1999), no. 3, 569–593. MR1680032 ↑187, 196
- [Kon79] V. S. Konjagin, *The number of solutions of congruences of the n th degree with one unknown*, Mat. Sb. (N.S.) **109(151)** (1979), no. 2, 171–187, 327. MR542556 ↑196
- [KV24] J. M. Kowalski and R. C. Vaughan, *Squarefree density of polynomials*, Acta Arith. **214** (2024), 215–233. MR4772284 ↑186
- [KV25] ———, *Squarefree density of polynomials II*, Math. Ann. **391** (2025), no. 4, 6037–6056. MR4884566 ↑186
- [Las17] A. Laskar, *On local monodromy filtrations attached to motives*, Int. J. Number Theory **13** (2017), no. 3, 801–817. MR3606955 ↑193
- [Leh56] D. H. Lehmer, *On the Diophantine equation $x^3 + y^3 + z^3 = 1$* , J. London Math. Soc. **31** (1956), 275–280. MR78397 ↑183
- [LMU24] J. Lyczak, V. Mitankin, and H. Uppal, *Cubic surfaces failing the integral Hasse principle*, arXiv:2311.10008 (2024). ↑184
- [Mil04] S. J. Miller, *One- and two-level densities for rational families of elliptic curves: evidence for the underlying group symmetries*, Compos. Math. **140** (2004), no. 4, 952–992. MR2059225 ↑186
- [Mor53] L. J. Mordell, *On the integer solutions of the equation $x^2 + y^2 + z^2 + 2xyz = n$* , J. London Math. Soc. **28** (1953), 500–510. MR56619 ↑183

- [MPPRW24] J. Miller, P. Patzt, D. Petersen, and O. Randal-Williams, *Uniform twisted homological stability*, [arXiv:2402.00354](#) (2024). [↑184](#)
- [Ng04] N. Ng, *The distribution of the summatory function of the Möbius function*, Proc. London Math. Soc. (3) **89** (2004), no. 2, 361–389. MR2078705 [↑188](#)
- [Pey95] E. Peyre, *Hauteurs et mesures de Tamagawa sur les variétés de Fano*, Duke Math. J. **79** (1995), no. 1, 101–218. MR1340296 [↑185](#)
- [Poo03] B. Poonen, *Squarefree values of multivariable polynomials*, Duke Math. J. **118** (2003), no. 2, 353–373. MR1980998 [↑186](#)
- [PSW16] L. B. Pierce, D. Schindler, and M. M. Wood, *Representations of integers by systems of three quadratic forms*, Proc. Lond. Math. Soc. (3) **113** (2016), no. 3, 289–344. MR3551849 [↑196](#)
- [RS94] M. Rubinstein and P. Sarnak, *Chebyshev’s bias*, Experiment. Math. **3** (1994), no. 3, 173–197. MR1329368 [↑187](#), [188](#)
- [Sai03] T. Saito, *Weight spectral sequences and independence of l* , J. Inst. Math. Jussieu **2** (2003), no. 4, 583–634. MR2006800 [↑193](#)
- [Sch12] P. Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313. MR3090258 [↑193](#)
- [Seg43] B. Segre, *A note on arithmetical properties of cubic surfaces*, J. London Math. Soc. **18** (1943), 24–31. MR0009471 [↑183](#)
- [Ser70] J.-P. Serre, *Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures)*, Séminaire Delange-Pisot-Poitou. 11e année: 1969/70. Théorie des nombres. Fasc. 1: Exposés 1 à 15; Fasc. 2: Exposés 16 à 24, 1970, pp. 15. MR3618526 [↑192](#), [193](#)
- [Sou09] K. Soundararajan, *Moments of the Riemann zeta function*, Ann. of Math. (2) **170** (2009), no. 2, 981–993. MR2552116 [↑204](#)
- [SST16] P. Sarnak, S. W. Shin, and N. Templier, *Families of L -functions and their symmetry*, Families of automorphic forms and the trace formula, 2016, pp. 531–578. MR3675175 [↑186](#), [188](#), [200](#), [201](#), [202](#)
- [ST14] S. W. Shin and N. Templier, *On fields of rationality for automorphic representations*, Compos. Math. **150** (2014), no. 12, 2003–2053. MR3292292 [↑195](#)
- [SX91] P. Sarnak and X. X. Xue, *Bounds for multiplicities of automorphic representations*, Duke Math. J. **64** (1991), no. 1, 207–227. MR1131400 [↑186](#), [222](#)
- [Sza24] B. Szabó, *High moments of theta functions and character sums*, Mathematika **70** (2024), no. 2, Paper No. e12242, 37 pp. MR4707854 [↑204](#)
- [Ter18] Y. Terakado, *The determinant and the discriminant of a complete intersection of even dimension*, Math. Res. Lett. **25** (2018), no. 1, 255–280. MR3818622 [↑189](#)
- [Vau20] R. C. Vaughan, *On some questions of partitio numerorum: Tres cubi*, Glasgow Mathematical Journal (2020), 1–22. [↑185](#)
- [Vuk23] A. Vuković, *Divisibility of discriminants of homogeneous polynomials*, Ph.D. Thesis, University of Waterloo, 2023. URL: <http://hdl.handle.net/10012/19767>. [↑186](#)
- [VW95] R. C. Vaughan and T. D. Wooley, *On a certain nonary cubic form and related equations*, Duke Math. J. **80** (1995), no. 3, 669–735. MR1370112 [↑185](#)
- [Wan22] V. Y. Wang, *Families and dichotomies in the circle method*, Ph.D. Thesis, Princeton University, 2022. URL: <http://arks.princeton.edu/ark:/88435/dsp01rf55zb86g>. [↑192](#), [220](#), [222](#), [227](#)
- [Wan23a] ———, *Dichotomous point counts over finite fields*, J. Number Theory **250** (2023), 1–34. MR4580450 [↑184](#), [189](#), [200](#), [219](#), [220](#)
- [Wan23b] ———, *Sums of cubes and the Ratios Conjectures*, [arXiv:2108.03398](#) (2023). [↑184](#)
- [Wan24a] ———, *Notes on zeta ratio stabilization*, [arXiv:2402.01214](#) (2024). [↑184](#)

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- [Wan24b] ———, *Special cubic zeros and the dual variety*, J. Lond. Math. Soc. (2) **110** (2024), no. 3, Paper No. e12975, 35 pp. ↑[184](#), [185](#), [187](#), [190](#), [191](#), [219](#), [220](#), [222](#)
- [Wan25] ———, *Diagonal cubic forms and the large sieve*, Mathematika **71** (2025), no. 1, Paper No. e70008, 29. MR4846860 ↑[190](#)
- [Woo00] T. D. Wooley, *Sums of three cubes*, Mathematika **47** (2000), no. 1-2, 53–61 (2002). MR1924487 ↑[184](#)
- [Woo15] ———, *Sums of three cubes, II*, Acta Arith. **170** (2015), no. 1, 73–100. MR3373831 ↑[184](#)
- [Woo95] ———, *Breaking classical convexity in Waring's problem: sums of cubes and quasi-diagonal behaviour*, Invent. Math. **122** (1995), no. 3, 421–451. MR1359599 ↑[184](#)

AUTHOR

Victor Y. Wang

Institute of Mathematics, Academia Sinica
6F, Astronomy-Mathematics Building
No. 1, Sec. 4, Roosevelt Road
Taipei 106319
Taiwan

vywang@alum.mit.edu

<https://web.math.princeton.edu/~vywang/>