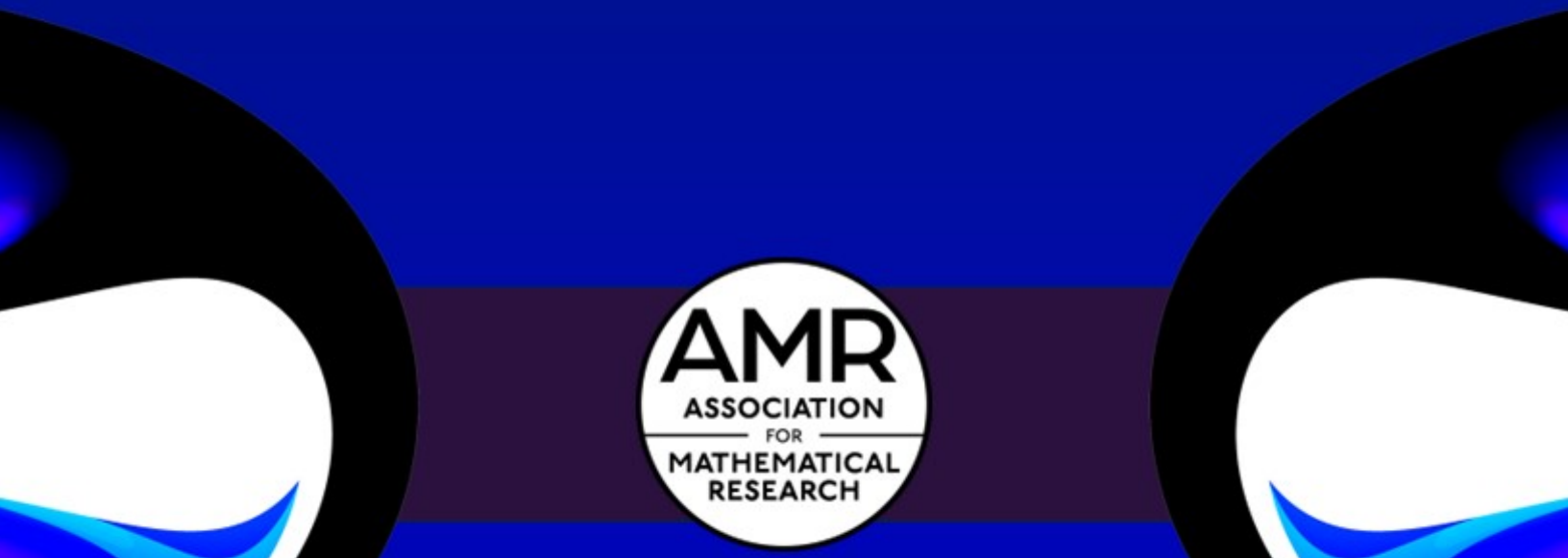
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Family Bauer–Furuta invariant, Exotic Surfaces and Smale conjecture

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Abstract:

We establish the existence of a pair of exotic surfaces in a punctured $K3$ which remains exotic after one external stabilization and have diffeomorphic complements. A key ingredient in the proof is a vanishing theorem of the family Bauer–Furuta invariant for diffeomorphisms on a large family of spin 4-manifolds, which is proved using the tom Dieck splitting theorem in equivariant stable homotopy theory. In particular, we prove that the S^1-equivariant family Bauer–Furuta invariant of any orientation-preserving diffeomorphism on S^4 is trivial and that the $\mathrm{Pin}(2)$-equivariant family Bauer–Furuta invariant for a diffeomorphism on $S^2 \times S^2$ is trivial if the diffeomorphism acts trivially on the homology. Therefore, these invariants do not detect exotic self-diffeomorphisms on S^4 or $S^2 \times S^2$. Furthermore, our theorem also applies to certain exotic loops of diffeomorphisms on S^4 (as recently discovered by Watanabe) and show that these loops have trivial family Bauer–Furuta invariants. En route, we observe a curious element in the $\mathrm{Pin}(2)$-equivariant stable homotopy group of spheres which could potentially be used to detect an exotic diffeomorphism on S^4.

Key words and phrases:
Exotic Surfaces; Stabilization; Bauer-Furuta Invariant

1 Introduction

The Bauer–Furuta invariant of smooth 4-manifolds is a stable homotopy refinement of the classical Seiberg–Witten invariant [BF04]. It contains subtle but important information about the underlying 4-manifolds which is invisible from other invariants. In particular, the Bauer–Furuta invariant can

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detect new exotic smooth structures (for example, on $K3\#K3$) and its family version [Szy10, Xu04] is a powerful tool in detecting exotic diffeomorphisms [KM20, Lin20]. For this reason, this invariant has been actively studied in recent years [BK19a, BK19b, Bry97, Iid19, KMT21, KM20, Sch03, Szy08, Szy12]. While the classical Seiberg–Witten invariant is an integer, the Bauer–Furuta invariant is an element in the (equivariant) stable homotopy group of spheres. This allows one to apply powerful techniques in equivariant topology (e.g. equivariant K-theory and equivariant homology) to extract information on the Bauer–Furuta invariiant and to study 4-manifolds. This idea has resulted in many remarkable theorems including Furuta's $\frac{10}{8}$-theorem [Fur01] (see also [FKMM07, HLSX18]) and Manolescu's disproof of the triangulation conjecture [Man16] (see also [Lin18]). In this paper, we follow this line of thought and use the equivariant stable homotopy theory to prove a general vanishing result for the family Bauer–Furuta invariant of diffeomorphisms on a large family of spin 4-manifolds (Theorem 9). This vanishing theorem has several applications, which we discuss first.

1.1 Exotic surfaces

The first application concerns exotic surfaces in 4-manifolds. Recall that a pair of compact surfaces Σ, Σ' in a 4-manifold X with $\partial\Sigma = \partial\Sigma' \subset \partial X$ are called *exotic surfaces* if there exists a topological isotopy of X relative to the boundary that takes Σ to Σ' but there is no such smooth isotopy. Understanding exotic surfaces in 4-manifolds is of great interest since they are closely related to exotic smooth structures on 4-manifolds. In recent years, various methods have been used to detect examples of exotic surfaces (e.g., Donaldson's invariants [FKV87], the Seiberg–Witten invariants [FS97], Heegaard-Floer homology [JMZ20], symplectic & contact geometry [Hay21] and Khovanov homology [HS21]). In this paper, we add the family Bauer–Furuta invariant to this list. As we will see, this method reveals a new phenomenon : some exotic surfaces can survive a stabilization.

To further explain this, we mention an important principle in 4-dimensional topology, discovered by Wall in the 1960s [Wal64]. This principle says that exotic phenomena are all eliminated by sufficiently many stabilizations, i.e., taking connected sum with $S^2 \times S^2$'s. Since then, it has been a fundamental problem to search for exotic phenomena that survive one stabilization. In the case of exotic surfaces, work of Perron [Per86] and Quinn [Qui86] showed that exotic surfaces always become smoothly isotopic after taking sufficiently many stabilizations (away from the surfaces). *It is a long standing open-question, whether a single stabilization is enough for this purpose?* (See [Akb15, AKMR15] for studies of this problem.) Using the 4-dimensional light bulb theorem (proved by Gabai [Gab20] and Schneiderman–Teichner [ST19]), Auckly–Kim–Melvin–Ruberman–Schwartz [AKM$^+$19] proves a surprising result in the positive direction: They showed that for exotic pairs of connected surfaces in non-characteristic homology classes with simply-connected complements, a single stabilization is enough. In this paper, we prove a result in the negative direction by producing the first pair of orientable exotic surfaces that remain non-isotopic after a single stabilization.

Theorem 1. *There exists a link L on the boundary S^3 of punctured $K3$ that bounds a pair of slice disks D_L and D'_L which are topologically isotopic relative to the boundary but not smoothly so. Moreover, they remain exotic after a single stabilization, i.e. connected sum with $S^2 \times S^2$, and their complements are diffeomorphic.*

Remark 2. As mentioned in Theorem 1, another interesting feature of our example is that the complements

of D_L and D'_L are diffeomorphic. Examples with this feature were produced by Schwartz [Sch19] (for closed manifolds, using non-triviality of an h-cobordism) and Akbulut [Akb21a] (for embedded disks in the 4-ball, using corks). Note that recent work of Juhász–Miller–Zemke [JMZ20] and Hayden [Hay21] give some fascinating examples of knots in S^3 that bound exotic surfaces in 4-ball. In those examples it is not clear if the complements of the surfaces are diffeomorphic or not.

We sketch the proof of Theorem 1 as follows (see Section 3 for details). Let $D_L = \bigsqcup_{22} D^2$ be the disjoint union of the cores of 2-handles in the punctured $K3$ and let D'_L be the image of D_L under the Dehn twist on the punctured $K3$ along its boundary. By a result in [KK21] and [OP22], this Dehn twist is topologically isotopic to the identity map relative to the boundary. Hence D_L and D'_L are topologically isotopic relative to the boundary. However, D_L and D'_L are not smoothly isotopic relative to the boundary even after a stabilization. Suppose they were. Then we show that there is a smooth isotopy between $\delta\#\mathrm{id}_{S^2\times S^2}$ and $\mathrm{id}_{K3\#K3}\#f$, where δ denotes the Dehn twist on $K3\#K3$ and f is a diffeomorphism $S^2\times S^2$. By the connected sum formula of the Bauer–Furuta invariant and our vanishing theorem (Theorem 9) in the case of $S^2\times S^2$, the $\mathrm{Pin}(2)$-equivariant family Bauer–Furuta invariant of $\mathrm{id}_{K3\#K3}\#f$ is trivial. This contradicts with a result of the first author [Lin20], which states that the corresponding invariant of $\delta\#\mathrm{id}_{S^2\times S^2}$ is nontrivial.

It is worth noting that the argument in the current paper is essentially different from the argument in [Lin20] by the first author. In [Lin20], there is a specific diffeomorphism $\delta\#\mathrm{id}_{S^2\times S^2}$. By partially computing its family Bauer–Furuta invariant, one can show that the invariant is non vanishing. However, in the current paper, we just have a map of the form $\mathrm{id}_{K3\#K3}\#f$ with f being a hypothetical diffeomorphism on $S^2\times S^2$. Since we don't have much control over f, we have to prove a general vanishing theorem for the family Bauer–Furuta invariant that works for arbitrary f that acts trivially on homology. This is where extra arguments in equivariant stable homotopy theory have to get involved.

We also emphasize that in the proof of Theorem 1, the $\mathrm{Pin}(2)$-equvariance is essentially used multiple times for different purposes. First, the $\mathrm{Pin}(2)$-symmetry is included in the definition of the equivariant Bauer–Furuta invariant, so that this invariant can be nonzero even after a stabilization. This special feature, which is different from many other gauge theoretic invariants (e.g., the Seiberg–Witten invariant, Donaldson's polynomial invariant, the S^1-equivariant and the non-equivariant Bauer–Furuta invariant), makes the $\mathrm{Pin}(2)$-equivariant Bauer–Furuta invariant particularly suitable for detecting exotic phenomena under stabilizations. Second, because of this $\mathrm{Pin}(2)$-symmetry, the map between spheres which underlies the Bauer–Furuta invariant must be a $\mathrm{Pin}(2)$-equivariant map. This actually imposes a very strong constraint on the possible values of the Bauer–Furuta invariant. This constraint is crucial in the proof of our vanishing theorem.

Remark 3. From the construction of D_L and D'_L, we see that L can be taken to be the attaching link in any Kirby diagram of $K3$ (without 1-handles and 3-handles). Different choices of L will give different exotic surfaces. The following proposition gives an upper bound of the stabilizations needed to make these surfaces smoothly isotopic. It is proved by combining the argument in [AKM+19] with the recent work of Schwartz [Sch21], which generalizes the light bulb theorem to embedded disks.

Proposition 4. *(1) Let $L_0 \subset L$ be a sublink and let $D_{L_0} \subset D_L$ and $D'_{L_0} \subset D'_L$ be the subsurfaces that contain all the components with boundary in L_0. Suppose L_0 is an unlink. Then D_{L_0} is smoothly isotopic to D'_{L_0} relative to the boundary (without any stabilizations).*

(2) Suppose L contains an m-component unlink ($m \geq 0$). Then after $22-m$ stabilizations, D_L and D'_L are smoothly isotopic relative to the boundary. In particular, for any L, D_L and D'_L are smoothly isotopic after 22 stabilizations.

In Section 3, we give an example of L with all components being the unknot. By Proposition 4, each component of D_L is smoothly isotopic to the corresponding component of D'_L. Therefore, in this example, the exotic phenomena that persists a stabilization is the knotting between different components. Proposition 4 also motivates the following question,

Question 1. *Given L, what is the minimal number of stablizations that can make D_L and D'_L smoothly isotopic?*

1.2 The Smale Conjecture

Our second motivation comes from a famous conjecture by Smale. In dimension 4, this conjecture states that the natural inclusion $O(5) \hookrightarrow \mathrm{Diff}(S^4)$ is a homotopy equivalence [Kirby's Problem List [Kir95], Problem 4.34, 4.126]. Recently, Watanabe [Wat19] disproved The Smale Conjecture by showing that the induced maps on higher homotopy groups are not surjective. Here we state a special case of his result which is most relevant to our discussion:

Theorem 5 (Watanabe, [Wat19])**.** *There is a non-torsion element $\gamma_1 \in \pi_1(\mathrm{Diff}(S^4))$ such that for any nonzero integer d, one has*

$$d\gamma_1 \notin \mathrm{image}(\pi_1(O(5)) \to \pi_1(\mathrm{Diff}(S^4))).$$

Furthermore, γ_1 is an exotic loop of diffeomorphisms. Namely, γ_1 is null-homotopic in the homeomorphism group of S^4.

We note that the following important case is still open:

Conjecture 1 (Smale)**.** *Any orientation preserving self-diffeomorphism of S^4 is smoothly isotopic to Identity.*

Watanabe's proof is not gauge theoretic. Instead, he used Kontsevich's characteristic classes to detect γ_1. Note that gauge theory has also been very successful in detecting exotic diffeomorphisms on 4-manifolds [Akb15, AKMR15, BK20, KM20, Lin20, Rub98]. In an upcoming paper by Auckly–Ruberman [AR], they succeeded in using gauge theory to detect exotic loops (and elements in higher homotopy groups) in the space of diffeomorphisms of a 4-manifold. Given these developments, one may ask the following two general questions:

Question 2. *Can γ_1 and its multiples be detected by gauge theoretic invariants?*

Question 3. *Can we use gauge theoretic invariants to attack Conjecture 1? (One can see Akbulut's recent work [Akb21b] towards this conjecture.)*

Note that a negative answer to Question 2 would also be very interesting because it implies that even on S^4, there exist exotic phenomena that are invisible from gauge theory.

Since the family Bauer–Furuta invariant is a homotopy refinment of the family Seiberg–Witten invariant [BK19b, BF04], it is natural to specialize these questions to the family Bauer–Furuta invariants. Our results (Theorem 7 and Theorem 6) provide partial negative answers to these questions.

Theorem 6. *Let γ be an exotic loop in* $\mathrm{Diff}(S^4)$ *then one has*

$$\mathrm{FBF}^{S^1}(S^4,\gamma)=0 \quad \textit{and} \quad \mathrm{FBF}^{\{e\}}(S^4,\gamma)=0.$$

If we further assume that γ is divisible by 2 *in* $\pi_1(\mathrm{Diff}(S^4))$*, then*

$$\mathrm{FBF}^{\mathrm{Pin}(2)}(S^4,\gamma)=0.$$

Theorem 6 implies that $2\gamma_1$,which can be detected by Kontsevich's characteristic classes, can not be detected by Seiberg-Witten theory. This is interesting for the following reason: Recall that a formally smooth manifold is a topological manifold equipped with a vector bundle structure on its tangent microbundle. Formally smooth 4-manifolds form an intermediate category between the category of topological manifolds and the category of smooth manifolds. So there are really two types of exotic phenomena in dimension 4: Type I phenomena arise from differences between the topological category and the formally smooth category, while Type II phenomena arise from differences formally smooth category and the smooth category. It is suggested in [KA23] to study them separately. In a recent paper by the first author and Xie [LX23], it is proved that Kontsevich's characteristic classes can only detect Type I phenomena. Theorem 6 implies that $2\gamma_1$ is an example of a Type I phenomena that can not be detected by Seiberg-Witten theory.

Theorem 7. *Let f be an orientation-preserving diffeomorphism of S^4. Let $\widetilde{f}$ be a lift of f to an automorphism of the spin structure of S^4. Then we have*

$$\mathrm{FBF}^{S^1}(S^4,f,\widetilde{f})=0 \quad \textit{and} \quad \mathrm{FBF}^{\{e\}}(S^4,f,\widetilde{f})=0.$$

Theorem 7 and Theorem 6 together imply that the S^1-equivariant version and the non-equivariant version of the family Bauer–Furuta invariant do not detect exotic (loops of) diffeomorphisms on S^4. Note that Theorem 7 has a generalization (Theorem 9) that applies to a large family of spin 4-manifolds. However, it is unknown whether the $\mathrm{Pin}(2)$-equivariant family Bauer–Furuta invariant of a diffeomorphism on S^4 can be nonzero. Actually, we discovered a curious 2-torsion element in the $\mathrm{Pin}(2)$-equivariant stable homotopy group of spheres. This element satisfies a basic requirement, called *"BF-type"*, of being realizable as $\mathrm{FBF}^{\mathrm{Pin}(2)}(S^4,f,\widetilde{f})$ for some exotic diffeomorphism f. Roughly speaking, this condition means that the equivariant homotopy class is trivial when restricted to the S^1-fixed points. We refer to Definition 30 for a precise definition.

Theorem 8. *There is a unique nonzero element $\alpha_0\in\pi_{1,\mathcal{U}}^{\mathrm{Pin}(2)}(S^0)$ that is of BF-type.*

Theorem 8 suggests that the $\mathrm{Pin}(2)$-equivariant family Bauer–Furuta invariant could potentially be useful disprove Conjecture 1. Recently, Konno-Mallick-Taniguchi [KMT23] proved that for any

contractible manifold C bounded by $\Sigma(2,3,13)$ and $\Sigma(2,3,25)$, the boundary Dehn twist t_C is an exotic diffeomorphism on C. Their proof uses the family Seiberg-Witten theory. Note that some of these examples do admit smooth embedding into S^4. So one may ask whether an exotic diffeomorphism on S^4 can be obtained by extending t_c by the identity map on $S^4 \setminus C$. Theorem 8 supports this strategy. By an explicit space level description of α_0 in Section 5, the element α_0 does admit a decomposition

$$S^{\mathbb{R}} \to S^0 \vee \mathrm{Pin}(2)_+ \to S^0$$

of stable maps. And the Seiberg-Witten-Floer homotopy type of $-\Sigma(2,3,13)$ is indeed $S^0 \vee \mathrm{Pin}(2)_+$ [Man07, Section 7.2]. We will prove that α_0 can not be detected by Borel (co)homology for various subgroups G of $\mathrm{Pin}(2)$. These homology theories correspond to various versions of monopole Floer homology. So the use of Floer homotopy theory (instead of Floer homology) is essential here.

1.3 The vanishing theorem

Now we state the vanishing theorem for the family Bauer–Furuta invariant of a diffeomorphism. As we mentioned, this theorem generalizes Theorem 7 and it plays an essential role in the proof of Theorem 1.

Theorem 9. *Let X be an oriented spin 4-manifold with $b_1(X) = 0$. We fix a homology orientation and a spin structure on X. Let $f : X \to X$ be a diffeomorphism which preserves the spin structure and acts trivially on $H^*(X;\mathbb{R})$. Then for any lift $\widetilde{f}$ of f to an automorphism on the spin structure, we have the following results:*

1. *Suppose $2\chi(X)+3\sigma(X) = 4$. Then the invariants*

$$\mathrm{FBF}^{S^1}(X,f,\widetilde{f}) \quad \textit{and} \quad \mathrm{FBF}^{\{e\}}(X,f,\widetilde{f})$$

   are both vanishing.

2. *Suppose $2\chi(X)+3\sigma(X) > 4$. Then the invariants*

$$\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f}), \quad \mathrm{FBF}^{S^1}(X,f,\widetilde{f}) \quad \textit{and} \quad \mathrm{FBF}^{\{e\}}(X,f,\widetilde{f})$$

   are all vanishing.

We note that for any spin 4-manifold, we have $4 \mid 2\chi(X)+3\sigma(X)$. We also note that the quantity $2\chi(X)+3\sigma(X)$, which equals $c_1(TX)^2[X]$ when X is equipped with an almost complex structure, plays an important role in the geography problem of irreducible and symplectic 4-manifolds (see, for example, [FS09]). Actually, all known examples of simply-connected, irreducible 4-manifolds satisfy the inequality

$$2\chi(X)+3\sigma(X) \geq 0. \tag{1}$$

Moreover, by a result of Taubes [Tau00], the inequality (1) holds for any irreducible symplectic 4-manifolds with $b^+(X) > 1$. In particular, this implies that Theorem 9 applies to any simply-connected, spin, symplectic 4-manifold that is not homeomorphic to an elliptic surface $E(2k)$. And it also applies

to many reducible 4-manifolds (e.g. $K3\#(S^2\times S^2)$). We refers to Remark 33 for examples where $2\chi(X)+3\sigma(X)=-4,0$ or 4 but $\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f})\neq 0$.

Theorem 9 is proved using tools from equivariant stable homotopy theory (in particular, the tom Dieck splitting theorem and the Adams isomorphism). The geometric input from gauge theory is the following observation: When restricted to the S^1-fixed points, the family Bauer–Furuta invariant can be defined via the linear operator $d^+\oplus d^*$, which only records the induced map on $H^2(X;\mathbb{R})$ and hence can not distinguish f with the identity map on X. As a result, the family Bauer–Furuta invariant becomes trivial once we restrict it to the S^1-fixed points. In a certain range of dimensions, this is enough to show that the family Bauer–Furuta invariant itself is trivial even before passing to the S^1-fixed points. So we can prove Theorem 9.

As a final remark, we mention that Theorem 9 doesn't say anything about those spin^c structures not coming from a spin structure. Actually, the family Bauer–Furuta invariants and the family Seiberg–Witten invariants for these spin^c structures do detect exotic diffeomorphisms (see [BK20, Rub98]). Note that when defining the family Bauer–Furuta invariant in the absence of a spin structure, there are some extra complications in choosing the framing of the moduli space (see [KM20, Section 3]).

Organization of the paper: In Section 2, we recall the definition of the $\mathrm{Pin}(2)$-equivariant family Bauer–Furuta invariant. Then in Section 3, we prove Theorem 1 (existence of exotic surfaces under a stabilization) by assuming that Theorem 9 (the vanishing theorem for the family Bauer–Furuta invariant) holds for $S^2\times S^2$. We also prove Proposition 4 in this section. The rest of the paper is devoted to the proof of the vanishing results Theorem 6, Theorem 8 and Theorem 9. In Section 4, we first review some standard constructions and results in equivariant stable homotopy theory. Then we prove several algebraic vanishing results for the $\mathrm{Pin}(2)$-equivariant stable homotopy groups (Proposition 24 $\sim$ Proposition 27). These vanishing results will be used in Section 5 to finish the proof of Theorem 6, Theorem 8 and Theorem 9. Section 4 and 5 mainly consist of algebraic topology.

2 The Family Bauer–Furuta Invariant.

Our proof of Theorem 1 heavily uses the family Bauer–Furuta invariant, which was defined by Bauer–Furuta [BF04] (for a single 4-manifold), Szymik [Szy10] and Xu [Xu04] (for families). So for completeness, we first recall its definition following the expositions in [KM20, Lin20]. We start by introducing some notations. Consider the 1-dimensional Lie group

$$\mathrm{Pin}(2)=\{e^{i\theta}\}\cup\{j\cdot e^{i\theta}\}\subset\mathbb{H}.$$

We are mainly interested in the following three representations of $\mathrm{Pin}(2)$:

- $\mathbb{H}$: the 4-dimensional representation, where $\mathrm{Pin}(2)$ acts as the left multiplication.
- $\widetilde{\mathbb{R}}$: the 1-dimensional representation, where the unique component $S^1\subset\mathrm{Pin}(2)$ acts trivially and the other component acts as multiplication by -1. Note that this is also the adjoint representation of $\mathrm{Pin}(2)$ on its Lie algebra.
- $\mathbb{R}$: the 1-dimensional trivial representation.

For a finite dimensional $\mathrm{Pin}(2)$-representation V, we use S^V to denote the one point compactification of V and use $S(V)$ to denote the unit sphere in V. The space S^V is called the "representation sphere" and it admits a $\mathrm{Pin}(2)$-action that fixes the point ∞, which we choose as the base point.

2.1 The invariant of a single diffeomorphism

Now we start the construction of the family Bauer–Furuta invariant of a self-diffeomorphism. Our exposition here follows closely with [KM20]. Since our setting is slightly different, we give some details for completeness. Let X be an oriented 4-manifold with $b_1(X) = 0$. We equip X with a cohomological orientation (i.e. an orientation of $H^2_+(X;\mathbb{R})$) and a spin structure. Here a spin structure in the absence of a Riemannian metric means a principal bundle

$$\widetilde{\mathrm{GL}^+(4)} \to P \to X$$

that lifts the oriented frame bundle

$$\mathrm{GL}^+(4) \to \mathrm{Fr} \to X.$$

Here $\mathrm{GL}^+(4)$ denotes the group of rank-4 real matrices with positive determinant, and $\widetilde{\mathrm{GL}^+(4)}$ denotes its universal cover. Let f be a self-diffeomorphism on X that acts trivially on $H^*(X;\mathbb{R})$. Then we form the mapping torus

$$Tf = ([0,1]\times X)/(0,x) \sim (1, f(x)).$$

We treat Tf a fiber bundle over S^1 with fiber X. For any $s \in S^1$, we use X_s denote the fiber over s. Let $\widetilde{f} : P \to P$ be a bundle automorphism of P which lifts the differential $df : \mathrm{Fr} \to \mathrm{Fr}$. (There are two choices of $\widetilde{f}$ and we pick one.) Then we form the $\widetilde{\mathrm{GL}^+(4)}$-bundle

$$\widetilde{P} = ([0,1]\times P)/(0,x) \sim (1, \widetilde{f}(x)).$$

This is a bundle that lifts the vertical frame bundle

$$\mathrm{GL}^+(4) \to \widetilde{\mathrm{Fr}} \to Tf$$

so it gives a family spin structure. Now we equip $\{X_s\}_{s\in S^1}$ with a smooth family of Riemannian metrics. Then the structure group of $\widetilde{\mathrm{Fr}}$ is reduced to $\mathrm{SO}(4)$. And the structure group of $\widetilde{P}$ is reduced to $\mathrm{Spin}(4) = \mathrm{Sp}(1)\times\mathrm{Sp}(1)$. This gives us the associated quaternionic bundles

$$\mathbb{H} \to S^{\pm} \to Tf.$$

For each $s \in S^1$, we denote $S^{\pm}|_{X_s}$ by $S^{\pm}_s$. Then we have the Clifford multiplication

$$\gamma : T^*X_s \otimes \mathbb{C} \to \mathrm{Hom}(S^+_s, S^-_s)$$

and the spin Dirac operator

$$D : \Gamma(S^+_s) \to \Gamma(S^-_s),$$

which is quaternionic linear. The Seiberg-Witten equations, together with the Coulomb gauge fixing condition, give the Seiberg-Witten map

$$SW_s = l_s + c_s : \Omega^1(X_s) \oplus \Gamma(S_s^+) \to \Omega^2_+(X_s) \oplus (\Omega^0(X_s)/\mathbb{R}) \oplus \Gamma(S_s^-).$$

Here

$$l_s(\alpha, \phi) = (d^+\alpha, d^*\alpha, D)$$

and

$$c_s(\alpha, \phi) = (q(\phi, \phi), 0, \gamma(i\alpha)\phi).$$

$q(\phi, \phi)$ is a quadratic term defined using γ. Furthermore, sw_s is $\mathrm{Pin}(2)$-equivariant under the $\mathrm{Pin}(2)$-action on $\Omega^1(X_s) \oplus \Gamma(S_s^+)$ defined by

$$e^{i\theta}(\alpha, \phi) = (\alpha, e^{i\theta}\phi), \quad j(\alpha, \phi) = (-\alpha, j \cdot \phi) \tag{2}$$

and the $\mathrm{Pin}(2)$-action on $\Omega^2_+(X_s) \oplus (\Omega^0(X_s)/\mathbb{R}) \oplus \Gamma(S_s^-)$ defined by

$$e^{i\theta}(\beta, \gamma, \psi) = (\beta, \gamma, e^{i\theta}\psi), \quad j(\beta, \gamma, \psi) = (-\beta, -\gamma, j \cdot \psi). \tag{3}$$

After suitable Sobolev completions, we obtain a map

$$SW_s : \mathcal{W}_s^+ \to \mathcal{W}_s^-$$

between Hilbert spaces. Putting the maps $\{SW_s\}_{s \in S^1}$ together, we get a map

$$SW = l + c : \mathcal{W}^+ \to \mathcal{W}^-$$

between $\mathcal{W}^\pm = \cup_s \mathcal{W}_s^\pm$, which are Hilbert bundles over S^1. To do finite dimensional approximation on SW, we carefully pick finite dimensional trivialized subbundles $W^\pm \subset \mathcal{W}^\pm$ as follows: Consider the decomposition

$$\mathcal{W}^\pm = \mathcal{U}^\pm \oplus \mathcal{V}^\pm,$$

where $\mathcal{U}^\pm$ denotes the component of differential forms and $\mathcal{V}^\pm$ denotes the components of spinors. We have a further decomposition $\mathcal{U}^- = H \oplus H_\perp$ into the bundle H of self-dual harmonic 2-forms and its orthogonal complement $H_\perp$. By Kuiper's theorem, the Hilbert bundle $\mathcal{U}^+$ has a trivialization

$$\mathcal{U}^+ \cong S^1 \times \widetilde{\mathbb{R}}^\infty.$$

Under this trivialization, the standard subspace $\widetilde{\mathbb{R}}^{n'} \subset \widetilde{\mathbb{R}}^\infty$ gives a trivialized subbundle U^+ of $\mathcal{U}^+$. We let $U_\perp^-$ be the image of U^+ under the isomorphism

$$(d^+, d^*) : \mathcal{U}^+ \to H_\perp$$

and we let $U^- = H \oplus U_\perp^-$. Note that H is a maximal positive subbundle of the trivial bundle $\{H^2(X_s; \mathbb{R})\}_s$, under the intersection form. Since the space of maximal positive subspaces of any inner product space is contractible, we have an isomorphism

$$H \cong S^1 \times H^2_+(X; \mathbb{R})$$

Up to homotopy, the homological orientation on X gives an unique isomorphism $H^2_+(X;\mathbb{R}) \cong \widetilde{\mathbb{R}}^{b_2^+(X)}$. This in turns gives the trivialization

$$H \cong S^1 \times \widetilde{\mathbb{R}}^{b_2^+(X)}.$$

Now we consider $\mathcal{V}^{\pm}$, which are actually quaternionoic bundles. We take a finite dimensional quaternionoic subbundle $V^- \subset \mathcal{V}^-$ that is fiberwisely transverse to the image of the Dirac operator D and we let $V^+ = D^{-1}(V^-)$. Since the Dirac operator has index $-\frac{\sigma(X)}{16}\mathbb{H}$, we have

$$\mathrm{rank}_{\mathbb{H}}(V^-) = \mathrm{rank}_{\mathbb{H}}(V^+) + \frac{\sigma(X)}{16} = m'$$

for some m'. Since $\mathrm{Sp}(m)$ is 2-connected for any m, we have canonical (up to homotopy) trivializations

$$V^+ \cong S^1 \times \mathbb{H}^{m' - \frac{\sigma(X)}{16}}, \quad V^- \cong S^1 \times \mathbb{H}^{m'}.$$

We let $W^{\pm} = U^{\pm} \oplus V^{\pm}$. Then the above trivializations together give trivializations

$$W^+ \cong S^1 \times (\widetilde{\mathbb{R}}^{n'} \oplus \mathbb{H}^{m' - \frac{\sigma(X)}{16}}), \quad W^- \cong S^1 \times (\widetilde{\mathbb{R}}^{n' + b_2^+(X)} \oplus \mathbb{H}^{m'}) \tag{4}$$

Let $W^-_{\perp}$ be the orthogonal complement of W^- in $\mathcal{W}^-$ and let $S_R(W^-_{\perp})$ be the sphere bundle of radius R. By the compactness property of the Seiberg-Witten equations [BF04, Lemma 2.5, Proposition 3.1], one has

$$SW(W^+) \cap S_R(W^-_{\perp}) = \emptyset$$

if n', V^- and R chosen to be large enough. Then we define the approximated Seiberg-Witten map

$$sw = \rho \circ SW : W^+ \to W^-$$

Here $\rho : \mathcal{W}^- \setminus S_R(W^-_{\perp}) \to W^-$ is a deformation retraction (see [BF04, Page 6]). The map sw is $\mathrm{Pin}(2)$-equivariant and proper. We ignore the S^1-component in W^- and use the trivialization (4) to rewrite sw into the map

$$\widetilde{sw} : S^1 \times (\widetilde{\mathbb{R}}^{n'} \oplus \mathbb{H}^{m' - \frac{\sigma(X)}{16}}) \to \widetilde{\mathbb{R}}^{n' + b_2^+(X)} \oplus \mathbb{H}^{m'}. \tag{5}$$

Since $\widetilde{sw}$ is a proper map, it extends to a map between one-point compactifications

$$\widetilde{sw}^+ : S^1_+ \wedge S^{(m' - \frac{\sigma(X)}{16})\mathbb{H} + n'\widetilde{\mathbb{R}}} \to S^{m'\mathbb{H} + (n' + b_2^+(X))\widetilde{\mathbb{R}}}.$$

Here S^1_+ denotes union of S^1 with a disjoint base point. To obtain a map between two spheres, we embed S^1 as the unit sphere $S(2\mathbb{R})$ in $\mathbb{R}^2$ and let $c_0 : S^{2\mathbb{R}} \to S^{\mathbb{R}} \wedge S(2\mathbb{R})_+$ be the Pontryagin-Thom collapsing map which collapses all the points outside a tubular neighborhood of $S(2\mathbb{R})$. Consider the composition

$$\begin{aligned} S^{2\mathbb{R}} \wedge S^{(m' - \frac{\sigma(X)}{16})\mathbb{H} + n'\widetilde{\mathbb{R}}} &\xrightarrow{c_0 \wedge \mathrm{id}} S^{\mathbb{R}} \wedge (S(2\mathbb{R})_+) \wedge S^{(m' - \frac{\sigma(X)}{16})\mathbb{H} + n'\widetilde{\mathbb{R}}} \\ &\xrightarrow{\mathrm{id} \wedge \widetilde{sw}^+} S^{\mathbb{R}} \wedge S^{m'\mathbb{H} + (n' + b_2^+(X))\widetilde{\mathbb{R}}}. \end{aligned} \tag{6}$$

It is a $\mathrm{Pin}(2)$-equivariant map between two spheres and thus it defines an element in the $\mathrm{Pin}(2)$-equivariant stable homotopy of spheres. This element is exactly the $\mathrm{Pin}(2)$-equivariant family Bauer–Furuta invariant of f, denoted by $\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f})$. More precisely, we have

$$\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f}) \in \begin{cases} \{S^{\mathbb{R}+(-\frac{\sigma(X)}{16})\mathbb{H}}, S^{b^+(X)\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} & \text{if } \sigma(X)\le 0 \\ \{S^{\mathbb{R}}, S^{b^+(X)\widetilde{\mathbb{R}}+\frac{\sigma(X)}{16}\mathbb{H}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} & \text{if } \sigma(X)>0 \end{cases}. \tag{7}$$

(See Definition 16 for a precise description of these groups.)

The S^1-equivariant Bauer–Furuta invariant $\mathrm{FBF}^{S^1}(X,f,\widetilde{f})$ and the non-equivariant Bauer–Furuta invariant $\mathrm{FBF}^{\{e\}}(X,f,\widetilde{f})$ are defined to be the restriction of $\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f})$ to the subgroups $S^1,\{e\}\subset \mathrm{Pin}(2)$ respectively. Therefore, we have

$$\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f})=0 \implies \mathrm{FBF}^{S^1}(X,f,\widetilde{f})=0 \implies \mathrm{FBF}^{\{e\}}(X,f,\widetilde{f})=0.$$

All these invariants are unchanged under smooth isotopy. In particular, when f is smoothly isotopic to id_X and $\widetilde{f}$ is the lift corresponding to the identity map on the spin bundle, all of them are zero. To get an invariant of f, we can consider the unordered pair

$$\{\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f}_1),\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f}_2)\}$$

for two possible lifts $\widetilde{f}_1,\widetilde{f}_2$. If f is smoothly isotopic to id_X, then at least one element in this pair must be vanishing.

Remark 10. We briefly explain the motivation of the composition (6). Since S^1_+ is stably homotopy equivalent to $S^1\vee S^0$, the map $\widetilde{sw}^+$ admits a splitting into $\widetilde{sw}^+_0\vee\widetilde{sw}^+_1$ in the stable homotopy category. The component $\widetilde{sw}^+_0$ represent the Bauer–Furuta invariant of the fiber X and the component $\widetilde{sw}^+_1$ captures information about f. Since we are studying exotic diffeomorphisms on a fixed smooth manifold X, our main interest is $\widetilde{sw}^+_1$. The effect of composing with a suspension of c_0 in (6) is exactly keeping $\widetilde{sw}^+_1$ and ignoring $\widetilde{sw}^+_0$. This also explains why the family Bauer–Furuta invariant can be trivial even when the Bauer–Furuta invariant of the fiber is nontrivial.

2.2 The invariant of a loop of diffeomorphisms

Let X be as before. Now we construct the family Bauer–Furuta invariant of a loop of diffeomorphisms on X. Let $\gamma\colon S^1\to \mathrm{Diff}(X)$ be a loop based at id_X. We use γ as a clutching function and form a smooth fiber bundle

$$X\hookrightarrow T\gamma\to S^2.$$

Definition 11. γ is *spin-liftable* if the second Stiefel–Whitney class $w_2(T\gamma)$ vanishes.

Note that this condition is always satisfied if γ is null-homotopic in the homeomorphism group of X. Because in this case, $T\gamma$ is homeomorphic to $X\times S^2$. By Wu's formula, the Seiefel–Whitney class is independent of the smooth structure, we have $w_2(T\gamma)=w_2(X\times S^2)=0$.

From now on, we assume γ is spin-liftable. Then $T\gamma$ carries a family spin structure. Since the inclusion $X\hookrightarrow T\gamma$ induces an isomorphism on $H^1(-;\mathbb{Z}/2)$, such a family spin structure is unique if we

require that its restriction the fiber is the given spin structure on X. (This is different from the previous case of a single diffeomorphism.)

Now we repeat the same construction as before and get a map

$$\widetilde{sw}^+ : S^2_+ \wedge S^{(m'-\frac{\sigma(X)}{16})\mathbb{H}+n'\widetilde{\mathbb{R}}} \to S^{m'\mathbb{H}+(n'+b_2^+(X))\widetilde{\mathbb{R}}}. \tag{8}$$

for some $m', n' \gg 0$. We embed S^2 as the unit sphere in $S^{3\mathbb{R}}$ and form the composition between (suspensions of) $\widetilde{sw}^+$ and the Pontryagin-Thom collapsing map $S^{3\mathbb{R}} \to S^{\mathbb{R}} \wedge S(3\mathbb{R})_+$. This gives us a map

$$S^{3\mathbb{R}+(m'-\frac{\sigma(X)}{16})\mathbb{H}+n'\widetilde{\mathbb{R}}} \to S^{\mathbb{R}+m'\mathbb{H}+(n'+b_2^+(X))\widetilde{\mathbb{R}}}.$$

The $\mathrm{Pin}(2)$-equivariant family Bauer–Furuta invariant $\mathrm{FBF}^{\mathrm{Pin}(2)}(X,\gamma)$ is defined as the equivariant stable homotopy class of this map. More precisely, we have

$$\mathrm{FBF}^{\mathrm{Pin}(2)}(X,\gamma) \in \begin{cases} \{S^{2\mathbb{R}+(-\frac{\sigma(X)}{16})\mathbb{H}}, S^{b^+(X)\widetilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}_{\mathcal{U}} & \text{if } \sigma(X) \le 0 \\ \{S^{2\mathbb{R}}, S^{b^+(X)\widetilde{\mathbb{R}}+\frac{\sigma(X)}{16}\mathbb{H}}\}^{\mathrm{Pin}(2)}_{\mathcal{U}} & \text{if } \sigma(X) > 0 \end{cases}.$$

The invariants $\mathrm{FBF}^{S^1}(X,\gamma)$ and $\mathrm{FBF}^{\{e\}}(X,\gamma)$ are defined by restricting $\mathrm{FBF}^{\mathrm{Pin}(2)}(X,\gamma)$ to the subgroups $\{e\}, S^1$ respectively. Again, these invariants are unchanged under any smooth isotopies.

Remark 12. As noted in [KM20], there are extra complications if one tries to generalize this construction and define invariants of elements in $\pi_k(\mathrm{Diff}(X))$ with $k \ge 2$. Because the symplectic group $\mathrm{Sp}(n)$ is 2-connected, every quaternion bundle over S^{k+1} has a trivialization which is unique up to homotopy when $k \le 1$. This allows us to define the canonical trivializations of the bundles involved in construction (see (4)). However, this argument breaks down when $k \ge 2$.

3 Exotic Surfaces and Stabilization

3.1 Exotic surfaces that survive a stabilization

We will now prove Theorem 1, which shows the existence of exotic surfaces that remains exotic after a single stabilization. The proof here relies on Theorem 9 in the case of $S^2 \times S^2$. Theorem 9 will be proved in Section 5 in full generality.

We start with the following technical lemma.

Lemma 13. *Let $D = \bigsqcup D^2 \hookrightarrow X$ be a smooth, proper embedding of a finite disjoint union of disks into a 4-manifold X with boundary. Consider a self-diffeomorphism $f : X \to X$ that fixes all points in a neighborhood of ∂X and all points on D. Then f is smoothly isotopic (relative to ∂X) to some f' which fixes all points in a neighborhood of $(\partial X) \cup D$.*

Proof. Let N be the normal bundle of D in X. By choosing a trivialization of this bundle, we get an identification $N = D \times \mathbb{R}^2$. Since f fixes D pointwise, its differential induces a bundle automorphism $l' : N \to N$. Let $\rho : N \to \nu(D)$ be a diffeomorphism to a tubular neighborhood of D. We use l to denote the composition

$$\rho \circ l' \circ \rho^{-1} : \nu(D) \to \nu(D).$$

For $\varepsilon > 0$, we use $\nu_\varepsilon(D)$ to denote $\rho(D \times D_\varepsilon)$, where $D_\varepsilon \subset \mathbb{R}^2$ is the ball centered at the origin with radius ε. We pick ε small enough such that $f(\nu_\varepsilon(D)) \subset \nu(D)$. We first construct a smooth isotopy $h_1 : [0,1] \times \nu_\varepsilon(D) \to \nu(D)$ between the maps

$$f|_{\nu_\varepsilon(D)} \colon \nu_\varepsilon(D) \to \nu(D) \quad \text{and} \quad l|_{\nu_\varepsilon(D)} \colon \nu_\varepsilon(D) \to \nu(D).$$

(Isotopy here means that for any t, the map $h_1(t,-)$ is a diffeomorphism onto its image.) This is done via the standard rescaling trick: For any $t > 0$, we use $\phi_t : \nu(D) \to \nu(D)$ to denote the self-diffeomorphism defined by

$$\phi_t(\rho(x,\vec{v})) = \rho(x,t\vec{v}) \quad \forall x \in D, \vec{v} \in \mathbb{R}^2.$$

Then the isotopy h_1 is given by

$$h_1(t,y) := \begin{cases} \phi_t^{-1} \circ f \circ \phi_t(y) & \text{if } t \in (0,1] \\ l(y) & \text{if } t = 0 \end{cases}.$$

Next, we construct an smooth isotopy $h_2 : [0,1] \times \nu_\varepsilon(D) \to \nu(D)$ from the map $l|_{\nu_\varepsilon(D)}$ to the inclusion map $i : \nu_\varepsilon(D) \to \nu(D)$. This is done via a simple obstruction argument: The map l' is linear on each fiber so it has the form

$$l'(x,\vec{v}) = (x, \tau(x) \cdot \vec{v})$$

for some $\tau : D \to \mathrm{GL}^+(\mathbb{R}^2)$ that equals 1 near ∂D. Since $\mathrm{GL}^+(\mathbb{R}^2)$ is homotopy equivalent to S^1 and $\pi_2(S^1) = 0$, there exists a smooth homotopy (relative to ∂D)

$$\widetilde{\tau} : [0,1] \times D \to \mathrm{GL}^+(\mathbb{R}^2)$$

from τ to the constant map 1. We define $\widetilde{l'} : [0,1] \times N \to N$ by

$$\widetilde{l'}(t,x,\vec{v}) = (x, \widetilde{\tau}(t,x) \cdot \vec{v}) \quad \text{for } t \in [0,1], x \in D, \vec{v} \in \mathbb{R}^2.$$

Then we define $h_2(t,y) := \rho(\widetilde{l'}(t, \rho^{-1}(y)))$. By compositing h_1 with h_2, we obtain a smooth isotopy between the maps

$$f|_{\nu_\varepsilon(D)} \colon \nu_\varepsilon(D) \to \nu(D) \quad \text{and} \quad i : \nu_\varepsilon(D) \to \nu(D).$$

Furthermore, this isotopy is trivial near ∂X. Therefore, we can apply the isotopy extension theorem to produce an isotopy of f to some f' that fixed all points in a neighborhood of $(\partial X) \cup D$. □

Next, we briefly recall the construction of the Dehn twist on the punctured $K3$. We pick an essential loop

$$\gamma : [0,1] \to SO(4) \tag{9}$$

such that $\gamma(t) = 1$ for t near $0, 1$. Then we define the self-diffeomorphism

$$\widetilde{\gamma} \colon [0,1] \times S^3 \to [0,1] \times S^3$$

by $\widetilde{\gamma}(t,x) = (t, \gamma(t)x)$. Let $K3^\circ$ be the punctured $K3$. We fix a diffeomorphism from $[0,1] \times S^3$ to a collar neighborhood of $K3^\circ$ and define the Dehn twist as follows

$$\delta_{K3^\circ} := \widetilde{\gamma} \cup \mathrm{id}_{K3^\circ \setminus ([0,1] \times S^3)} : K3^\circ \to K3^\circ. \tag{10}$$

Now we are ready to show the existence of exotic surfaces that survive even after a single stabilization.

Proof of Theorem 1. Recall that $K3^\circ$ has a handle decomposition with 22 2-handles and a single 4-handle. This means that one can construct $K3^\circ$ by starting with $S^3 \times I$ and then adding a bunch of 2-handles along $S^3 \times \{1\}$ and cap it off with a 4-ball. We pick such a decomposition and let L be the set of attaching spheres on S^3. Then L is a link with 22 components. Let D_L be the union

$$(L \times I) \cup_{L \times \{1\}} \{\text{cores of the two handles}\}.$$

Then D_L is a disjoint union of 22 slice disks bounded by $L \subset \partial(K3^\circ)$.

Consider the Dehn-twist δ_{K3° defined in (10). In [KM20], Kronheimer–Mrowka showed that δ_{K3° is not smoothly isotopic to id_{K3° relative to the boundary. Let $D'_L = \delta_{K3^\circ}(D_L)$, then D'_L and D_L are topologically isotopic relative to the boundary, since a result of Krannich–Kupers [KK21, Remark 6] (see also [BK19a]) and Orson-Powell [OP22, Theorem A] implies δ_{K3° is indeed topologically isotopic to id_{K3° relative to the boundary.

We do a single stabilization in the interior of the 4-handle. The claim is that D_L and D'_L are not smoothly isotopic even after this stabilization. Suppose this were not true. Denote $\delta \# \mathrm{id}_{S^2 \times S^2}$ by $\delta_{K3^\circ \# (S^2 \times S^2)}$. By the isotopy extension theorem, there exists a self-diffeomorphism $g_1 : K3^\circ \# (S^2 \times S^2) \to K3^\circ \# (S^2 \times S^2)$ that is smoothly isotopic to the identity relative to the boundary and sends D'_L to D_L. Then $g_1 \circ \delta_{K3^\circ \# (S^2 \times S^2)}$ fixes all points on D_L and is smoothly isotopic to $\delta_{K3^\circ \# (S^2 \times S^2)}$ relative to the boundary. By Lemma 13, we can further isotope $g_1 \circ \delta_{K3^\circ \# (S^2 \times S^2)}$ relative to the boundary to a diffeomorphism

$$g_2 : K3^\circ \# (S^2 \times S^2) \to K3^\circ \# (S^2 \times S^2)$$

that is identity on the neighbourhood of $D_L \cup S^3 \times \{0\}$. Notice that the complement of this set in $K3^\circ \# (S^2 \times S^2)$ is a once stabilized 4-disk, i.e., a punctured $S^2 \times S^2$. Thus g_2 can be decomposed as $\mathrm{id}_{K3^\circ} \# f$ for some $f \in \mathrm{Diff}(S^2 \times S^2)$ which acts trivially on the cohomology and fixes B pointwise. Here $B \subset S^2 \times S^2$ is the small disk to which we apply the connected sum construction.

So we have shown that $\delta_{K3^\circ \# S^2 \times S^2}$ is smoothly isotopic to $\mathrm{id}_{K3^\circ} \# f$. This further implies that the Dehn twist

$$\delta_{K3\#K3\#(S^2 \times S^2)} : K3\#K3\#(S^2 \times S^2) \to K3\#K3\#(S^2 \times S^2)$$

along the neck between the two $K3's$ is smoothly isotopic to $(\mathrm{id}_{K3\#K3})\# f$.

Let $\mathrm{id}_{\mathfrak{s}_{K3\#K3}}$ be the identity map on the spin bundle over $K3\#K3$. There are two lifts of f to the spin bundle over $S^2 \times S^2$. We set $\widetilde{f}$ to be the one that is identity on all the fibers over B. Then we can form the connected sum $\mathrm{id}_{\mathfrak{s}_{K3\#K3}} \# \widetilde{f}$, which is a lift of $\mathrm{id}_{K3\#K3} \# f$ to an automorphism on the spin bundle.

It follows the connected sum formula [Lin20, Proposition 2.26] that

$$\begin{aligned} &\mathrm{FBF}^{\mathrm{Pin}(2)}(K3\#K3\#S^2 \times S^2, \mathrm{id}_{K3\#K3} \# f, \mathrm{id}_{\mathfrak{s}_{K3\#K3}} \# \widetilde{f}) \\ &\quad = BF^{\mathrm{Pin}(2)}(K3\#K3) \wedge \mathrm{FBF}^{\mathrm{Pin}(2)}(S^2 \times S^2, f, \widetilde{f}). \end{aligned}$$

But Theorem 9 implies that $\mathrm{FBF}^{\mathrm{Pin}(2)}(S^2 \times S^2, f, \widetilde{f}) = 0$, and hence

$$\mathrm{FBF}^{\mathrm{Pin}(2)}(K3\#K3\#S^2 \times S^2, \mathrm{id}_{K3\#K3} \# f, \mathrm{id}_{\mathfrak{s}_{K3\#K3}} \# \widetilde{f}) = 0.$$

But in [Lin20, Proof of Theorem 1.2], the first author proves that

$$\mathrm{FBF}^{\mathrm{Pin}(2)}(K3\#K3\#S^2 \times S^2, \delta_{K3\#K3\#(S^2 \times S^2)}, \widetilde{\delta}) \neq 0$$

for any lift $\widetilde{\delta}$ of $\delta_{K3\#K3\#(S^2\times S^2)}$ to the spin structure of $K3\#K3\#S^2\times S^2$. [1] Since $\delta_{K3\#K3\#(S^2\times S^2)}$ is smoothly isotopic to $\mathrm{id}_{K3\#K3}\#f$, this contracts the isotopy invariance of the family Bauer–Furuta invariant. Thus D_L and D'_L are not smoothly isotopic relative to the boundary in $K3^\circ\#S^2\times S^2$. □

3.2 An upper bound on the number of stabilizations

Now we move on to the proof of Proposition 4, which gives an upper bound on the stabilizations needed to make D_L and D'_L smoothly isotopic. We start with part (1).

Proof of Proposition 4 (1). Since L_0 is an unlink, there is a diffeomorphism $f : S^3 \to S^3$ that is smoothly isotopic to id_{S^3} and sends L_0 to the standard unlink

$$L'_0 = \{(\cos\theta, \sin\theta, i) \mid \theta \in [0, 2\pi], i = 1, \cdots m\} \subset \mathbb{R}^3 \cup \{\infty\} = S^3.$$

By taking the rotations along the z-axis, we can construct an essential loop $\gamma' : [0,1] \to SO(4)$ such that $\gamma'(t)$ fixes L'_0 as a subset for any t. Then we consider the self-diffeomorphism

$$\widetilde{\gamma'} : [0,1]\times S^3 \to [0,1]\times S^3$$

defined by

$$\widetilde{\gamma'}(t,x) := (t, f^{-1}\circ\gamma'(t)\circ f(x)).$$

By extending $\widetilde{\gamma'}$ with the identity map on $K3^\circ \setminus ([0,1]\times S^3)$, we obtain a self-diffeomorphism δ'_{K3° on $K3^\circ$. Since γ' is homotopic to the loop γ in (9), and f is smoothly isotopic to id_{S^3}, we see that δ'_{K3° is smoothly isotopic to δ_{K3° relative to the boundary. Therefore, $\delta'_{K3^\circ}(D_{L_0})$ and D'_{L_0} are smoothly isotopic relative to the boundary. Furthermore, since $f^{-1}\circ\gamma'(t)\circ f$ fixes L_0 as a subset, the embedded surfaces $\delta'_{K3^\circ}(D_{L_0})$ and D_{L_0} differ from each other by a Dehn twist on the domain $\bigsqcup_m D^2$. Since this Dehn twist is smoothly isotopic the identity map relative to the boundary, the surfaces D_{L_0} and $\delta'_{K3^\circ}(D_{L_0})$ are also smoothly isotopic relative to the boundary. □

Figure 1 is a Kirby diagram of $K3$ [GS99, Page 305]. The attaching link has 21 unknotted components and a single knotted component (which is a trefoil). By doing a handle slide of this knotted component along one of the unknoted components, we can change the middle crossing and make it into the unknot. This gives an example of L whose all components are unknots. By Proposition 4 (1), all components of D_L are smoothly isotopic to the corresponding components of D'_L relative to the boundary.

Now we move on to the proof of Proposition 4 (2). We start by the following lemma.

Lemma 14. *Given a sublink L' of L, let $\nu(D_{L'})$ be an open tubular neighborhood of $D_{L'}$ in $K3^\circ$. Then we have the following results.*

(i) $K3^\circ \setminus \nu(D_{L'})$ and $K3^\circ \setminus \nu(D'_{L'})$ are simply connected.

[1] There is a subtle difference between the setting of the current paper and [Lin20]. Here we are using an incomplete universe $\mathbb{H}^\infty \oplus \widetilde{\mathbb{R}}^\infty \oplus \mathbb{R}^\infty$ to set up an equivariant stable homotopy category, while in [Lin20], a complete universe is used. (See Remark 22 for further explanations.) This difference does not cause a problem because by definition, $\mathrm{FBF}^{\mathrm{Pin}(2)}(K3\#K3\#S^2\times S^2, \delta_{K3\#K3\#(S^2\times S^2)}, \widetilde{\delta}) \neq 0$ in a complete universe implies the same result in any universe.

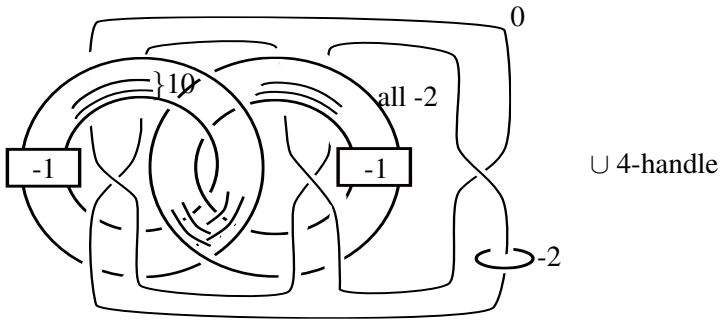


Figure 1: A Kirby Diagram of K3.

(ii) For any $n \geq 0$, the inclusion $(K3^\circ \setminus \nu(D_{L'})) \# n(S^2 \times S^2) \hookrightarrow K3^\circ \# n(S^2 \times S^2)$ induces an injection on the second homotopy group $\pi_2(-)$.

Proof. (i) We prove the case of $D_{L'}$ and the other case is similar. For each component K of L', let $m_k \subset \partial(K3^\circ \setminus \nu(D_{L'}))$ be the meridian of the embedded disk D_K. Since $K3^\circ$ is simply-connected, we see that $\pi_1(K3^\circ \setminus \nu(D_{L'}))$ is normally generated by $\{m_K\}$. So it suffices to show that m_K is null-homotopic in $K3^\circ \setminus \nu(D_{L'})$. To prove this, we consider the cocore of the 2-handle attached to K. This cocore becomes a cylinder in $K3^\circ \setminus \nu(D_{L'})$ and it provides a homotopy between m_K and a knot on the boundary of the 4-handle D^4 (which doesn't intersect $\nu(D_{L'})$). Since D^4 is contractible, this knot (and hence m_K) is null-homotopic.

(ii) By the Mayer-Vietoris sequence, the map

$$H_2((K3^\circ \setminus \nu(D_{L'})) \# n(S^2 \times S^2); \mathbb{Z}) \to H_2(K3^\circ \# n(S^2 \times S^2); \mathbb{Z})$$

induced by the inclusion is injective. By the Hurewicz theorem, this map is also the induced map on $\pi_2(-)$. □

The next proposition is a variation of the main theorem in [AKM+19].

Proposition 15. *Let X be a simply-connected, smooth 4-manifold with boundary. Let D_K and D'_K be two smoothly embedded disks bounded by the same knot $K \subset \partial X$. Suppose the following conditions are all satisfied:*

(i) $X \setminus D_K$ and $X \setminus D'_K$ are simply-connected.

(ii) D_K and D'_K are homotopic relative to the boundary.

(iii) $[D_K] \in H_2(X, \partial X; \mathbb{Z}/2)$ is ordinary (i.e. it is not the Poincaré dual of $w_2(X)$).

Then in $X \# (S^2 \times S^2)$, D_K and D'_K are smoothly isotopic relative to the boundary.

Proof. The proof is adapted from the corresponding proof in [AKM+19]. For completeness, we sketch the argument, with an emphasize on the adaptions needed here. First, since $X \setminus D_K$ is simply connected,

we can find an immersed sphere Σ that intersects D_K transversely at a single point. Using condition (iii), one can modify Σ so that it has an even self-intersection number. Since D'_K is homotopic to D_K relative to the boundary, the algebraic intersection number $\Sigma \cdot D'_K$ equals 1. After suitable finger moves and the (immersed) Whitney moves, we can isotope D'_K and Σ so that Σ intersects both D_K and D'_K transversely at a single point. To arrange for Σ to be embedded, we take a connected sum with $S^2 \times S^2$ away from $D_K \cup D'_K \cup \Sigma$. Take coordinate 2-spheres $S = S^2 \times \{\mathrm{pt}\}$ and $T = \{\mathrm{pt}\} \times S^2$. By replacing Σ with its connected sum with S along a tube, we may assume that Σ has a geometric dual T (i.e., T intersects transverse Σ at a single point). Then we can eliminate the double points of Σ by the Norman trick, tubing to parallel copies of T. This gives us an embedded Σ with even self-intersection number. After tubing additional copies of T, we can arrange the self-intersection number of Σ to be zero. Therefore, D_K and D'_K have a common geometric dual Σ in $X\#(S^2 \times S^2)$. No essential changes are necessary until step. Then we apply the light bulb theorem for embedded disks [Sch21, Corollary 3.3], which states that D_K and D'_K are smoothly isotopic relative to the boundary if the following three conditions are verified:

(1) D_K and D'_K are homotopic in $X\#(S^2 \times S^2)$ relative to the boundary.

(2) The Dax invariant $\mathrm{Dax}(D_K, D'_K)$ vanishes.

(3) The map $\pi_1((X\#(S^2 \times S^2)) \setminus \Sigma) \to \pi_1(X\#(S^2 \times S^2))$ induced by the inclusion is an isomorphism. (I.e. $(X\#(S^2 \times S^2)) \setminus \Sigma$ is simply connected.)

(1) follows from our assumption (ii). (2) is trivial since any surfaces in a simply connected 4-manifold have trivial Dax invariant. (3) is satisfied because Σ has an geometric dual T (so the meridian of Σ bounds a disk in $X \setminus \Sigma$). This finishes the proof. □

Proof of Proposition 4 (2). Let $L_0 \subset L$ be the m-component unlink. Denote the components of $L \setminus L_0$ by $K_1, K_2, \cdots, K_{22-m}$. For $1 \leq n \leq 22-m$, we set

$$L_n = L_0 \bigsqcup K_1 \bigsqcup \cdots \bigsqcup K_n \quad \text{for} \quad 1 \leq n \leq 22-m.$$

We will prove the following claim by induction.

Claim 1. *For any all $0 \leq n \leq 22-m$, D_{L_n} and D'_{L_n} are smoothly isotopic relative to the boundary in $K3^\circ \# n(S^2 \times S^2)$.*

The case $n = 0$ is implied by Proposition 4 (1). Now suppose the case $n = i$ has been proved. Denote $K3^\circ \# i(S^2 \times S^2)$ by X_i. Then by our induction hypothesis, there exists a diffeomorphism $f : X_i \to X_i$ which is smoothly isotopic to id_{X_i} relative to the boundary and satisfies $f(D'_{L_i}) = D_{L_i}$. Note that $D_{K_{i+1}}$ and $f(D'_{K_{i+1}})$ are disks embedded in $X_i \setminus \nu(D_{L_i})$ bounded by the same knot K_{i+1}. To prove the case $n = i+1$, it suffices to show that $D_{K_{i+1}}$ and $f(D'_{K_{i+1}})$ are smoothly isotopic relative to the boundary in $(X_i \setminus \nu(D_{L_i}))\#(S^2 \times S^2)$. (The connected sum is taken near ∂X_i so that the isotopy between f and id_{X_i} can be trivially extended.) By Proposition 15, we just need to check the following three conditions:

1. $X_i \setminus \nu(D_{L_i})$, $X_i \setminus \nu(D_{L_i} \sqcup D_{K_{i+1}})$ and $X_i \setminus \nu(D_{L_i} \sqcup f(D'_{K_{i+1}}))$ are all simply connected.
2. The homology class $[D_{K_{i+1}}] \in H_2(X_i \setminus \nu(D_{L_i}); \mathbb{Z}/2)$ is ordinary.

3. $D_{K_{i+1}}$ and $f(D'_{K_{i+1}})$ are homotopic in $X_i \setminus \nu(D_{L_i})$ relative to the boundary.

Note that

$$X_i \setminus \nu(D_{L_i} \bigsqcup D_{K_{i+1}}) \cong (K3^\circ \setminus \nu(D_{L_{i+1}}))\#i(S^2 \times S^2),$$
$$X_i \setminus \nu(D_{L_i} \bigsqcup f(D'_{K_{i+1}})) \cong (K3^\circ \setminus \nu(D'_{L_{i+1}}))\#i(S^2 \times S^2).$$

Therefore, (1) is directly implied by Lemma 14 (i).

To prove (2), we consider the cocore of the two-handle h_2^{i+1} attached to K_{i+1}. The boundary of this cocore (i.e., the belt sphere) is a knot on the boundary of the 4-handle D^4. Hence it bounds a properly embedded surface F in D^4. We consider the surface

$$F' = \{\text{cocore of } h_2^{i+1}\} \cup F \hookrightarrow K3^\circ \setminus \nu(D_{L_i}) \subset K3^\circ.$$

Since $K3^\circ$ has an even intersection form, the self-intersection number of F' is even. Note that

$$D_{K_{i+1}} \cdot F' = 1 \not\equiv F' \cdot F' \mod 2.$$

So $[D_{K_{i+1}}]$ is ordinary.

To prove (3), we note that $D_{K_{i+1}}$ and $D'_{K_{i+1}}$ are topologically isotopic relative to the boundary in $K3^\circ$. Hence they are also topologically isotopic relative to the boundary in X_i. Since $D'_{K_{i+1}}$ is smoothly isotopic to $f(D'_{K_{i+1}})$ relative to the boundary, we see that $D_{K_{i+1}}$ and $f(D'_{K_{i+1}})$ are topologically isotopic relative to the boundary in X_i. In particular, they are homotopic to each other relative to the boundary in X_i. Consider the map $\tau : S^2 \to X_i$ which sends the upper hemisphere and the lower hemisphere to $D_{K_{i+1}}$ and $f(D'_{K_{i+1}})$ respectively. Since $D_{K_{i+1}}$ and $f(D'_{K_{i+1}})$ are homotopic in X_i relative to the boundary, $\tau(S^2)$ is null-homotopic in X_i. By Lemma 14 (2), $\tau(S^2)$ is also null-homotopic in $X_i \setminus \nu(D_{L_i})$. Therefore, $D_{K_{i+1}}$ and $f(D'_{K_{i+1}})$ are homotopic in $X_i \setminus \nu(D_{L_i})$ relative to the boundary. This proves (3). □

4 Equivariant Stable Homotopy Theory

In this section, we briefly recall the definitions of equivariant stable homotopy groups and state some important properties about them. In particular, we state a version of the tom Dieck splitting theorem for an incomplete universe, which turns the computation of the equivariant stable group of a G-space into the computation of certain non-equivariant stable homotopy groups. Then we use this theorem to prove some algebraic propositions about the $\mathrm{Pin}(2)$-equivariant stable homotopy groups of spheres. These results will be essentially used in Section 5 to prove the vanishing theorem for the family Bauer–Furuta invariant. Our exposition here follows the classical method of Adams [Ada84] (which assumes G is finite). We refer to [Sch14, LMSM86] for the general case of compact Lie groups and more complete discussions using the language of equivariant spectra.

4.1 The tom Dieck splitting theorem

Let G be a compact Lie group. We consider an infinite dimensional G-representation $\mathcal{U}$ that satisfies the following two properties:

- There is a canonical embedding from the infinite dimensional trivial representation $\bigoplus_{\infty}\mathbb{R}$ to $\mathcal{U}$.
- For any finite dimensional subrepresentation V of $\mathcal{U}$, there exists an embedding from $\bigoplus_{\infty}V$ to $\mathcal{U}$.

We call such $\mathcal{U}$ a "G-universe". If all finite dimensional representation of G can embedded into $\mathcal{U}$, then we call $\mathcal{U}$ a complete universe. Otherwise, we say $\mathcal{U}$ is incomplete. For our topological applications, it is necessary to consider incomplete universes. (See Remark 22.)

We equip $\mathcal{U}$ with an G-invariant inner product. For any pointed G-spaces X,Y and any finite dimensional subrepresentations $V\subset W\subset\mathcal{U}$, there is a map

$$[S^V\wedge X,S^V\wedge Y]^G\xrightarrow{\mathrm{id}_{S^{W-V}}\wedge -}[S^W\wedge X,S^W\wedge Y]^G,$$

where $\mathrm{id}_{S^{W-V}}$ denotes the identity map on the representation sphere of the orthogonal complement of V in W.

Definition 16. For any pointed spaces X,Y, the equivariant stable homotopy group is defined as

$$\{X,Y\}^G_{\mathcal{U}}:=\varinjlim_{V\subset\mathcal{U}}[S^V\wedge X,S^V\wedge Y]^G,$$

where the limit is taken over all finite dimensional G-invariant subspaces $V\subset\mathcal{U}$. (Just like the non-equivariant setting, $\{X,Y\}^G_{\mathcal{U}}$ has a abelian group structure.) For $k\in\mathbb{N}$, we use $\pi^G_{k,\mathcal{U}}(X)$ to denote $\{S^k,X\}^G_{\mathcal{U}}$.

When G is the trivial group, there is essentially a unique choice of $\mathcal{U}$. In this case, we denote $\{X,Y\}^G_{\mathcal{U}}$ by $\{X,Y\}$ and denote $\pi^G_{k,\mathcal{U}}(X)$ by $\pi^s_k(X)$.

Given a closed subgroup H of G, we use $\mathrm{res}^G_H(\mathcal{U})$ to denote the same space $\mathcal{U}$, regarded as an H-representation space. Then there is a well-defined restriction map

$$\mathrm{res}^G_H:\{X,Y\}^G_{\mathcal{U}}\to\{X,Y\}^H_{\mathrm{res}^G_H(\mathcal{U})}$$

defined by only remembering the H-action.

Also, let N_GH be the normalizer of H in G and let $W_GH=N_GH/H$ be the Weyl group. Then the H-fixed points of $\mathcal{U}$, which we denote by $\mathcal{U}^H$, is an W_GH-universe. There is a well-defined map

$$(-)^H_{Y,X,G}:\{Y,X\}^G_{\mathcal{U}}\to\{Y^H,X^H\}^{W_GH}_{\mathcal{U}^H},\tag{11}$$

which sends a map to its restriction to the H-fixed points. In the special case $Y=S^k$, we have the map

$$(-)^H_{S^k,X,G}:\pi^G_{k,\mathcal{U}}(X)\to\pi^{W_GH}_{k,\mathcal{U}^H}(X^H).$$

When X,Y,G are obvious from the context, we will suppress them from our notations and simply write $(-)^H$.

Next, for any pointed G-space X, we introduce a map

$$\zeta^G_H:\pi^{W_GH}_{k,\mathcal{U}^H}(EW_GH_+\wedge X^H)\to\pi^G_{k,\mathcal{U}}(X)\tag{12}$$

which is crucial in the tom Dieck splitting theorem. The precise definition of ζ_H^G can be found in [Sch14, Section 6, Page 60]. For our purpose, it suffices to consider the case that H is a normal subgroup of G. In this case, we have $N_G H = G, W_G H = G/H$ and the map ζ_H^G has a simple description: We have taken an element $[f] \in \pi_{k,\mathcal{U}^H}^{G/H}((EG/H)_+ \wedge X^H)$, represented by a G/H-equivariant map

$$f : S^V \wedge S^k \to S^V \wedge (EG/H)_+ \wedge X^H$$

for some G/H-representation $V \subset \mathcal{U}^H \subset \mathcal{U}$. We regard f as a G-equivariant map and consider the composition

$$\widetilde{f} : S^V \wedge S^k \xrightarrow{f} S^V \wedge (EG/H)_+ \wedge X^H \xrightarrow{1\wedge p_0 \wedge 1} S^V \wedge S^0 \wedge X^H = S^V \wedge X^H \xrightarrow{1\wedge i} S^V \wedge X. \tag{13}$$

Here $p_0 : (EG/H)_+ \to S^0$ is the map that preserves the base point and collapses the whole EG/H to the other point in S^0, and the map $i : X^H \to X$ is the inclusion map. Note that both p_0 and i are G-equivariant because H is normal. We define $\zeta_H^G[f] := [\widetilde{f}]$.

Now we state the tom Dieck splitting theorem for an incomplete universe. This theorem was originally proved in [tD75] for a complete universe. The version given below is a special case of [Lew00, Theorem 3.8]. See [BH21, LMSM86] for various generalizations to equivariant spectra.

Theorem 17. *Let X be a pointed G-space. Then the map*

$$\sum_{[H]} \zeta_H^G : \bigoplus_{[H]} \pi_{k,\mathcal{U}^H}^{W_G H}(EW_G H_+ \wedge X^H) \to \pi_{k,\mathcal{U}}^G(X) \tag{14}$$

is an isomorphism for any $k \geq 0$. Here the direct sum is taken over all conjugacy classes of subgroups H that arises as the isotropy group of some point in $\mathcal{U}$.

To further simplify (14), we recall the Adams isomorphism theroem [LMSM86, Corollary II.7.2] (see [Lew00, Theorem 12.2] for a version with an incomplete universe) as follows: Let K be a compact Lie group and let $\mathrm{Ad}(K)$ be the adjoint representation of K on its Lie algebra. Let $\mathcal{U}'$ is a K-universe. We assume that K can be embedded into $\mathcal{U}'$ and that $\mathcal{U}'$ contains $\mathrm{Ad}(K)$. Let Y be a K-CW complexes with a trivial K-action and let Z be a K-CW complex with a K-action which is free away from the base point. Then the Adams isomorphism can be written as

$$\{Y, (S^{\mathrm{Ad}(K)} \wedge Z)/K\} \xrightarrow{\cong} \{Y, Z\}_{\mathcal{U}'}^K \tag{15}$$

Now we let $K = W_G H$, $\mathcal{U}' = \mathcal{U}^H$, $Y = S^{k\mathbb{R}}$ and let

$$Z = (EW_G H)_+ \wedge X^H.$$

Then (15) can be written as

$$\pi_k^s((S^{\mathrm{Ad}(W_G H)} \wedge (EW_G H)_+ \wedge X^H)/W_G H) \xrightarrow{\cong} \pi_{k,\mathcal{U}^H}^{W_G H}(EW_G H_+ \wedge X^H). \tag{16}$$

Using (16), we can rewrite the isomorphism (14) as the isomorphism

$$\bigoplus_{[H]} \pi_k^s((S^{\mathrm{Ad}(W_G H)} \wedge (EW_G H)_+ \wedge X^H)/W_G H) \xrightarrow{\cong} \pi_{k,\mathcal{U}}^G(X). \tag{17}$$

4.2 Some additional lemmas

We introduce a few additional lemmas that will be useful in our later computations.

Lemma 18. *Let X be a pointed space and Y be a pointed G-space. Then $\{G_+ \wedge X, Y\}^G_{\mathcal{U}} \cong \{X, Y\}$.*

Proof. This comes from the standard adjunction $(G_+ \wedge -) \rightleftharpoons \mathrm{res}^G_{\{e\}}(-)$. (See [LMSM86, Theorem 4.7] and [Ada84, Theorem 5.1]). □

Lemma 19. *Let X be a d-dimensional G-CW complex and let Y a pointed G-space. Suppose the G-action on X is free away from the base point and that the non-equivariant stable homotopy group $\pi^s_m(Y)$ is trivial for each $0 \leq m \leq d$. Then we have $\{X, Y\}^G_{\mathcal{U}} \cong 0$.*

Proof. This follows from a standard induction argument. Recall that a G-CW complex is obtained by attaching the G-cells $(G/H) \times D^k$ for various closed subgroups H inductively over k. Since the G-action on X is free away from the base point, all cells (except the base point) are of the form $G \times D^k$. Let X_k be the k-skeleton of X. Then

$$X_k/X_{k-1} = G_+ \wedge (\bigvee_{A(k)} S^{k\mathbb{R}})$$

where $A(k)$ denotes the set of k-cells in X. For any $k \leq d$, by Lemma 18, we have

$$\{G_+ \wedge (\bigvee_{A(k)} S^{k\mathbb{R}}), Y\}^G_{\mathcal{U}} \cong \{\bigvee_{A(k)} S^{k\mathbb{R}}, Y\} \cong 0.$$

Therefore, we can inductively prove that $\{X_k, Y\}^G_{\mathcal{U}} \cong 0$ using the long exact sequence

$$\cdots \to \{X_k/X_{k-1}, Y\}^G_{\mathcal{U}} \to \{X_k, Y\}^G_{\mathcal{U}} \to \{X_{k-1}, Y\}^G_{\mathcal{U}} \to \cdots.$$

induced by the cofiber sequence $X_{k-1} \to X_k \to X_k/X_{k-1}$. □

Lemma 20. *Let X be a space with a free G-action and let V be a n-dimensional G-representation. Consider the vector bundle $\widetilde{V} := (V \times X)/G$ over the space X/G. Suppose $\widetilde{V}$ is a trivial bundle. Then there is a base-point preserving, G-equivariant homeomorphism from $S^V \wedge X_+$ to $S^{n\mathbb{R}} \wedge X_+$.*

Proof. Since $\widetilde{V}$ is a trivial bundle, there exists a map $\rho_0 : (V \times X)/G \to \mathbb{R}^n$ which restricts to a linear isomorphism on any fiber. We compose this map with the quotient map $V \times X \to (V \times X)/G$ to get a G-equivariant map $\rho_1 : V \times X \to \mathbb{R}^n$. Then we consider the G-equivariant homeomorphism

$$\rho : V \times X \to \mathbb{R}^n \times X$$

defined by $\rho(\vec{v}, x) := (\rho_1(\vec{v}, x), x)$. After adding the base point at infinity, ρ induces the homeomorphism from $S^V \wedge X_+$ to $S^{n\mathbb{R}} \wedge X_+$. □

4.3 Pin(2)-equivariant stable homotopy groups

From now on, we work with the incomplete universe

$$\mathcal{U} = \mathbb{H}^\infty \oplus \widetilde{\mathbb{R}}^\infty \oplus \mathbb{R}^\infty.$$

From $\mathcal{U}$, we can define four other universes $\mathrm{res}^{\mathrm{Pin}(2)}_{S^1}(\mathcal{U})$, $\mathrm{res}^{\mathrm{Pin}(2)}_{\{e\}}(\mathcal{U})$, $\mathcal{U}^{S^1}$ and $\mathcal{U}^{\mathrm{Pin}(2)}$. We denote them by $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ and $\mathcal{U}_4$ respectively. The corresponding Lie groups are $S^1, \{e\}, C_2$ and $\{e\}$ respectively. Here $C_2 = \mathrm{Pin}(2)/S^1$ denotes the cyclic group of order 2.

We will use $S(\mathbb{H}^\infty)$ as a specific model for $E\mathrm{Pin}(2)$. Then $B\mathrm{Pin}(2)$ is the quotient of $BS^1 = \mathbb{CP}^\infty$ under the involution

$$[(z_0, z_1, z_2, z_3, \cdots)] \to [(-\overline{z_1}, \overline{z_0}, -\overline{z_3}, \overline{z_2}, \cdots)].$$

We start with the following lemmas:

Lemma 21. *Let X be a $\mathrm{Pin}(2)$-CW complex. Then for any $k \geq 0$, the kernel of the map*

$$(-)^{S^1}_{S^{k\mathbb{R}},X,\mathrm{Pin}(2)} : \pi^{\mathrm{Pin}(2)}_{k,\mathcal{U}}(X) \to \pi^{C_2}_{k,\mathcal{U}_3}(X^{S^1}) \tag{18}$$

is exactly the image of the injective map $\zeta^{\mathrm{Pin}(2)}_{\{e\}}$. In particular, the kernel of the map (18) is isomorphic to the non-equivariant stable homotopy group

$$\pi^s_k((S^{\widetilde{\mathbb{R}}} \wedge E\mathrm{Pin}(2)_+ \wedge X)/\mathrm{Pin}(2)).$$

Proof. Consider the groups

$$\begin{aligned} G_1 &= \pi^{\mathrm{Pin}(2)}_{k,\mathcal{U}}(E\mathrm{Pin}(2)_+ \wedge X), \\ G_2 &= \pi^{C_2}_{k,\mathcal{U}_3}((EC_2)_+ \wedge X^{S^1}), \\ G_3 &= \pi^{\{e\}}_{k,\mathcal{U}_4}(X^{\mathrm{Pin}(2)}). \end{aligned}$$

By Theorem 17, we have the isomorphism

$$\zeta^{\mathrm{Pin}(2)}_{\{e\}} \oplus \zeta^{\mathrm{Pin}(2)}_{S^1} \oplus \zeta^{\mathrm{Pin}(2)}_{\mathrm{Pin}(2)} : G_1 \oplus G_2 \oplus G_3 \xrightarrow{\cong} \pi^{\mathrm{Pin}(2)}_{k,\mathcal{U}}(X) \tag{19}$$

and the isomorphism

$$\zeta^{C_2}_{C_2} \oplus \zeta^{C_2}_{\{e\}} : G_2 \oplus G_3 \xrightarrow{\cong} \pi^{C_2}_{k,\mathcal{U}_3}(X^{S^1}). \tag{20}$$

Consider the following diagram

$$\begin{array}{ccc} G_1 \oplus G_2 \oplus G_3 & \xrightarrow{\zeta^{\mathrm{Pin}(2)}_{\{e\}} \oplus \zeta^{\mathrm{Pin}(2)}_{S^1} \oplus \zeta^{\mathrm{Pin}(2)}_{\mathrm{Pin}(2)}} & \pi^{\mathrm{Pin}(2)}_{k,\mathcal{U}}(X) \\ \downarrow \pi & & \downarrow (-)^{S^1}_{S^k,X,\mathrm{Pin}(2)} \\ G_2 \oplus G_3 & \xrightarrow{\zeta^{C_2}_{C_2} \oplus \zeta^{C_2}_{\{e\}}} & \pi^{C_2}_{k,\mathcal{U}_3}(X^{S^1}) \end{array},$$

where π is the projection to the second and third summand. By checking the explicit description of ζ_H^G in (13), it is straightforward to see that this diagram commutes. By Theorem 17, both horizontal maps are isomorphisms. Therefore, the kernel of $(-)^{S^1}_{S^{k\mathbb{R}},X,\mathrm{Pin}(2)}$ is exactly the image of $\zeta^{\mathrm{Pin}(2)}_{\{e\}}$. We then apply (16) to finish the proof. □

Remark 22. Only $\{e\}, S^1, \mathrm{Pin}(2)$ arise as the isotropy groups of points in the incomplete universe $\mathcal{U}$. However, if one sets $\mathcal{U}$ to be the complete universe, then there will be some other isotropy groups involved in the decomposition (19) (e.g. the cyclic group C_4 generated by j). As a result, the kernel of the map (18) will be strictly larger than the image of $\zeta^{\mathrm{Pin}(2)}_{\{e\}}$. This is the main reason that we use the incomplete universe $\mathbb{H}^\infty \oplus \widetilde{\mathbb{R}}^\infty \oplus \mathbb{R}^\infty$.

Using the same argument, one can prove the following lemma.

Lemma 23. *Let X be an S^1-CW complex. Then the kernel of the map*

$$(-)^{S^1}_{S^k,X,S^1} : \pi^{S^1}_{k,\mathcal{U}_1}(X) \to \pi^s_k(X^{S^1})$$

equals the image of the injective map $\zeta^{S^1}_{\{e\}} : \pi^{S^1}_{k,\mathcal{U}_1}(ES^1_+ \wedge X) \to \pi^{S^1}_{k,\mathcal{U}_1}(X)$. In particular, the kernel is isomorphic to $\pi^s_{k-1}((ES^1_+ \wedge X)/S^1)$.

The following propositions (Proposition 24∼ Proposition 26) are the key results in this section. They are the homotopy theoretic version of Theorem 9.

Proposition 24. *Suppose $m,n \geq 0$ and m,n are not both 0. Then the map*

$$(-)^{S^1}_{S^{\mathbb{R}},S^{m\mathbb{H}+n\widetilde{\mathbb{R}}},\mathrm{Pin}(2)} : \pi^{\mathrm{Pin}(2)}_{1,\mathcal{U}}(S^{m\mathbb{H}+n\widetilde{\mathbb{R}}}) \to \pi^{C_2}_{1,\mathcal{U}_3}(S^{n\widetilde{\mathbb{R}}})$$

is injective.

Proposition 25. *Suppose $n > 4m \geq 0$. Then the map*

$$(-)^{S^1}_{S^{\mathbb{R}+m\mathbb{H}},S^{n\widetilde{\mathbb{R}}},\mathrm{Pin}(2)} : \{S^{\mathbb{R}+m\mathbb{H}}, S^{n\widetilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}_{\mathcal{U}} \to \pi^{C_2}_{1,\mathcal{U}_3}(S^{n\widetilde{\mathbb{R}}})$$

is injective.

Proposition 26. *Suppose $n = 4m \geq 0$. Then the map*

$$(-)^{S^1}_{S^{\mathbb{R}+m\mathbb{H}},S^{n\widetilde{\mathbb{R}}},\mathrm{Pin}(2)} : \{S^{\mathbb{R}+m\mathbb{H}}, S^{n\widetilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}_{\mathcal{U}} \to \pi^{C_2}_{1,\mathcal{U}_3}(S^{n\widetilde{\mathbb{R}}})$$

has a kernel isomorphic to $\mathbb{Z}/2$. Furthermore, let α be the nonzero element in this kernel. Then we have $\mathrm{res}^{\mathrm{Pin}(2)}_{S^1}(\alpha) = 0$.

Proposition 27. *The map*

$$(-)^{S^1}_{S^{2\mathbb{R}},S^0,\mathrm{Pin}(2)} : \pi^{\mathrm{Pin}(2)}_{2,\mathcal{U}}(S^0) \to \pi^{C_2}_{2,\mathcal{U}_3}(S^0)$$

has a kernel isomorphic to $\mathbb{Z}/2$. Furthermore, let α' be the nonzero element in this kernel. Then we have $\mathrm{res}^{\mathrm{Pin}(2)}_{S^1}(\alpha') = 0$.

Proof of Proposition 24. By Lemma 21, we have

$$\mathrm{Ker}(-)^{S^1}_{S^{\mathbb{R}},S^{m\mathbb{H}+n\widetilde{\mathbb{R}}},\mathrm{Pin}(2)} \cong \pi_1^s((S^{m\mathbb{H}+(n+1)\widetilde{\mathbb{R}}} \wedge E\mathrm{Pin}(2)_+)/\mathrm{Pin}(2)). \tag{21}$$

Let $\xi_{\widetilde{\mathbb{R}}}$ (resp. $\xi_{\mathbb{H}}$) be the bundle associated to the universal bundle

$$\mathrm{Pin}(2) \hookrightarrow E\mathrm{Pin}(2) \to B\mathrm{Pin}(2)$$

and the representations $\widetilde{\mathbb{R}}$ (resp. $\mathbb{H}$). Then we have

$$(S^{m\mathbb{H}+(n+1)\widetilde{\mathbb{R}}} \wedge E\mathrm{Pin}(2)_+)/\mathrm{Pin}(2) = \mathrm{Th}(B\mathrm{Pin}(2), m\xi_{\mathbb{H}} \oplus (n+1)\xi_{\widetilde{\mathbb{R}}}).$$

Here $\mathrm{Th}(-)$ denotes the Thom space.

Note that $m\xi_{\mathbb{H}} \oplus (n+1)\xi_{\widetilde{\mathbb{R}}}$ is a vector bundle of real dimension $4m+n+1$. Therefore, other than the base point, the space $\mathrm{Th}(B\mathrm{Pin}(2), m\xi_{\mathbb{H}} \oplus (n+1)\xi_{\widetilde{\mathbb{R}}})$ has a decomposition for which there are no cells of dimension ≤ 1. By the skeletal approximation theorem, we have

$$\mathrm{Ker}(-)^{S^1}_{S^{\mathbb{R}},S^{m\mathbb{H}+n\widetilde{\mathbb{R}}},\mathrm{Pin}(2)} \cong \pi_1^s(\mathrm{Th}(B\mathrm{Pin}(2), m\xi_{\mathbb{H}} \oplus (n+1)\xi_{\widetilde{\mathbb{R}}})) \cong 0.$$

□

Proof of Proposition 25. The case $m = 0$ has been covered by Proposition 24 so we assume $m \geq 1$, which implies that $n \geq 5$. Take

$$\beta \in \mathrm{Ker}((-)^{S^1}_{S^{\mathbb{R}+m\mathbb{H}},S^{n\widetilde{\mathbb{R}}},\mathrm{Pin}(2)}).$$

Take the inclusion $i_0 : S^{\mathbb{R}} \to S^{\mathbb{R}+m\mathbb{H}}$. We also use i_0 to denote the corresponding element in $\{S^{\mathbb{R}}, S^{\mathbb{R}+m\mathbb{H}}\}^{\mathrm{Pin}(2)}_{\mathcal{U}}$. Then we have

$$(-)^{S^1}_{S^{\mathbb{R}},S^{n\widetilde{\mathbb{R}}},\mathrm{Pin}(2)}(\beta \cdot i_0) = (-)^{S^1}_{S^{\mathbb{R}+m\mathbb{H}},S^{n\widetilde{\mathbb{R}}},\mathrm{Pin}(2)}(\beta) \cdot (-)^{S^1}_{S^{\mathbb{R}},S^{\mathbb{R}+m\mathbb{H}},\mathrm{Pin}(2)}(i_0) = 0. \tag{22}$$

But $\mathrm{Ker}((-)^{S^1}_{S^{\mathbb{R}},S^{n\widetilde{\mathbb{R}}},\mathrm{Pin}(2)})$ is trivial by Proposition 24. Hence we see that $\beta \cdot i_0 = 0$.

Next, we consider the long exact sequence

$$\begin{aligned} \cdots \to \{S^{2\mathbb{R}} \wedge (S(m\mathbb{H})_+), S^{n\widetilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}_{\mathcal{U}} &\to \{S^{\mathbb{R}+m\mathbb{H}}, S^{n\widetilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}_{\mathcal{U}} \\ &\xrightarrow{-\cdot i_0} \{S^{\mathbb{R}}, S^{n\widetilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}_{\mathcal{U}} \to \cdots \end{aligned} \tag{23}$$

induced by the cofiber sequence

$$S^0 \to S^{m\mathbb{H}} \to S^{\mathbb{R}} \wedge S(m\mathbb{H})_+. \tag{24}$$

Note that $\mathrm{Pin}(2)$ acts freely on $S^{2\mathbb{R}} \wedge (S(m\mathbb{H})_+)$ away from the base point, and the quotient is a CW complex of dimension $4m$. Since $n > 4m$, by Lemma 19, we see that $\{S^{2\mathbb{R}} \wedge (S(m\mathbb{H})_+), S^{n\widetilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}_{\mathcal{U}} \cong 0$. Hence the map $- \cdot i_0$ is injective. Since $\beta \cdot i_0 = 0$, we get $\beta = 0$. □

We need some preparations in order to prove Proposition 26.

Lemma 28. *(1) For $m > 0$, we have*

$$\{S^{2\mathbb{R}} \wedge S(m\mathbb{H})_+, S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} \cong \mathbb{Z}/2.$$

(2) Let γ be the unique nonzero element in $\{S^{2\mathbb{R}} \wedge S(m\mathbb{H})_+, S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}$. Then $\mathrm{res}_{S^1}^{\mathrm{Pin}(2)}(\gamma) = 0$.

Proof. (1) The $\mathrm{Pin}(2)$-action on $S^{2\mathbb{R}} \wedge S((m-1)\mathbb{H})_+$ is free away from the base point and the quotient is a CW-complex of dimension $4m-4$. (Here we set $S(0\mathbb{H}) = \emptyset$.) By Lemma 19, we have

$$\begin{aligned} &\{S^{2\mathbb{R}} \wedge S((m-1)\mathbb{H})_+, S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} \\ \cong &\{S^{3\mathbb{R}} \wedge S((m-1)\mathbb{H})_+, S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} \\ \cong &0. \end{aligned} \tag{25}$$

Consider the cofiber sequence

$$S((m-1)\mathbb{H})_+ \hookrightarrow S(m\mathbb{H})_+ \to S(m\mathbb{H})/S((m-1)\mathbb{H}).$$

we have the induced long exact sequence of homotopy groups

$$\begin{aligned} \cdots \to &\{S^{3\mathbb{R}} \wedge S((m-1)\mathbb{H})_+, S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} \\ \to &\{S^{2\mathbb{R}} \wedge S(m\mathbb{H})/S((m-1)\mathbb{H}), S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} \\ \to &\{S^{2\mathbb{R}} \wedge S(m\mathbb{H})_+, S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} \\ \to &\{S^{2\mathbb{R}} \wedge S((m-1)\mathbb{H})_+, S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} \to \cdots. \end{aligned}$$

Note that $S(m\mathbb{H})/S((m-1)\mathbb{H}) = S^{(m-1)\mathbb{H}} \wedge (S(\mathbb{H})_+)$. By the vanishing result (25), we get

$$\{S^{2\mathbb{R}} \wedge S(m\mathbb{H})_+, S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} \cong \{S^{2\mathbb{R}+(m-1)\mathbb{H}} \wedge (S(\mathbb{H})_+), S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}.$$

To compute this group, we consider the vector bundles $\xi'_{\mathbb{H}}$ (resp. $\xi'_{4\widetilde{\mathbb{R}}}$) associated to the principal bundle

$$\mathrm{Pin}(2) \hookrightarrow S(\mathbb{H}) \to \mathbb{RP}^2$$

for the representations $\mathbb{H}$ (resp. $4\widetilde{\mathbb{R}}$). Since the structure group of both bundles can be lifted to $SU(2)$ and any $SU(2)$ bundle is trivial on a 2-dimensional CW complex, we see that $\xi'_{\mathbb{H}}$ and $\xi'_{4\widetilde{\mathbb{R}}}$ are both trivial. By Lemma 20, we have the following $\mathrm{Pin}(2)$-equivariant homeomorphisms

$$S^{4\mathbb{R}+(m-1)\mathbb{H}} \wedge (S(\mathbb{H})_+) \cong S^{4m\mathbb{R}} \wedge (S(\mathbb{H})_+) \cong S^{4m\widetilde{\mathbb{R}}} \wedge (S(\mathbb{H})_+).$$

They give the following isomorphism

$$\begin{aligned}\{S^{2\mathbb{R}+(m-1)\mathbb{H}}\wedge(S(\mathbb{H})_+),S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} &\cong \{S^{4\mathbb{R}+(m-1)\mathbb{H}}\wedge(S(\mathbb{H})_+),S^{4m\widetilde{\mathbb{R}}+2\mathbb{R}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}\\ &\cong \{S^{4m\widetilde{\mathbb{R}}}\wedge(S(\mathbb{H})_+),S^{4m\widetilde{\mathbb{R}}+2\mathbb{R}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}\\ &\cong \{S(\mathbb{H})_+,S^{2\mathbb{R}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}\\ &\cong \{\mathbb{RP}^2_+,S^2\}\\ &\cong \mathbb{Z}/2.\end{aligned}$$

(2) The element $\mathrm{res}^{\mathrm{Pin}(2)}_{S^1}(\gamma)$ belongs to the group

$$\{S^{2\mathbb{R}}\wedge S(m\mathbb{H})_+,S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}_3}^{S^1}\cong\{S(m\mathbb{H})_+,S^{(4m-2)\widetilde{\mathbb{R}}}\}_{\mathcal{U}_3}^{S^1}.$$

Since S^1 acts freely on $S(m\mathbb{H})$ and trivially on $S^{(4m-2)\widetilde{\mathbb{R}}}$, we have

$$\{S(m\mathbb{H})_+,S^{(4m-2)\widetilde{\mathbb{R}}}\}_{\mathcal{U}_3}^{S^1}\cong\{(S(m\mathbb{H})_+)/S^1,S^{(4m-2)\widetilde{\mathbb{R}}}\}\cong\{\mathbb{CP}^{2m-1}_+,S^{4m-2}\}\cong\mathbb{Z}.$$

Since γ is 2-torsion, $\mathrm{res}^{\mathrm{Pin}(2)}_{S^1}(\gamma)$ must be zero. □

Lemma 29. *For $m>0$, the map*

$$q^*:\{S^{2\mathbb{R}}\wedge(S(m\mathbb{H})_+),S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}\to\{S^{\mathbb{R}+m\mathbb{H}},S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} \tag{26}$$

induced by the quotient map $q:S^{m\mathbb{H}}\to S^{m\mathbb{H}}/S^0=S^{\mathbb{R}}\wedge(S(m\mathbb{H})_+)$ is injective.

Proof. We consider the long exact sequence

$$\cdots\to\{S^{2\mathbb{R}},S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}\xrightarrow{p_1^*}\{S^{2\mathbb{R}}\wedge(S(m\mathbb{H})_+),S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}\xrightarrow{q^*}\{S^{\mathbb{R}+m\mathbb{H}},S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}\to\cdots$$

induced by the cofiber sequence $S(m\mathbb{H})_+\xrightarrow{p_1}S^0\xrightarrow{i_1}S^{m\mathbb{H}}$, where i_1 is the inclusion map and $p_1:S(m\mathbb{H})_+\to S^0$ is the map that collapses the whole $S(m\mathbb{H})$ to the non-base point in S^0. It suffices to show that the map p_1^* is trivial. By (17), we have

$$\begin{aligned}\{S^{2\mathbb{R}},S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}\cong&\{S^2,(S^{(4m+1)\widetilde{\mathbb{R}}}\wedge E\mathrm{Pin}(2)_+)/\mathrm{Pin}(2)\}\oplus\\ &\{S^2,(S^{4m\widetilde{\mathbb{R}}}\wedge(EC_2)_+)/C_2\}\oplus\{S^2,S^0\}.\end{aligned}$$

Since $m>0$, the first two summands vanish by the skeletal approximation theorem. Hence we have

$$\{S^{2\mathbb{R}},S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}\cong\{S^2,S^0\}\cong\mathbb{Z}/2,$$

and it is generated by $\zeta^{\mathrm{Pin}(2)}_{\mathrm{Pin}(2)}(\eta^2)$. (Here $\eta\in\pi_1(S^0)$ is the Hopf map.) We need to show that $p_1^*(\zeta^{\mathrm{Pin}(2)}_{\mathrm{Pin}(2)}(\eta^2))=0$. By the definition of $\zeta^{\mathrm{Pin}(2)}_{\mathrm{Pin}(2)}$ in (13), $\zeta^{\mathrm{Pin}(2)}_{\mathrm{Pin}(2)}(\eta^2)$ is equal to the following composition of stable maps

$$S^2\xrightarrow{\eta^2}S^0\xrightarrow{i_1}S^{4m\widetilde{\mathbb{R}}},$$

where $i_1 : S^0 \to S^{4m\widetilde{\mathbb{R}}}$ is the inclusion. Therefore, $p_1^*(\zeta_{\mathrm{Pin}(2)}^{\mathrm{Pin}(2)}(\eta^2))$ is represnted by the composition

$$S^{2\mathbb{R}} \wedge S(m\mathbb{H}) \xrightarrow{\eta^2 \wedge \mathrm{id}} S(m\mathbb{H}) \xrightarrow{p_1} S^0 \xrightarrow{i_1} S^{4m\widetilde{\mathbb{R}}}.$$

So it suffices to show that the composition

$$S(m\mathbb{H})_+ \xrightarrow{p_1} S^0 \xrightarrow{i_1} S^{4m\widetilde{\mathbb{R}}} \tag{27}$$

is stably null-homotopic. By the exact sequence

$$\cdots \to \{S^{m\mathbb{H}}, S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} \to \{S^0, S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} \xrightarrow{p_1^*} \{S(m\mathbb{H})_+, S^{4m\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)} \to \cdots, \tag{28}$$

the composition (27) is trivial if $i_1 : S^0 \to S^{4m\widetilde{\mathbb{R}}}$ can be extended to a map $\tilde{i} : S^{m\mathbb{H}} \to S^{4m\widetilde{\mathbb{R}}}$. Note that there is a $\mathrm{Pin}(2)$-equivariant map $\varphi : S^{\mathbb{H}} \to S^{3\widetilde{\mathbb{R}}}$ defined by

$$\varphi(z + wj) := (|z|^2 - |w|^2, \mathrm{Re}(z\bar{w}), \mathrm{Im}(z\bar{w})).$$

By smashing m copies of φ and the inclusion map $S^0 \hookrightarrow S^{m\widetilde{\mathbb{R}}}$, we get the desired extension $\widetilde{i}$. □

Proof of Proposition 26. We first consider the case $m = n = 0$. By (21), we get

$$\mathrm{Ker}(-)_{S^{\mathbb{R}}, S^0, \mathrm{Pin}(2)}^{S^1} \cong \pi_1^s(\mathrm{Th}(B\mathrm{Pin}(2), \xi_{\widetilde{\mathbb{R}}})).$$

Note that the 3-skeleton of $B\mathrm{Pin}(2)$ is $\mathbb{RP}^2$ and the restriction of $\xi_{\widetilde{\mathbb{R}}}$ to the 3-skeleton is exactly the tautological line bundle over it. Therefore, the 4-skeleton of $\mathrm{Th}(B\mathrm{Pin}(2), \xi_{\widetilde{\mathbb{R}}})$ is $\mathbb{RP}^3$. This implies that

$$\mathrm{Ker}(-)_{S^1, S^0, \mathrm{Pin}(2)}^{S^1} \cong \pi_1^s(\mathbb{RP}^3) \cong \mathbb{Z}/2.$$

Let α be the nonzero element in this kernel. By definition, $\mathrm{res}_{S^1}^{\mathrm{Pin}(2)}(\alpha)$ belongs to the kernel of the map

$$(-)_{S^1, S^{\widetilde{\mathbb{R}}}, S^1}^{S^1} : \pi_{1, \mathcal{U}_2}^{S^1}(S^0) \to \pi_1^s(S^0).$$

By Lemma 23, we have

$$\mathrm{Ker}(-)_{S^1, S^{\widetilde{\mathbb{R}}}, S^1}^{S^1} \cong \pi_0^s(\mathbb{CP}_+^\infty) \cong \mathbb{Z}.$$

Since α is 2-torsion, we must have $\mathrm{res}_{S^1}^{\mathrm{Pin}(2)}(\alpha) = 0$.

Now we consider the case $n = 4m > 0$. We take a element

$$\beta \in \mathrm{Ker}(-)_{S^{\mathbb{R}+m\mathbb{H}}, S^{n\widetilde{\mathbb{R}}}, \mathrm{Pin}(2)}^{S^1}.$$

Similar to the proof of Proposition 25, we consider the inclusion

$$i_0 : S^{\mathbb{R}} \to S^{\mathbb{R}+m\mathbb{H}}.$$

Then $\beta \cdot i_0$ belongs $\mathrm{Ker}((-)^{S^1}_{S^{\mathbb{R}},S^{n\widetilde{\mathbb{R}}},\mathrm{Pin}(2)})$, which is trivial by Proposition 24. So we have $\beta \cdot i_0 = 0$. By the long exact sequence (23) again, β belongs to the image of the map

$$\{S^{2\mathbb{R}} \wedge (S(m\mathbb{H})_+), S^{n\widetilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}_{\mathcal{U}} \xrightarrow{q^*} \{S^{\mathbb{R}+m\mathbb{H}}, S^{n\widetilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}_{\mathcal{U}}.$$

(See (26).) By Lemma 28, $\{S^{2\mathbb{R}} \wedge (S(m\mathbb{H})_+), S^{n\widetilde{\mathbb{R}}}\}^{\mathrm{Pin}(2)}_{\mathcal{U}} = \{0, \gamma\}$. So β is either 0 or $q^*(\gamma)$. By Lemma 29, $q^*(\gamma) \neq 0$. Also note that $(-)^{S^1}(\gamma) = 0$ because the S^1-action on its domain $S^{2\mathbb{R}} \wedge S(m\mathbb{H})_+$ is free away from the base point. This implies that $q^*(\gamma)$ actually belongs to $\mathrm{Ker}((-)^{S^1}_{S^{\mathbb{R}+m\mathbb{H}},S^{n\widetilde{\mathbb{R}}},\mathrm{Pin}(2)})$. To this end, we see that

$$\mathrm{Ker}((-)^{S^1}_{S^{\mathbb{R}+m\mathbb{H}},S^{n\widetilde{\mathbb{R}}},\mathrm{Pin}(2)}) = \{0, q^*(\gamma)\} \cong \mathbb{Z}/2.$$

Since $\mathrm{res}^{\mathrm{Pin}(2)}_{S^1}(\gamma) = 0$ by Lemma 28, we have $\mathrm{res}^{\mathrm{Pin}(2)}_{S^1}(q^*_1(\gamma)) = 0$. □

Proof of Proposition 27. By Lemma 21, we have

$$\mathrm{Ker}(-)^{S^1}_{S^{2\mathbb{R}},S^0,\mathrm{Pin}(2)} \cong \pi^s_2(\mathrm{Th}(B\mathrm{Pin}(2), \xi_{\widetilde{\mathbb{R}}})) \cong \pi^s_2(\mathbb{RP}^3).$$

Stably, $\mathbb{RP}^3$ splits into $S^3 \vee \mathbb{RP}^2$. Note that $\mathbb{RP}^2$ fits in to the cofiber sequence

$$S^1 \xrightarrow{\cdot 2} S^1 \to \mathbb{RP}^2$$

Using the induced long exact sequence of stable homotopy groups, we get an isomorphism

$$\mathrm{Ker}(-)^{S^1}_{S^{2\mathbb{R}},S^0,\mathrm{Pin}(2)} \cong \pi^s_2(\mathbb{RP}^3) \xrightarrow[\eta\cdot -]{\cong} \pi^s_1(\mathbb{RP}^3) \cong \mathbb{Z}/2,$$

where $\eta \in \pi^s_1(S^0)$ denotes the Hopf map. Let α' be the nonzero element in this kernel. Then since the smash product with η commutes with the Adams isomorphism (16), we have $\alpha' = \eta \cdot \alpha$, where $\alpha \in \mathrm{Ker}(-)^{S^1}_{S^{\mathbb{R}},S^0,\mathrm{Pin}(2)}$ is the element established in Proposition 26. By that proposition, we get

$$\mathrm{res}^{\mathrm{Pin}(2)}_{S^1}(\alpha') = \eta \cdot \mathrm{res}^{\mathrm{Pin}(2)}_{S^1}(\alpha) = 0.$$

□

5 Properties of The Family Bauer–Furuta Invariant

In this section, we use the homotopy theoretic results that we proved in Section 4 to study the family Bauer–Furuta invariant. We will start with proving the vanishing properties of the family Bauer–Furuta invariant. Then we give a space level construction of the stable homotopy class α_0 given in Theorem 8, which provides a potential tool to detect exotic diffeomorphisms on S^4.

5.1 Proof of the vanishing results.

To prove Theorem 9, we give the following definition, which is motivated by the construction of the family Bauer–Furuta map defined in Section 2.

Definition 30. Let α be an element in one of the following stable homotopy groups:

- $\{S^{k\mathbb{R}} \wedge S^{m\mathbb{H}}, S^{n\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}$ for some integers $k,m,n \geq 0$, or
- $\{S^{k\mathbb{R}}, S^{n\widetilde{\mathbb{R}}+(-m)\mathbb{H}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}$ for some integers $k,n \geq 0$ and $m < 0$.

We say α is an element *of BF-type* if it satisfies the following condition

$$(-)^{S^1}(\alpha) = 0 \in \{S^{k\mathbb{R}}, S^{n\widetilde{\mathbb{R}}}\}_{\mathcal{U}_3}^{C_2}.$$

Namely, the restriction of α to the S^1-fixed points is trivial as a C_2-equivariant stable homotopy class.

Lemma 31. *Let X be a spin 4-manifold with $b_1(X) = 0$. Then we have the following results:*

(i) Let f be an orientation preserving self-diffeomorphism on X that preserves the spin structure and acts trivially on $H^*(X;\mathbb{R})$. Let $\widetilde{f}$ be an automorphism of the spin structure on X that lifts f. Then $\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f})$ is an element of BF-type.

(ii) Let $\gamma : S^1 \to \mathrm{Diff}(X)$ be a loop based at id_X. Suppose γ is spin-liftable (see Definition 11). Then $\mathrm{FBF}^{\mathrm{Pin}(2)}(X,\gamma)$ is an element of BF-type.

Proof. We focus on the proof of (i) since (ii) is similar. Consider the approximated Seiberg-Witten map

$$sw : W^+ \to W^-$$

(see (5)). By the explicit description of the $\mathrm{Pin}(2)$-action in (2) and (3), the S^1-fixed points of $W^{\pm}$ are given by

$$(W^{\pm})^{S^1} = U^{\pm}.$$

When restricted to U^+, the quadratic part c of the Seiberg-Witten map SW equals 0. By the explicit description of sw in Section 2, we see that the map

$$sw^{S^1} : U^+ \to U^-$$

is just the linear map (d^+, d^*). Under our preferred choice of the trivialization of $U^{\pm}$ in Section 2, this map correspond to the standard inclusion $i_2 : \widetilde{\mathbb{R}}^{n'} \hookrightarrow \widetilde{\mathbb{R}}^{n'+b_2^+(X)}$ into the first n' components. From this, we see that the map

$$(\widetilde{sw}^+)^{S^1} : S^1_+ \wedge S^{n'\widetilde{\mathbb{R}}} \to S^{(n'+b_2^+(X))\widetilde{\mathbb{R}}}$$

equals

$$(S(2\mathbb{R})_+) \wedge S^{n'\widetilde{\mathbb{R}}} \xrightarrow{p_2 \wedge i_2^+} S^0 \wedge S^{(n'+b_2^+(X))\widetilde{\mathbb{R}}} \tag{29}$$

where $p_2 : S(2\mathbb{R})_+ \to S^0$ preserves the base point and collapses $S(2\mathbb{R})$ to the other point in S^0, and $i_2^+ : S^{n'\widetilde{\mathbb{R}}} \to S^{(n'+b_2^+(X))\widetilde{\mathbb{R}}}$ is one-point compactification of i_2.

Recall that $\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f})$ is defined via the composition (6). From our discussion above, the element $(-)^{S^1}(\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f}))$ can be represented by the following composition

$$S^{2\mathbb{R}} \wedge S^{n'\widetilde{\mathbb{R}}} \xrightarrow{c_0 \wedge \mathrm{id}_{S^{n'\widetilde{\mathbb{R}}}}} S^{\mathbb{R}} \wedge (S(2\mathbb{R})_+) \wedge S^{n'\widetilde{\mathbb{R}}} \xrightarrow{\mathrm{id}_{S^{\mathbb{R}}} \wedge p_2 \wedge i_2^+} S^{\mathbb{R}} \wedge S^0 \wedge S^{((b_2^+(X)+n')\widetilde{\mathbb{R}}}.$$

Note that $(\mathrm{id}_{S^{\mathbb{R}}} \wedge p_2) \circ c_0 : S^{2\mathbb{R}} \to S^{\mathbb{R}}$ is null-homotopic. Thus we have

$$(-)^{S^1}(\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f})) = 0.$$

□

The following proposition is just a restatement of Proposition 24 ∼ Proposition 27.

Proposition 32. *(i) Suppose $m,n \geq 0$ and are not both zero. Then $\pi_{1,\mathcal{U}}^{\mathrm{Pin}(2)}(S^{m\mathbb{H}+n\widetilde{\mathbb{R}}})$ doesn't contain any nonzero element of BF-type.*

(ii) Suppose $n > 4m \geq 0$. Then $\{S^{\mathbb{R}+m\mathbb{H}}, S^{n\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}$ doesn't contain any nonzero element of BF-type.

(iii) Suppose $n = 4m \geq 0$. Then $\{S^{\mathbb{R}+m\mathbb{H}}, S^{n\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}$ contains a unique nonzero element of BF-type, denoted by α_0. Furthermore, we have $\mathrm{res}_{S^1}^{\mathrm{Pin}(2)}(\alpha_0) = 0$.

(iv) $\pi_2^{\mathrm{Pin}(2)}(S^0)$ contains an unique nonzero element of BF-type, denoted by α_1. Furthermore, we have $\mathrm{res}_{S^1}^{\mathrm{Pin}(2)}(\alpha_1) = 0$.

Proof of Theorem 6. Let γ an exotic loop in $\mathrm{Diff}(S^4)$. Then γ is spin-liftable. By Lemma 31, $\mathrm{FBF}^{\mathrm{Pin}(2)}(S^4,\gamma)$ is an element in $\pi_2^{\mathrm{Pin}(2)}(S^0)$ of BF-type. By Proposition 32 (iv), $\mathrm{FBF}^{\mathrm{Pin}(2)}(S^4,\gamma)$ is either 0 or α_1. So we have

$$\mathrm{FBF}^{S^1}(S^4,\gamma) = \mathrm{res}_{S^1}^{\mathrm{Pin}(2)}(\mathrm{FBF}^{\mathrm{Pin}(2)}(S^4,\gamma)) = 0.$$

This further implies that $\mathrm{FBF}^{\{e\}}(S^4,\gamma) = 0$.

Now suppose γ is homotopic to $2\gamma'$ for some loop γ' in $\mathrm{Diff}(S^4)$. We may assume that γ' is also spin-liftable by compositing it with an essential loop in $\mathrm{SO}(5) \subset \mathrm{Diff}(S^4)$. By Proposition 32 (iv) again, $\mathrm{FBF}^{\mathrm{Pin}(2)}(S^4,\gamma')$ is either 0 or α_1. Since α_1 is 2-torsion, we have

$$\mathrm{FBF}^{\mathrm{Pin}(2)}(S^4,\gamma) = 2\,\mathrm{FBF}^{\mathrm{Pin}(2)}(S^4,\gamma') = 0.$$

□

Proof of Theorem 8. This is a special case of Proposition 32 (iii) for $m = n = 0$. □

Proof of Theorem 9: Assume the intersection form of X is isomorphic to $-2mE8 \oplus n\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. (Here $E8$ is positive definite.) Then $\chi(X) = 2+16|m|+2n$ and $\sigma(X) = -16m$. Therefore, we have

$$2\chi(X)+3\sigma(X) = \begin{cases} 4+4\cdot(n-4m) & \text{if } m \geq 0 \\ 4+4\cdot(n-20m) & \text{if } m < 0 \end{cases}.$$

Suppose $2\chi(X)+3\sigma(X) = 4$. Then $m \geq 0$ and $n = 4m$. By (7) and Lemma 31, $\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f})$ is an element in $\{S^{\mathbb{R}+m\mathbb{H}}, S^{n\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}$ of BF-type. Therefore, by Proposition 32 (iii), we have

$$\mathrm{FBF}^{S^1}(X,f,\widetilde{f}) = \mathrm{res}_{S^1}^{\mathrm{Pin}(2)}(\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f})) = 0.$$

This proves (1).

Suppose $2\chi(X)+3\sigma(X) > 4$. There are two cases to consider.

- If $m \geq 0$, then $n > 4m$. Then $\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f})$ is an element of BF-type in the group $\{S^{\mathbb{R}+m\mathbb{H}}, S^{n\widetilde{\mathbb{R}}}\}_{\mathcal{U}}^{\mathrm{Pin}(2)}$. By Proposition 32 (ii), we have $\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f}) = 0$.
- If $m < 0$, then $\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f})$ is an element of BF-type in the group $\pi_{1,\mathcal{U}}^{\mathrm{Pin}(2)}(S^{-m\mathbb{H}+n\widetilde{\mathbb{R}}})$. By Proposition 32 (i), we have
$$\mathrm{FBF}^{\mathrm{Pin}(2)}(X,f,\widetilde{f}) = 0.$$

This finishes the proof of (2). □

Remark 33. We mention various examples of nonvanishing family Bauer-Furuta invariants. These examples show that various conditions in Theorem 9 are indeed necessary.

- Consider the Dehn twist $\delta_{K3\#K3}$ along the separating neck in $K3\#K3$. Then Kronheimer–Mrowka [KM20, Proof of Theorem 1.1] prove that
$$\mathrm{FBF}^{\{e\}}(K3\#K3, \delta_{K3\#K3}, \widetilde{\delta}) = \eta^3 \neq 0$$
for any spin bundle automorphism $\widetilde{\delta}$ which lifts $\delta_{K3\#K3}$. Here $\eta \in \pi_1^s(S^0)$ denotes the Hopf element. Note that $2\chi(K3\#K3)+3\sigma(K3\#K3) = -4$.
- Consider the covering transformation on the spin bundle over $K3$. This is a lift of id_{K3}, denoted by $\widetilde{\mathrm{id}}_{K3}$. Then Kronheimer–Mrowka [KM20, Corollary 4.2] prove that
$$\mathrm{FBF}^{\{e\}}(K3, \mathrm{id}_{K3}, \widetilde{\mathrm{id}}_{K3}) = \eta^2 \neq 0.$$
Note that $2\chi(K3)+3\sigma(K3) = 0$.
- Consider the covering transformation on the spin bundle over $K3\#(S^2\times S^2)$, denoted by $\widetilde{\mathrm{id}}_{K3\#(S^2\times S^2)}$. Then by [Lin20, Proof of Theorem 1.2], we have
$$\mathrm{FBF}^{\mathrm{Pin}(2)}(K3\#(S^2\times S^2), \mathrm{id}_{K3\#(S^2\times S^2)}, \widetilde{\mathrm{id}}_{K3\#(S^2\times S^2)}) \neq 0.$$
Note that $2\chi(K3\#(S^2\times S^2))+3\sigma(K3\#(S^2\times S^2)) = 4$. Hence by Theorem 9, we have
$$\begin{aligned}&\mathrm{FBF}^{S^1}(K3\#(S^2\times S^2), \mathrm{id}_{K3\#(S^2\times S^2)}, \widetilde{\mathrm{id}}_{K3\#(S^2\times S^2)}) \\ =&\mathrm{FBF}^{\{e\}}(K3\#(S^2\times S^2), \mathrm{id}_{K3\#(S^2\times S^2)}, \widetilde{\mathrm{id}}_{K3\#(S^2\times S^2)}) = 0.\end{aligned}$$

5.2 A space level description of α_0

The element $\alpha_0 \in \pi_{1,\mathfrak{U}}^{\mathrm{Pin}(2)}(S^0)$ given in Theorem 8 is of special interest because it could arises as the $\mathrm{Pin}(2)$-equivariant family Bauer–Furuta invariant of an exotic diffeomorphism on S^4. In this section, we explicitly construct a map

$$l : S^{\mathbb{R}+\mathbb{H}} \to S^{\mathbb{H}}$$

which represents α_0.

We treat $\mathrm{Pin}(2)$ as a submanifold of $\mathbb{H}$, with the standard embedding

$$\mathrm{Pin}(2) = \{e^{i\theta}\} \cup \{je^{i\theta}\} \hookrightarrow \mathbb{H}.$$

Denote the tangent bundle and the normal bundle of $\mathrm{Pin}(2)$ by $T_*\mathrm{Pin}(2)$ and $N_*\mathrm{Pin}(2)$ respectively. Both bundles are $\mathrm{Pin}(2)$-bundles. Namely, there are $\mathrm{Pin}(2)$-actions on both bundles that cover the left-multiplication action of $\mathrm{Pin}(2)$ on itself. Furthermore, we have trivailizations

$$T_*\mathrm{Pin}(2) \cong \mathbb{R} \times \mathrm{Pin}(2), \tag{30}$$

$$T_*\mathrm{Pin}(2) \oplus N_*\mathrm{Pin}(2) \cong \mathbb{H} \times \mathrm{Pin}(2). \tag{31}$$

Here (30) is induced by the Lie group structure on $\mathrm{Pin}(2)$ and (31) is restricted from the trivialization of $T_*\mathbb{H}$. These two trivializations induce equivariant homeomorphisms

$$\varphi_1 : S^{\mathbb{R}} \wedge \mathrm{Th}(\mathrm{Pin}(2), N_*\mathrm{Pin}(2)) \xrightarrow{\cong} \mathrm{Th}(\mathrm{Pin}(2), T_*\mathrm{Pin}(2) \oplus N_*\mathrm{Pin}(2)),$$

$$\varphi_2 : \mathrm{Th}(\mathrm{Pin}(2), T_*\mathrm{Pin}(2) \oplus N_*\mathrm{Pin}(2)) \xrightarrow{\cong} S^{\mathbb{H}} \wedge (\mathrm{Pin}(2)_+).$$

Let $\nu(\mathrm{Pin}(2))$ be a $\mathrm{Pin}(2)$-equivariant, closed tubular neighborhood of $\mathrm{Pin}(2)$ in $\mathbb{H}$. Then there exists an equivariant homeomorphism

$$\varphi_3 : \nu(\mathrm{Pin}(2))/\partial\nu(\mathrm{Pin}(2)) \xrightarrow{\cong} \mathrm{Th}(\mathrm{Pin}(2), N_*\mathrm{Pin}(2)).$$

Then we consider the following composition

$$\begin{aligned} l : S^{\mathbb{R}} \wedge S^{\mathbb{H}} &\xrightarrow{\mathrm{id}_{S^{\mathbb{R}}} \wedge c_1} S^{\mathbb{R}} \wedge (\nu(\mathrm{Pin}(2))/\partial\nu(\mathrm{Pin}(2))) \\ &\xrightarrow{\mathrm{id}_{S^{\mathbb{R}}} \wedge \varphi_3} S^{\mathbb{R}} \wedge \mathrm{Th}(\mathrm{Pin}(2), N_*\mathrm{Pin}(2)) \\ &\xrightarrow{\varphi_1} \mathrm{Th}(\mathrm{Pin}(2), T_*\mathrm{Pin}(2) \oplus N_*\mathrm{Pin}(2)) \\ &\xrightarrow{\varphi_2} S^{\mathbb{H}} \wedge (\mathrm{Pin}(2)_+) \\ &\xrightarrow{\mathrm{id}_{S^{\mathbb{H}}} \wedge p_3} S^{\mathbb{H}} \end{aligned} \tag{32}$$

Here $c_1 : S^{\mathbb{H}} \to \nu(\mathrm{Pin}(2))/\partial\nu(\mathrm{Pin}(2))$ is the Pontryagin-Thom collapsing map. The map $p_3 : \mathrm{Pin}(2)_+ \to S^0$ preserves the base point and collapses the whole $\mathrm{Pin}(2)$ into the other point in S^0.

Proposition 34. *α_0 is represented by the map l.*

Proof. By Lemma 21, $\alpha_0 = \zeta_{\{e\}}^{\mathrm{Pin}(2)}(\alpha_1)$, where

$$\zeta_{\{e\}}^{\mathrm{Pin}(2)} : \pi_{1,\mathcal{U}}^{\mathrm{Pin}(2)}(E\mathrm{Pin}(2)_+) \to \pi_{1,\mathcal{U}}^{\mathrm{Pin}(2)}(S^0)$$

is the map defined in (12) and α_1 is any generator of $\pi_{1,\mathcal{U}}^{\mathrm{Pin}(2)}(E\mathrm{Pin}(2)_+)$. We first give a space level description of α_1. Consider the following commutative diagram

$$\begin{array}{ccc} \pi_{1,\mathcal{U}}^{\mathrm{Pin}(2)}(\mathrm{Pin}(2)_+) & \xrightarrow{\quad i_* \quad} & \pi_{1,\mathcal{U}}^{\mathrm{Pin}(2)}(E\mathrm{Pin}(2)_+) \\ \uparrow{\scriptstyle j}\,{\scriptstyle\cong} & & \uparrow{\scriptstyle j'}\,{\scriptstyle\cong} \\ \pi_1^s((S^{\widetilde{\mathbb{R}}} \wedge \mathrm{Pin}(2)_+)/\mathrm{Pin}(2)) & \xrightarrow{i'_*} & \pi_1^s((S^{\widetilde{\mathbb{R}}} \wedge E\mathrm{Pin}(2)_+)/\mathrm{Pin}(2)) \end{array} . \tag{33}$$

Here i_* and i'_* are both induced by the inclusion $i : \mathrm{Pin}(2)_+ \hookrightarrow E\mathrm{Pin}(2)_+$. And j, j' are the Adams' isomorphisms (16). As we discussed in the proof of Proposition 26, the 4-skeleton of $(S^{\widetilde{\mathbb{R}}} \wedge E\mathrm{Pin}(2)_+)/\mathrm{Pin}(2)$ is homotopy equivalent to $\mathbb{RP}^3$ (with the unique 0-cell being the basepoint). Since $\pi_1^s(\mathbb{RP}^3) \cong \mathbb{Z}/2$ is generated by the inclusion of the 1-skeleton

$$S^1 \cong \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^3,$$

we see that the group $\pi_1^s((S^{\widetilde{\mathbb{R}}} \wedge E\mathrm{Pin}(2)_+)/\mathrm{Pin}(2))$ is isomorphic to $\mathbb{Z}/2$ and it is also generated by the inclusion of the 1-skeleton:

$$S^{\mathbb{R}} \cong (S^{\widetilde{\mathbb{R}}} \wedge \mathrm{Pin}(2)_+)/\mathrm{Pin}(2) \hookrightarrow (S^{\widetilde{\mathbb{R}}} \wedge E\mathrm{Pin}(2)_+)/\mathrm{Pin}(2).$$

Therefore, the map i'_* is just the surjective map $\mathbb{Z} \to \mathbb{Z}/2$.

By the diagram (33), we see that $\alpha_1 = i_* \circ j(1)$. By the explicit description of the Adams' isomorphism [LMSM86, Page 84], we see that $j(1) \in \pi_{1,\mathcal{U}}^{\mathrm{Pin}(2)}(\mathrm{Pin}(2)_+)$ is represented by the composition

$$l_1 := \varphi_2 \circ \varphi_1 \circ (\mathrm{id}_{S^{\mathbb{R}}} \wedge \varphi_3) \circ (\mathrm{id}_{S^{\mathbb{R}}} \wedge c_1) : S^{\mathbb{R}+\mathbb{H}} \to S^{\mathbb{H}} \wedge (\mathrm{Pin}(2)_+). \tag{34}$$

(i.e., the composition of the first four maps in (32)). Hence α_1 is presented by the composition

$$l_2 : S^{\mathbb{R}+\mathbb{H}} \xrightarrow{l_1} S^{\mathbb{H}} \wedge (\mathrm{Pin}(2)_+) \xrightarrow{\mathrm{id}_{S^{\mathbb{H}}} \wedge i} S^{\mathbb{H}} \wedge (E\mathrm{Pin}(2)_+).$$

By the construction of $\zeta_{\{e\}}^{\mathrm{Pin}(2)}$ in (13), $\alpha_0 = \zeta_{\{e\}}^{\mathrm{Pin}(2)}(\alpha_1)$ is represented by the composition

$$l_3 : S^{\mathbb{R}+\mathbb{H}} \xrightarrow{l_2} S^{\mathbb{H}} \wedge (E\mathrm{Pin}(2)_+) \xrightarrow{\mathrm{id}_{S^{\mathbb{H}}} \wedge p_0} S^{\mathbb{H}}.$$

Since $p_0 \circ i : \mathrm{Pin}(2)_+ \to S^0$ is exactly the map p_3, we see that the maps l_3 and l are the same. □

Remark 35. It is worth noting that l actually represents an element in set $\{S^1, S^0\}_{\mathcal{U}'}^{\mathrm{Pin}(2)}$, where $\mathcal{U}' = \mathbb{H}^\infty \oplus \widetilde{\mathbb{R}}^\infty$ is a universe that only contains the nontrivial representations $\widetilde{\mathbb{R}}$ and $\mathbb{H}$. Compared to the universe $\mathcal{U} = \mathbb{H}^\infty \oplus \widetilde{\mathbb{R}}^\infty \oplus \mathbb{R}^\infty$ (which we have been working with so far), this smaller universe actually fits better

with our setting in gauge theory. This is because the trivializations (4) only involve $\widetilde{\mathbb{R}}$ and $\mathbb{H}$. However, since $\mathcal{U}'$ doesn't contain the trivial representation $\mathbb{R}$, the corresponding stable homotopy category behaves badly. (For example, for general X, Y, the set $\{X,Y\}^{\mathrm{Pin}(2)}_{\mathcal{U}'}$ does not even carry a group structure.) This is the main reason why we choose to work with $\mathcal{U}$ instead of $\mathcal{U}'$. We also note that when restricting to the subgroup S^1, the universes $\mathcal{U}$ and $\mathcal{U}'$ are isomorphic.

We end the paper with the following proposition. It states that α_0 cannot be detected by Borel (co)homology, which corresponds to various versions of monopole Floer (co)homology.

Proposition 36. *Let A be an abelian group and let G be a closed subgroup of* $\mathrm{Pin}(2)$. *Then the induced maps*

$$\alpha_{0,*} : \tilde{H}^G_*(S^{\mathbb{R}+\mathbb{H}};A) \to \tilde{H}^G_*(S^{\mathbb{H}};A),$$
$$\alpha_0^* : \tilde{H}^*_G(S^{\mathbb{H}};A) \to \tilde{H}^*_G(S^{\mathbb{R}+\mathbb{H}};A).$$

between the G-equivariant reduced Borel (co)homology groups (with coefficient A) are both trivial.

Proof. α_0 is represented by the map l. Since l factors through the map l_1 in (34). It suffices to show that the induced maps

$$l_{1,*} : \tilde{H}^G_*(S^{\mathbb{R}+\mathbb{H}};A) \to \tilde{H}^G_*(S^{\mathbb{H}} \wedge (\mathrm{Pin}(2)_+);A),$$
$$l_1^* : \tilde{H}^*_G((\mathrm{Pin}(2)_+) \wedge S^{\mathbb{H}};A) \to \tilde{H}^*_G(S^{\mathbb{R}+\mathbb{H}};A).$$

are trivial. To prove this, we consider the fiber bundle

$$SU(2)/G \hookrightarrow BG \to BSU(2),$$

which makes $H^*(BG;A)$ a module over

$$H^*(BSU(2);\mathbb{Z}) = H^*(\mathbb{HP}^\infty;\mathbb{Z}) = \mathbb{Z}[v].$$

An elementary argument using the Serre spectral sequence shows that the map

$$v\cdot - : H^m(BG;A) \to H^{m+4}(BG;A)$$

is an isomorphism for any $m \geq 0$. By the definition of Borel cohomology, $\tilde{H}^*_G(S^{\mathbb{R}+\mathbb{H}};A)$ is a module over $H^*(BG;\mathbb{Z})$. Hence $\tilde{H}^*_G(S^{\mathbb{R}+\mathbb{H}};A)$ is also over $\mathbb{Z}[v]$. By the Thom isomorphism theorem, we have an isomorphism

$$H^*(BG;A) \cong \tilde{H}^{*+5}_G(S^{\mathbb{R}+\mathbb{H}};A)$$

between two $\mathbb{Z}[v]$-modules. Therefore, $\tilde{H}^*_G(S^{\mathbb{R}+\mathbb{H}};A)$ is supported in degrees greater than or equal to 5 and the map

$$v\cdot - : \tilde{H}^m_G(S^{\mathbb{R}+\mathbb{H}};A) \to \tilde{H}^{m+4}_G(S^{\mathbb{R}+\mathbb{H}};A)$$

is an isomorphism for any $m \geq 5$. Moreover, since the G-action on $S^{\mathbb{H}} \wedge (\mathrm{Pin}(2)_+)$ is free away from the base point, the group

$$\tilde{H}^*_G(S^{\mathbb{H}} \wedge (\mathrm{Pin}(2)_+);A) = \tilde{H}^*((S^{\mathbb{H}} \wedge (\mathrm{Pin}(2)_+)/G;A)$$

is supported in degrees less than or equal to 5 (less than or equal to 4 if G is not a finite group).

To this end, we see that l_1^* can only be nontrivial in degree 5. However, suppose $l_1^*(x) \neq 0$ for some $x \in \tilde{H}_G^5(S^{\mathbb{H}} \wedge (\mathrm{Pin}(2)_+); A)$. Then

$$0 = l_1^*(0) = l_1^*(v \cdot x) = v \cdot l_1^*(x) \neq 0.$$

This contradiction shows that $l_1^* = 0$ in all degrees. Hence we have $\alpha_0^* = 0$. The proof that $\alpha_{0,*} = 0$ is essentially the same. □

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