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Scalar Curvature, Injectivity Radius and Immersions with Small Second Fundamental Forms

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Abstract:

We prove special cases of the following.

$\bullet_{Sc}$ Bounds on the *injectivity radii* of "topologically complicated" Riemannian n-manifolds X, where the scalar curvatures of X are bounded from below, $Sc(X) \geq \sigma > 0$.

$\bullet_{curv}$ Lower bounds on *focal radii* of smooth immersions from k-manifolds, e.g. homeomorphic to the k-torus, to certain Riemannian manifolds of dimensions $n = k + m$, e.g. to the cylinders $S^{n-1} \times \mathbb{R}^1$.

$\bullet_{mean}$ Topological lower bounds on the mean curvatures of domains in Riemannian manifolds. e.g. in the Euclidean n-space $\mathbb{R}^n$.

At the present moment, our results are mainly limited by the *spin condition* and the $n \leq 8$ *restriction* with additional difficulties in the case of foliations. The corresponding problems and questions are presented in shades of red.

Key words and phrases:
Global geometric and topological methods; differential geometric analysis on metric spaces

1 Definitions, Conjectures and Illustrative Examples

Terminology and Notation. Given a Riemannian manifold $X = (X, g)$, possibly non-complete and/or with a boundary let

$$inrad_x(X) = dist(x, \partial_{metr} X),\ x \in X,$$

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be the radius R of the maximal open ball $B = B_x(R) \subset X$), which doesn't intersect the boundary $\partial X \subset X$ as well as the "infinity" of X, i.e. the boundary of X in the metric completion of X, where our $\partial_{metr} X$ stands for the topological boundary of the interior of X in the metric completion of X.

For instance $inrad_x(X) = dist(x, \partial X)$ for (metrically) complete manifolds with boundaries.

The conjugacy radius of X,

$$conj.rad_x(X) \leq inrad_x(X),$$

is the maximal r, such that the map $\exp_x$ is *a local diffeomorphisms* on the open r-ball $BT_x(r) \subset T_x(X)$, where, in fact, "$\exp_x$" is defined on the ball $BT_x(R = inrad_x(X)) \subset T_x(X)$.

Example. If the sectional curvature of X is bounded from below by a positive constant, $sect.curv(X) \geq \kappa > 0$, then $conj.rad_x(X) \leq \frac{\pi}{\sqrt{X}}$ for all $x \in X$ and if X is complete and $sect.curv(X) \leq \kappa > 0$, then $conj.rad_x(X) \geq \frac{\pi}{\sqrt{X}}$.

The injectivity radius

$$inj.rad_x(X) \leq conj.rad_x(X)$$

is the maximal r, such that the map $\exp_x : T_x(X) \to X$ is a *diffeomorphism* from the open r-ball $BT_x(r) \subset T_x(X)$ to its image in X, where, in fact, "$\exp_x$" is defined on the ball $BT_x(R = inrad_x(X))$.

Observe, that the *injectivity radius of the universal covering* $\tilde{X}$ of X *mediates* between the two above radii,

$$conj.rad_x(X) \geq inj.rad_x(\tilde{X}) \geq inj.rad_x(X),$$

where $inj.rad_x(\tilde{X})$ stands for $inj.rad_{\tilde{x}}(\tilde{X})$, where $\tilde{x} \in \tilde{X}$ is a lift of x to $\tilde{X}$ and where the latter radius is independent of the lift.

Remark. The ratio $conj.rad(X)/inj.rad(\tilde{X})$ can be arbitrarily large. For instance, *all 3-manifolds X with no boundaries, e.g. $X = S^3$, admit sequences of complete Riemannian metrics g_i, $i = 1, 2, \ldots$, such that $sect.curv(g_i) \leq 1/i$, (this makes $conj.rad(X, g_i) \to \infty$) and such that $inj.rad_x(X, g_i) \leq 1$ for all $i = 1, 2, \ldots$ and all $x \in X$. Moreover, if an X is compact one can make $diam(X, g_i) \leq 1$.*

(Probably, similar metrics exist on all manifolds of dimensions $n \geq 3$.)

The focal radius of a smooth immersion[1] map $f : Z \to X$, denoted

$$foc.rad_z(Z) = foc.rad_z(Z \overset{f}{\hookrightarrow} X)$$

is the maximal r, for which the differential of the *normal exponential map*

$$\exp_z^\perp = \exp_{f(z)}^\perp : BT_{f(z)}^\perp(Z) \to X,$$

is an immersion.

The scalar curvature of a Riemannian manifold $X = (X, g)$, denoted by

$$Sc = Sc(X, x) = Sc(X, g) = Sc(g) = Sc_g(x),$$

[1]"Immersions" are *locally* (smooth) *embeddings*, or, equivalently, smooth maps with *non-vanishing differentials* (on the non-zero tangent vectors).

is a function on X, which is equal to *the sum of the values of the sectional curvatures at the* $n(n-1)$ *ordered bivectors of an orthonormal frame in* X,

$$Sc(X,x) = \sum_{i,j} sect.curv_x(\tau_{i,j}),\ i \neq j = 1,...,n,$$

where this sum doesn't depend on the choice of the frame by the Pythagorean theorem and where a *characteristic property* of the scalar curvature is *additivity under Cartesian-Riemannian products:*

$$Sc(X_1 \times X_2, g_1 + g_2) = Sc(X_1, g_1) + Sc(X_2, g_2).$$

Examples. (a) The scalar curvature of the n-sphere of radius R is

$$Sc(S^n(R)) = n(n-1)sect.curv(S^n(R)) = \frac{n(n-1)}{R^2}.$$

(b) The scalar curvature of products of spheres by Euclidean spaces,

$$X = S^m(R) \times \mathbb{R}^k$$

satisfy

$$Sc(X) = \frac{m(m-1)}{R^2},$$

where, observe, the sectional curvatures of these products are pinched between 0 and $\frac{Sc(X)}{m(m-1)}$,

$$0 \leq sect.curv(X) \leq \frac{Sc(X)}{m(m-1)} = \frac{1}{R^2},$$

and their conjugacy and injectivity radii are

$$conj.rad(X) = inj.rad(X) = inj.rad(S^m(R)) = R\pi.$$

(c) If $X \subset \mathbb{R}^{n+1}$ is a smooth cooriented hypersurface with principal curvatures $\lambda_i = \lambda_i(x)$, $i = 1,...,n = dim(X)$, then, by *the Gauss formula*, the scalar curvature of the Riemannian metric g on X induced from the Euclidean/Riemannian one is

$$Sc_g(x) = \sum_{i \neq j} \lambda_i \lambda_j = \left(\sum_i \lambda_i\right)^2 - \sum_i \lambda_i^2.$$

Leon Green's Conjugate Radius Inequality (theorem 5.1 in [Gre63]). The *minimal conjugacy radius* of a complete Riemannian manifold X with *finite volume* and with the Ricci curvature bounded from below, $Ricci(X) \geq R > -\infty$[2] of dimension n is bounded by the *mean scalar curvature of* X as follows:

$$\inf_{x \in X} (conj.rad_x(X))^2 \cdot vol(X)^{-1} \int_X Sc(X,x)dx \leq \pi^2 \cdot n(n-1),$$

[2] In the original paper [Gre63], X is assumed compact, but if X is non-compact and geodesically complete, then $Vol(X) < \infty$ and $\inf Ricci(X) > -\infty$ suffice for the proof.

where the equality holds *only* for (necessarily compact) manifolds with constant sectional curvatures >0 and where, observe,

$$vol(x)^{-1}\int_X Sc(X,x)dy \geq infSc(X) = \inf_{x\in X} Sc(X,x).$$

Consequently,

$$[Sc|conjr]_n \qquad infSc(X)\cdot(\inf conj.rad(X))^2 \leq n(n+1)\pi^2$$

for all compact Riemannian n-manifolds X.

Below is a conjectural topological sharpening of $[Sc|conjr]_n$.

1.A. $[Sc|conjr]_{n-k}$**-Conjecture.** Let X be a complete Riemannian n-manifold, such that *there exists a homology class $h\in H_k(X)$, such that no non-zero multiple of h is representable by a continuous map from a compact oriented k-manifold with positive scalar curvature*[3], where this is expressed in writing by the inequality

$$Sc(X) \not>_{H_{k,\mathbb{Q}}} 0.$$

(Informally speaking, the *"$\mathbb{Q}$-homological scalar curvature"* of X is not *"fully positive in dimension k"*.)

Then

$$[Sc|conjr]_{n-k} \qquad infSc(X)\cdot(conj.rad(X))^2 \leq \pi^2 m(m-1), m=n-k.$$

Moreover, if X is *connected non-compact complete*, then

$$infSc(X)\cdot(\inf conj.rad(X))^2 \leq \pi^2(m-1)(m-2).$$[4]

Remark. These two inequalities are sharp as the following equalities show.

If $X = S^m\times\mathbb{T}^{n-m}$, then $Sc(X)\cdot(conj.rad(X))^2 = \pi^2 m(m-1)$.

If $X = S^{m-1}\times\mathbb{T}^{n-m}\times\mathbb{R}$, then $Sc(X)\cdot(conj.rad(X))^2 = \pi^2(m-1)(m-2)$.

Two Obvious, Remaining Conjectural, Corollaries. *to 1.A.* Since $inj.rad \leq conj.rad$ and

$$conjrad(X) \geq \frac{\pi}{\sqrt{\max(0,\sup sect.curv(X))}},$$

where $\sup sect.curv(X)$ is the supremum of the sectional curvatures over all tangent planes in X, the $[Sc|conjr]_m$-inequality $infSc(X)\cdot(conj.rad(X))^2 \leq \pi^2 m(m-1)$ implies that

$$[ScInjr]_m \qquad infSc(X)\cdot(inj.rad(\tilde{X}))^2 \leq \pi^2 m(m-1)$$

[3] Products $Y^m\times\mathbb{T}^k$ are basic examples of such X, see [SY79b, GL80]

[4] If X has *subexponential volume growth* and $\inf Ricci(X) > \infty$, then the inequality $infSc(X)\cdot(conj.rad(X))^2 \leq \pi^2 n(n-1)$ follows by Green's integration argument.

But no bound $infSc(X)\cdot(inj.rad(X))^2 < C = C_n$ (not even, for $C=\infty$) is known for general non-compact complete manifolds, except for $n\leq 5$, where quantifying the uniform contractibility argument from [CL20] (compare with [CL20], 3.10.3 in [Gro21] and 1.1.A in the next section) shows that $infSc(X)\cdot(inj.rad(X))^2 \leq 10^{100}$ for complete Riemannian manifolds X of dimensions $n\leq 5$.

where $\tilde{X}$ is the universal covering of X, and that

$$[Sc/sect]_m \qquad \frac{inf Sc(X)}{\sup sect.curv(X)} \le m(m-1).$$

Although, both inequalities remain conjectural, we shall show in section 4 that some manifolds X must satisfy at last one of the two.

Below is an instance of this.

1.B. Two Examples. Let X be a Riemannian manifold of dimension $n = m+k$, let Z be a closed connected orientable manifold Z, which admits a metric with *non-positive sectional curvature* and let

$$\phi : X \to Z,$$

be a continuous map.

(For instance, $X \underset{homeo}{=} Y \times \mathbb{T}^k$, where $\mathbb{T}^k$ is the k-torus and $\phi : (y,t) \mapsto t$.)

Let the scalar curvature of X be bounded from below by

$$Sc(X) \ge \sigma > 0.$$

Let the sectional curvature of X be bounded from above by

$$sect.curv(X) \le \kappa,\ \ \kappa > 0,$$ [5]

1.B(i). Compact Case. Let X be compact without boundary, let $dim(Z) = k$ and suppose the k-dimensional induced homology homomorphism

$$\phi_* : H_k(X) \to H_k(Z) = \mathbb{Z}$$

doesn't vanish.

Let

$$m(m-1)\kappa \le \sigma.$$

If

$\bullet_{spin}$ *either $m \le 3$ or the universal covering $\tilde{X}$ of X is spin*[6]

$\bullet_{\le 8}$ $n = dim(X) \le 8$,

then the universal covering of X satisfies

$$inj.rad(\tilde{X}) \le \pi\sqrt{\frac{1}{\kappa}}.$$

Remark. Albeit highly constrained, this inequality is non-vacuous. For instance, when applied to

$$X_\lambda = (S^m \times Z, g_{S^m} + \lambda^2 g_Z),\ \lambda \to \infty,$$

[5] This is a rather annoying (possibly redundant for complete X) condition, where, necessarily $\kappa \ge \frac{\sigma}{n(n-1)}$.

[6] This means that the restrictions of the tangent bundle $T(\tilde{X})$ to surfaces in $\tilde{X}$ are trivial, for which vanishing of the second homotopy group $\pi_2(X)$ is sufficient.

it shows that *compact Riemannian manifolds* $Z=(Z,g_Z)$ *with* $Sc(g_Z)>0$ *of dimension* $k=8-m$ *admit no metrics with* $sect.curv\leq 0$. (Of course, this is known for all k.)

1.B(ii). Non-Compact Case. Let $dim(Z)=k-1$, let $\phi: X\to Z$ be a fibration (e.g. a topological bundle of balls), let $\hat{\psi}:\tilde{Z}\to X$ be a "lift" of the universal covering of Z to X, i.e. $\phi\circ\hat{\psi}=\pi$. (For instance, if the fibration $\phi: X\to Z$ admits a section $Z\hookrightarrow X$, then the universal covering of the image of this section may be taken for $\hat{\psi}$.) Let the scalar and the sectional curvatures of X be bounded as earlier by

$$Sc(X)\geq\sigma \text{ and } sect.curv(X)\leq\kappa.$$

Let $\hat{R}$ be the distance from the image of $\hat{\psi}$ to the metric boundary of X,

$$\hat{R}=dist_{metr}(\hat{\psi}(\tilde{Z}),\partial X),$$

(where, by definition, $\hat{R}=\infty$ for geodesically complete X), let

$$\hat{U}=U_{\frac{5}{6}\hat{R}}(\hat{\psi}(\tilde{Z}))\subset X$$

be the $\frac{5}{6}\hat{R}$-neighbourhood of $\hat{\psi}(\tilde{Z})\subset X$, and let

$$\hat{\sigma}=\sigma-\frac{36(n-1)\pi^2}{n\hat{R}^2}\geq m(m-1)\kappa.$$

If

$\bullet_{spin}$ *either* $m\leq 3$ *or the universal covering of* X *is spin,*

$\bullet_{\leq 8}$ $n=dim(X)\leq 8$,

then the injectivity radius of the universal covering $\tilde{X}$ *of* X *is bounded in* U *as follows:*

$$\inf_{x\in\hat{U}} inj.rad_x(\tilde{X})\leq\pi\sqrt{\frac{m(m-1)}{\hat{\sigma}}}.$$

1.C. Corollary: Focal Radius Bound. Let X be a complete Riemannian n-manifold such that

$$Sc(X)\geq\sigma>0,\ sect.curv(X)\leq\kappa \text{ and } inj.rad(X)\geq r\geq\pi/\sqrt{\kappa}.$$

Let Z be a compact $(n-m)$-dimensional manifold, which admits a metric with $sect.curv\leq 0$ and let $Z\hookrightarrow X$ be a smooth immersion.

If *If* $m>0$ *and*

$\bullet_{spin}$ *either* $m\leq 4$ *or the universal covering of* X *is spin,*

$\bullet_{\leq 8}$ $n=dim(X)\leq 8$,

Then

$$foc.rad(Z)\leq 6\pi\sqrt{\frac{n-1}{n}}\sqrt{\frac{1}{\sigma-(m-1)(m-2)\kappa}}.$$

Proof. Let $T^\perp(Z)\to Z$ be the normal bundle of $Z\hookrightarrow X$ and let $BT^\perp(R)\subset T^\perp(Z)$ be the open R-ball subbundle.

Observe that the exponential map $\exp^{\perp} : BT^{\perp}(R) \to X$ is defined for $R \leq foc.rad(Z)$ and is an immersion for these R.

Then 1.B(ii) applies to $X^{\perp} = BT^{\perp}(foc.rad(Z))$ with the metric induced by $\exp^{\perp}$ from X and the proof follows.

MEAN CURVATURE INEQUALITIES. All of the above generalizes to Riemannian manifolds with *mean convex boundaries*, where the simplest instance of this is as follows.

1.D. Narrow Tunnel Theorem. *Let $X \subset \mathbb{R}^n$, $n \geq 3$, be a smooth (not necessarily connected and possibly infinite) domain with*

$$mean.curv(\partial X) \geq n-2 = mean.curv\left(S^{n-1}(R_+)\right) \text{ for } R_+ = 1 + \tfrac{1}{n-2}.$$

If $n \leq 8$, then the diameters of all connected components of the subset

$$X_{-(1+\varepsilon)} = X \smallsetminus U_{1+\varepsilon}(\partial X) = \{x \in X\}_{dist(x,\partial X)>1+\varepsilon} \subset X,\ \varepsilon > 0,$$

are bounded by $const_{n,\varepsilon} < \infty$. (This confirms conjecture 10 in [Gro19] for $n_1 = 1$.)

Remark/Conjecture. Probably, the connected components of the subset $X_{-1} \subset X$ are *bounded*.[7]

PLAN OF THE PAPER AND ORGANIZATION OF THE PROOFS. In the next (sub)section, we discuss our results in the general context of the metric scalar curvature geometry.

In section 2 we introduce a (not quite) numerical invariant on homology of a Riemannian manifold X, denoted $Sc^{\exists\exists\times}_{\widetilde{sp}}(h)$, $h \in H_m(X)$.

We represent h in some cases by *stable μ-bubbles* $Y^m_{bbl} \subset X$ with a properly tailored function $\mu(x)$[8] and thus bound $Sc^{\exists\exists}_{\widetilde{sp}}(h)$ in terms of $Sc(X)$ in manifolds X, where the universal covering $\tilde{X}$ is spin.

(These Y^m_{bbl} are constructed as in section 5 of [Gro21] by using the inductive dimension descent scheme originated in [SY79b] and combined with *symmetrization-by-* $\mathbb{T}^{n-m}$*-warping* from [GL83], also see [SY17], following an idea from [FCS80]. A presence of possible stable singularities in Y^m_{bbl} entails here the probably unnecessary constraint $dim(X) \leq 8$.)

In section 3 we introduce another invariant, denoted $Rad(h/S^N)$, which concerns area decreasing maps $X \to S^N$ and show that

$$Rad(h/S^N) \leq \sqrt{m(m-1)/Sc^{\exists\exists\rtimes}_{\widetilde{sp}}(h)}.$$

(This is done essentially, but not quite, as in [GL83] by relying on the *relative index theorem for families of twisted Dirac operators* over spin manifolds[9] combined with the *"twisted" Lichnerowitz formula*, which is sharpened in the present case by *Llarull's inequality,*[10] compare with section 3.4.3 in [Gro21]).

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[7] Embarrassingly, I see no direct elementary proof of this even for $X_{-(1+\varepsilon)}$, where $\varepsilon > 0$.

[8] Compare with sections 5.3, 5.5 in [Gro21].

[9] This theorem generalizes to manifolds, where the universal coverings are spin, see §§$9\frac{1}{9}$, $9\frac{1}{8}$ in [Gro96] and section 10 in [Gro19].

[10] Mario Llarull was Blaine Lawson's student who, by Blaine's suggestion, algebraically evaluated the bottom of the spectrum of the the curvature operator on the spinors on S^n, (formula 4.6 in [Lla98]) and thus obtained the *optimal* bound for the area dilations of maps from manifolds with $Sc \geq \sigma$ to spheres.

In section 4, we show that the inequalities $sect.curv(X) \leq \kappa$ and $inj.rad(X) \geq \pi/\sqrt{\kappa}$ imply that $Rad(h/S^N) \geq 1/\sqrt{\kappa}$ for $N \geq 2dim(X)$.

We combine this with the above and obtain the inequality

$$\kappa \geq Sc^{\exists\exists\rtimes}_{\widetilde{sp}}(h)/m(m-1)$$

for manifolds X with the universal coverings spin, which is a generalized quantitative version of the following *non-existence theorem* (see 13.8 in [GL83]).

❖ *Non-torsion* homology classes[11] in complete manifolds with non-positive sectional curvatures are not representable by continuous maps from oriented spin manifolds X with positive scalar curvatures.[12]

Finally, for $n \leq 8$, we combine the inequality $\kappa \geq Sc^{\exists\exists\rtimes}(h)/m(m-1)$ with a lower bound on $Sc^{\exists\exists\rtimes}(h)$ obtained with μ-bubbles Y^m_{bbl} in section 2, [13] which is essential to gain the correct $m(m-1)/n(n-1)$ factor in our inequalities.

Thus we arrive at our **main theorem** 4.C for distance decreasing maps from manifolds with lower bounds on their $\mathbb{T}^{\rtimes}$-*stabilized scalar curvatures* to manifolds with "large" metric balls.

In section 5 we say more on focal radii and discuss lower bounds on curvatures of immersions.

In section 6 we generalize 4.C to manifolds with mean convex boundaries, for which we adapt the results from [GS02] and [Lot21], which we slightly improve along the lines of [Lla98] and [Lis10].

In section 7 we prove a version of 1.B for Riemannian foliations (defined in 1.1.B below) on 4-manifolds.

Everywhere, we indicate limitations of our results and arguments, (most pronounced for Riemannian foliations with $Sc > 0$) and try to formulate conceivable generalizations and improvements.

1.1 Corollaries, Questions, Generalizations

(a) Most known inequalities obtained by combining (Dirac) index theoretic and (μ-bubble) geometric measure theoretic arguments have been generalized and proved with the index theorems for *Callias type operators* with no use of the geometric measure theory.[14]

However, for all I know, there are no such proofs of 1.B, 1.C, 1.D at the present moment and the constraint $n \leq 8$ remains intact.

(b) The $\square^{\exists\exists}(n, m = 2, N)$-*inequality* in 2.B of the next section and the (weakened) $\mathbb{T}^{\rtimes}$-*stable 2d Bonnet-Myers diameter inequality for surfaces Y with* $Sc^{\rtimes} \geq 2$ imply the following.

$[\circ\bullet]_{m=2}$ *Pairs of Riemannian metrics g and* $\underline{g} \leq g$ *on compact n-manifolds X homeomorphic to* $S^2 \times Z$,

[11] I am not certain on what is known in this respect about l-torsion, say for $l \neq 2$.

[12] The origin of the argument used in [GL83] can be traced to the proof of the *Novikov conjecture* for spaces with $sect.surv \leq 0$ in [Mis74].

[13] The extra $\mathbb{T}^{n-m}$-warping factor associated with Y^m_{bbl} is handled by twisting the relevant (already twisted) Dirac operator with a *family of flat bundles* over $\mathbb{T}^N$ or with a single *almost flat bundle* as in [GL80] and in [Gro96]. (See [Gro21] for a pedestrian presentation of these topics).

[14] See [Zei19, Zei20, Cec20, CZ21, WXY21].

where Z supports Riemannian metrics $sect.curv \leq 0$, satisfy, for $n \leq 8$, the following:

$$inf Sc(X,g) \cdot inj.rad(\tilde{X},\underline{\tilde{g}})^2 \leq 2\pi^2 \frac{2(n-1)}{n},$$

where $\tilde{X}$ is the universal covering of X.[15]

Furthermore, if $\inf Sc(g)/\sup sect.curv(\underline{g}) \geq n(n-1)$, then *Zhu's area inequality*[16] implies 1.B(i) for $m = 2$, that is :

$$inf Sc(X,g) \cdot inj.rad(\tilde{X},\underline{\tilde{g}})^2 \leq 2\pi^2.$$

Here, the Dirac operator is not used and the spin issue doesn't arise.

Besides, granted a mild generalization of theorem 4.6 from [SY17], the condition $n \leq 8$ can be removed,[17] while 1.B in the spin case may follow for all m and n by adopting the techniques/ideas of J. Lohkamp [Loh18b, Loh18a]. (One has no idea what to do with spin for $m \geq 4$.)

(d) *Contractibility λ-Radius.* Given a function $\lambda(r) \geq r$, $r \geq 0$, define $contr_\lambda rad(X)$ for a metric space X as the *supremum of the radii of the balls $B_x(r) \subset X$, which are contractible in the concentric ball $B_x(\lambda(r)) \supset B_x(r)$ for all $x \in X$.*

Observe that $contr_1 rad(X) \geq inj.rad(X)$ and that if X is a contractible manifold with a cocompact isometry group, e.g. X is the universal covering of a compact aspherical manifold, then $contr_\lambda rad(X) = \infty$ for some function $\lambda(r)$.

Similarly to $[\circ\bullet]_{m=2}$ one shows that if $n \leq 8$, then

the metrics $g \geq \underline{g}$ from $[\circ\bullet]_{m=2}$ satisfy

$$[\circ_\lambda\bullet]_{m=2} \qquad contr_\lambda rad(\tilde{X},\underline{\tilde{g}}) \leq \lambda\left(2\pi\sqrt{\frac{n-1}{n \cdot inf Sc(X,g)\cdot}}\right),$$

for all function $\lambda(r)$. (This generalizes $[\circ\bullet]_{m=2}$ for $\lambda(r) = r$.)

Next let $m = 3$ and recall (see 3.10.1 in [Gro21]) that the filling radii of 3-manifolds are bounded in terms of their $\mathbb{T}^{\rtimes}$-*stabilized scalar curvature* $Sc^{\rtimes}(X) \geq inf Sc(X)$ (the definition is in the next section) as follows

$$fillrad(X) \leq \frac{6\pi}{\sqrt{Sc^{\rtimes}(X)}}.$$

If $n = dim(X) \leq 8$, then the $\square^{\exists\exists}(n, m = 3, N)$-inequality implies (by an easy argument, see (see [Gro83]) that

$$[\circ_\Lambda\bullet]_{m=3} \qquad contr_\lambda rad(\tilde{X},\underline{\tilde{g}}) \leq \Lambda\left(\frac{6\pi}{\sqrt{Sc^{\rtimes}(X,g)}}\right),$$

[15]In fact, $inf Sc(X,g) \cdot inj.rad(\tilde{X},\underline{\tilde{g}})^2 < 2\pi^2 \frac{2(n-1)}{n}$, since, by the proof of the formula [BMD] in [Gro21], the inequality $Sc^{\rtimes}(Y) \geq 2$ implies that Y contains a pair of points with $dist(y_1, y_2) < 2\pi^2 \frac{2(n-1)}{n}$.

Probably, it is not hard to show that $inf Sc(X,g) \cdot inj.rad(\tilde{X},\underline{\tilde{g}})^2 \leq 2\pi^2 \frac{2(n-1)}{n} - 0.01$, but the corresponding sharp inequality $inf Sc(X,g) \cdot inj.rad(\tilde{X},\underline{\tilde{g}})^2 \leq 2\pi^2$ remain problematic.

[16]See [Zhu19, Zhu20] and 2.8 in [Gro21]).

[17]I can't claim this as a theorem, since I haven't mastered the argument in [SY17].

for all $\lambda = \lambda(r)$ and

$$\Lambda(r) = 2\lambda(2\lambda(2\lambda(r)))$$

(Desingularization of μ-bubles needed for $n = dim(X) \geq 8$ seems here more difficult than that for $m = 2$.)

(e) *Stable Quasi-conjugate λ-Radii.* There are several candidates for a (bi)Lipschitz stable version of conjugate radius, where our definition below is, probably, provisional.

This "radius" of a Riemannian manifold $X = (X, g)$,

$$conj_\lambda rad(X) \geq conj.rad(X),$$

is the supremum of the numbers r, such that *there exist the following*:

($\circ$) a smooth Riemannian manifold Θ of dimension $2n = 2dim(X)$ (instead of the space of tangent r-balls, $TB(r)(X) = \{B_{0(x)}(r) \subset T_x(X)\}_{x \in X}$),

($\circ$) smooth maps

$$\Pi : \Theta \to X,\ \Upsilon : \Theta \to X,\ O : X \to \Theta,$$

($\circ$) a *homotopy* $\mathcal{I}_t : \Theta \to \Theta$, $0 \leq t < \infty$, with the following properties.

$\bullet_\Pi$ The map Π is a smooth *submersion*, the fibers of which are denoted $\Theta_x = \Pi^{-1}(x) \subset \Theta$.

$\bullet_\Upsilon$ The maps

$$\Upsilon_x = \Upsilon_{|\Theta_x} : \Theta_x \to X$$

are locally diffeomorphic on all Θ_x, $x \in X$.

$\bullet_O$ The map $O : X \to \Theta$ (which corresponds to the zero section $\mathbf{0} : X \to T(X)$) satisfies: $O(x) \in \Theta_x = \Pi^{-1}(x) \subset \Theta$ for all $x \in X$.

$\bullet_\mathcal{I}$ The homotopy starts from the identity map, $\mathcal{I}_0 = id$, it preserves all Θ_x, i.e. $\Theta_x \overset{\mathcal{I}_t}{\to} \Theta_x$ for all t and all $x \in X$ and it fixes $O(X)$, i.e. $O \circ \mathcal{I}_t(x) = x$ for all $x \in X$.

Moreover, the maps $\Theta \overset{\mathcal{I}_t}{\to} \Theta$ are homeomorphisms[18] for sufficiently large t, and they converges to O for $t \to \infty$, that is

$$\mathcal{I}_t(\theta) \to O(x) \text{ for } \theta \in \Theta_x \text{ and } t \to \infty.$$

Now let us turn to metric properties of O and $\mathcal{I}_t$ with respect to the induced metrics $g_x = \Upsilon_x^*(g)$ on Θ_x.

$\bullet_r$ The distances from $O(x) \in \Theta_x$ to the boundary $\partial\Theta_x$ satisfy

$$dist(x, \partial\Theta_x) > r.^{19}$$

$\bullet_\lambda$ For all $t \geq 0$, all $\rho > 0$ and all $x \in X$, the map

$\mathcal{I}_t : \Theta_x \to \Theta_x$ *sends ρ-balls around $O(x) \in \Theta_x$ to the concentric $\lambda(r)$-balls.*

1.1.A. ***Observation.*** The arguments in [CL20] and [Gro20], as expounded in section 3.10.3 of [Gro21], show that

[18] This bizarre condition is need to justify 1.2.A below.

[19] Here we assume that X is complete with no boundary and this inequality means that the r-ball in Θ_x) around $O(x)$ is compact.

for all functions $\lambda = \lambda(r) \geq r$, there exists a positive number $R = R_\lambda < \infty$, such that all compact Riemannian manifolds X of dimensions $n \leq 5$ with $Sc^{\rtimes}(X) \geq 0$ satisfy the following λ-version of Green's $[Sc|conjr]_n$:

$conj_{\leq 5}$
$$conj_\lambda rad(X) \leq \frac{R_\lambda}{\sqrt{Sc^{\rtimes}(X)}}.$$

Notice that this does not yield the corresponding inequality for the contractibility radius of (the universal covering of) X, since our definition *doesn't make* $conj_\lambda rad(X) \geq contr_\lambda rad(\tilde{X})$. However, the inequality
$$contr_\lambda rad(\tilde{X}) \leq \frac{R_\lambda}{\sqrt{Sc^{\rtimes}(X)}}$$
holds for $dim(X) \leq 5$ with the same R_λ and for the same reason as $conj_{\leq 5}$ does.

QUESTIONS. All of the above, including the $[Sc|conjr]_{n-k}$-conjecture, tells us precious little about bounds on "radii" $rad^*(X)$ of a Riemannian manifold $X = (X, g)$ with $Sc(g) > 1$, where such a bound must depend on the topology and/or metric geometry of X and where rad^* may stand for $inj.rad$, $conj.rad$, $inj.rad(\tilde{X})$, the contractibility radius $contr_\lambda rad$ of X or $\tilde{X}$ for some $\lambda(r)$ or for $conj_\lambda rad$.

Here are a few specific questions.

Let (X, g_0) be a compact Riemannian n-manifold with $Sc(g_0) > 0$.

When can g_0 be $[Sc > 0]$-*homotoped*[20] to a metric g_1 with $Sc(g_1) \geq n(n-1) = Sc(S^n)$ and $inj.rad(X, g_1) \geq \rho$ for a given $\rho > 0$?

(I) *For instance*, – this seems unlikely but not impossible – does such $[Sc > 0]$-homotopy g_t exist for *all* $\rho < \pi$ and all *simply connected* manifolds X?

(II) Or, does the inequality $Sc(g) \geq n(n-1)$ for a metric g on X imply that $conj.rad(X, g_1) \leq \pi - \varepsilon_n$ for some universal $\varepsilon_n > 0$, e.g. for $\varepsilon_n = 1/10^n$, unless g is $[Sc > 0]$-homotopic to (is closed to?) a metric with constant sectional curvature on X?

(III) What are *$Scalrad^*$-extremal* manifolds (X, g_0), i.e. such that
$$\inf Sc(g_0) \cdot rad^*(X, g_0)^2 \geq \inf Sc(g) \cdot rad^*(X, g)^2$$
for all Riemannian metrics g on X?

For instance, are compact irreducible symmetric spaces $Scalrad^*$-extremal, say for $rad^* = inj.rad$?[21]

(IV) How much does a bound on the diameter of X contribute to the inequality $rad^*(X) \geq \rho$ for manifolds with $Sc(X) \geq 1$?

For instance, what can be said about the *topological complexity of an X which admits a metric g, such that*
$$Sc(g) \geq 1,\ inj.rad(X, g) \geq 0.01,\ diam(X, g) \leq 1?$$

Do these inequalities limit some complexity of the $[Sc > 0]$-homotopy class of g?

[20]This means a homotopy g_t of Riemannian metrics with $Sc(g_t) > 0$, for all $t \in [0, 1]$.

[21]This may be also true for some reducible spaces such as $S^2 \times S^2$, for instance.

Remark. If we drop the inequality $Sc(g_1) \geq 1$, the remaining two, which amounts to $\frac{diam}{inj.rad} \leq 100$, don't much restrict the topology of X.

For instance all surfaces X admit Riemannian metrics with $\frac{diam}{inj.rad} \leq 4$ obtained with ramified coverings $X \to \mathbb{T}^2$ and/or $X \to S^2$ (see [Bav96]).

Similarly, by ramifying $X^n \to S^n$ one can produce topologically complicated manifolds with small ratios $\frac{diam}{inj.rad}$.

In fact, using *Alexander's braid-knot theorem + Hilden's 3-fold branched covering theorem* (applied to S^3 preliminary very strongly ramified over two linked Hopf circles) one can show that

all compact 3-manifolds admit Riemannian metrics with $\frac{diam}{inj.rad} \leq 100$.

I am not certain if this is true for all n-manifolds.

1.1.B. Riemannian Foliations. Let

$$\mathcal{X} = (\mathcal{X}, g) = (Q, \mathcal{X}, g)$$

be a smooth Riemannian n-foliation, that is a smooth manifold Q foliated into smooth n-dimensional manifolds X, and g is a smooth positive definite quadratic form on the (sub)bundle $T(\mathcal{X}) \subset T(Q)$ tangent to the leaves.

Let $X_q \subset Q$, $q \in Q$ denotes the leaf passing trough the point q and let $Sc(\mathcal{X}, q)$, $conj.rad_q(\mathcal{X})$, etc. be the corresponding invariants of X_q at q defined in section 1, where the corresponding inf-invariant refer to $\inf_{q \in Q}$.

Observe that Green's integration argument applies to Riemannian foliations with *transversal measures* on compact manifolds[22] and that $[Sc|conjr]_{n-k}$-Conjecture 1.A makes sense for foliations.

2 $T^{\rtimes}$-Stabilized Curvatures $Sc^{\rtimes}$, $Sc^{\exists\rtimes}$, $Sc^{\exists\exists\rtimes}$ and Multispreads $\tilde{\square}^{\perp}(h)$ on Homology

$\mathbb{T}^{\rtimes}$-*Extensions.* A "warped" $\mathbb{T}^N$-extension, $N = 0, 1, \ldots.$, where $\mathbb{T}^N$ is the N-torus, of a Riemannian manifold $Y = (Y, g)$, where Y may have a boundary, is a Riemannian manifold denoted $Y_N^{\rtimes} = Y \rtimes \mathbb{T}^N$, that is the product $Y \times \mathbb{T}^N$ with a Riemannian metric $g_N^{\rtimes}$ of the following form:

$$g_N^{\rtimes} = dy^2 + \sum_{i=1}^{N} \varphi_i^2 dt_i^2$$

for some smooth positive functions $\varphi_i(y) \geq 0$ on Y, which are strictly positive (>0) in the interior $Y_N \smallsetminus \partial Y$ of Y.

Clearly, this $g_N^{\rtimes}$ is invariant under the obvious action of $\mathbb{T}^N$ on $Y_N^{\rtimes}$ and $\varphi_i(y)$ are equal to the $g^{\rtimes}$-lengths of the $\mathbb{T}_i^1$-orbits of the points $y^{\rtimes} = (y, t) \in Y^{\rtimes}$ under this action.

$\mathbb{T}^{\rtimes}$-*Stabilized Scalar Curvature(s).* Let us agree that the inequality

$$Sc_N^{\exists\rtimes}(Y, y) > \sigma(y)$$

for a given function $\sigma(y)$ on Y signifies that

[22] Here one needs only transversal continuity of $\mathcal{X}$ and g.

there exists a $\mathbb{T}^N$-extension $Y_N^{\rtimes}$ of Y, the scalar curvature of which satisfies

$$Sc(Y_N^{\rtimes}, y) > \sigma(y),$$

where this scalar curvature is a function $Y = Y_N^{\rtimes}/\mathbb{T}^N$, since the metric $g_N^{\rtimes}$ is $\mathbb{T}^N$-invariant.[23]

Clearly

$$Sc(Y) \le Sc_1^{\exists\rtimes}(Y) \le \ldots \le Sc_N^{\exists\rtimes}(Y) \le \ldots .$$

We agree that $Sc^{\exists\rtimes}(Y)$ stands for $Sc_\infty^{\exists\rtimes}(Y) = \sup_N Sc_N^{\exists\rtimes}(Y)$, i.e. $Sc_N^{\exists\rtimes}$ with an arbitrarily large N and observe that $Sc^{\exists\rtimes}(Y)$ (and even $Sc_1^{\exists\rtimes}(Y)$) of manifolds Y with boundaries (see below) can be significantly greater than $Sc(X)$. Yet, most known geometric properties of manifolds with $Sc > \sigma$ are also enjoyed by those with $Sc^{\exists\rtimes} > \sigma$.[24]

To have a single non-ambiguous number, let

$$Sc^{\rtimes}(Y)$$

be the supremum of the numbers σ, such that $Sc^{\exists\rtimes}(Y) > \sigma$.

Examples. It is not hard to show the following (see [Gro23]).

(a) *Rectangular solids* satisfy:

$$Sc^{\rtimes}\left(\bigtimes_{i=1}^{n}[-a_i, b_i]\right) = 4\lambda_1\left(\bigtimes_{i=1}^{n}[-a_i, b_i]\right) = \sum_{i=1}^{n}\frac{4\pi^2}{(b_i - a_i)^2}.$$

where this λ_1 is the first eigenvalue of the Laplace operator with the Dirichlet boundary condition in $\bigtimes_{i=1}^{n}[-a_i, b_i]$.

(b) Unit hemispheres satisfy

$$Sc^{\rtimes}(S_+^n) = n(n+3) = Sc(S_+^n) + 4\lambda_1(S_+^n).$$

(c) Unit balls satisfy

$$Sc^{\rtimes}(B^n) = 4\lambda_1(B^n) > n^2 + 5n + n^{4/3}.$$

Definition of $\mathbf{Sc}^{\exists\exists}$. Given a homology class in a Riemannian manifold X,

$$h \in H_m(X),$$

write

$$Sc_N^{\exists\exists\rtimes}(h) = Sc_N^{\exists\exists\rtimes}(h)_{dist} > \psi \text{ for a function } \psi = \psi(x) \text{ on } X,$$

[23] In fact, $Sc\left(g + \sum_{i=1}^N \varphi_i^2 dt_i^2\right) = Sc(g) - 2\Delta\Phi - ||\nabla\Phi||^2 - \sum_i ||\nabla \log \varphi_i||^2$ for $\Phi(y) = \log(\varphi_1(y) \cdot \ldots \cdot \varphi(y)_N)$, see [GL83, SY17, Zhu19], or compute yourself.

[24] The proofs of the bounds on the 2-waists of 3-manifolds Y with $Sc(Y) \ge \sigma > 0$ (see [MN11], section 3.10 in [Gro21] and [LM21]) do not apply, at least not immediately, to Y with $Sc^{\rtimes}(Y) \ge \sigma$.

Also the $\mathbb{T}^{\rtimes}$-stabilization of the known geometric 4d-inequalities obtained with the Seiberg-Witten equations (see [LeB21]) remains problematic.

if *there exits* a compact oriented (a priori disconnected) Riemannian k-manifold Y and a 1-Lipschitz (i.e. distance non-increasing) map

$$f : Y \to X, \text{ such that } f_*[Y] = h,$$

where $[Y] \in H_k(Y)$ is the fundamental class of Y, and such that the $\mathbb{T}^N$-stabilized scalar curvature of Y is bounded from below by ψ, i.e.

$$Sc_N^{\exists\rtimes}(Y) > \psi \circ f$$

for the composed function $\psi \circ f(y) = \psi(f(y))$.

$[\mathbf{Sc}^{\exists\exists\rtimes}(\mathbf{X})]$. Here $Sc^{\exists\exists\rtimes}(X) = Sc^{\exists\exists\rtimes}[X]$ for the fundamental homology class $[X] \in H_{dim(X)}(X)$ of an oriented manifold X.

Observe that

$$Sc^{\exists\exists\rtimes}(X) \geq Sc^{\rtimes}(X),$$

where this inequality is strict, for instance, for products $X = X_1 \times X_2$, where X_1 has constant scalar curvature $\sigma_1 > 0$ and $Sc(X_2) < 0$.

On the other hand the equality $Sc^{\exists\exists\rtimes}(X) = Sc^{\rtimes}(X)$, is expected for many manifolds with positive (sectional, Ricci?) curvatures, but at the present moment the proof is available only for spheres S^3, S^4 and closely related manifolds of dimensions 3 and 4. Yet higher dimensional equalities of this kind e.g. for compact symmetric spaces with non-zero Euler characteristics, are available (seeGS2000] and section 8.1) for the *spinor versions of* $Sc^{\exists\exists\rtimes}$ defined below.

Open manifolds and Homology with Infinite Support. The above definition of $Sc^{\exists\exists\rtimes}$ naturally generalizes to homology classes with *infinite supports* in non-compact manifolds, denoted

$$h \in H_m(X, \partial_\infty X),$$

where $\partial_\infty X$ stands for the complement of an unspecified "arbitrarily large" compact subset in X.

In other words, $H_m(X, \partial_\infty X)$ is the projective limit of $H_m(X, X_i)$ over all $X_i \subset X$ with *compact complements.*

Here the manifolds Y may be non-compact and the maps $f : Y \to X$ are required to be proper, that is f sends $\partial_\infty Y \to \partial_\infty X$, which means that the f-pullbacks of compact subsets in X are compact.

$\exists\exists^{\rtimes}$-**Spin.** One defines two spin versions of $Sc^{\exists\exists\rtimes}$:
$Sc_{sp}^{\exists\exists\rtimes}(h)$, where the manifolds Y are *spin*,
$Sc_{\widetilde{sp}}^{\exists\exists\rtimes}(h)$, where the *universal coverings* $\tilde{Y}$ of Y are spin.
Clearly,

$$Sc_{sp}^{\exists\exists\rtimes}(h) \leq Sc_{\widetilde{sp}}^{\exists\exists\rtimes}(h) \leq Sc^{\exists\exists\rtimes}(h).$$[25]

Homological Pullbacks. Let $f : V \to W$, where $dim(V) = dim(W) + k$, be a proper map between orientable manifolds, and recall that the cohomology homomorphism $f^* : H^l(W) \to H^l(V)$ composed with the Poincare duality isomorphisms defines the "pullback homomorphism" $H_m(W) \to H_{m+k}(V)$ for $m = dim(W) - l$, which, for generic smooth maps, is implemented by taking f-pullbacks of m-cycles in W.

[25] Conceivably, some nonzero multiples of all homology classes h satisfy $Sc_{sp}^{\exists\exists\rtimes}(i \cdot h) = Sc^{\exists\exists\rtimes}(i \cdot h)$.

Also, such a homomorphism is defined for topological fibrations and topological submersions $V \to W$ with manifold fibers. [26]

Multi-Spreads. (Compare with section 7.1 in [Gro21].) Let V be an n-dimensional Riemannian manifold (possibly non-compact) with a boundary, and let $h \in H_m(V, \partial_\infty V)$ be a homology class (with infinite, i.e. non-compact, support if ∂_∞ is non-empty, i.e. V is non-compact).

The $\square^\perp$*-spread* $\square(h)$ is the *supremum of the numbers* $d \geq 0$, for which *there exists* a continuous boundary-to-boundary map from V to the k-cube for $k = n - m$, $\psi = (\psi_1, \dots \psi_i, \dots \psi_k)$: $V \to \square = [-1,1]^k$, $\psi_i : V \to [-1,1]$, such that the following two conditions are satisfied.

(1) The class h is equal to the ψ-pullback of the generator of $h_0 \in H_0([-1,1]^k)$, that is, if ψ is smooth, then h is represented by the ψ-pullback of a generic point p in the interior of the cube.

(2) The distances between the pullbacks of the opposite faces in the cube $[-1,1]^k$,

$$d_i = dist_V(\psi_i^{-1}(-1), \psi_i^{-1}(1)), i = 1, \dots, k,$$

are bounded from below by the following inequality

$$\left(\frac{1}{k}\sum_{i=1}^{k}\frac{1}{d_i^2}\right)^{-\frac{1}{2}} \geq d,$$

that is

$$\frac{1}{k}\sum_{i=1}^{k}\frac{1}{d_i^2} \leq \frac{1}{d^2},$$

(e.g. if $d_i \geq d$ for all $i = 1, \dots, k$).

(Equivalently, one could require the functions $\psi_i : V \to [-1,1]$ to be $\frac{2}{d_i}$-Lipschitz.)

Next define $\tilde{\square}^\perp(h) \geq \square^\perp(h)$ of a homology class $h \in H_m(X, \partial_\infty X)$ for an n-dimensional Riemannian manifold X (possibly non-compact) with a boundary, as the supremum of the numbers $\tilde{d} \geq 0$, such that there exist

(i) a Riemannian manifold $\tilde{V}$ of dimension $n = dim(X)$

(ii) a homology class $\tilde{h} \in H_m(\tilde{V}, \partial_\infty \tilde{V})$ with $\square^\perp(\tilde{h}) \geq d$,

(iii) a proper locally isometric map[27] $\phi : \tilde{V} \to X$, such that the induced homology homomorphism $\phi_* : H_m(\tilde{V}; \partial_\infty \tilde{V}) \to H_m(X; \partial_\infty X)$ sends $\tilde{h}$ to h, in writing: $\phi_*(\tilde{h}) = h$.

2.A. *Enlargeable Manifolds and* $sect.curv \leq 0$. A Riemannian manifold Z is called enlargeable if there exist zero dimensinal homology classes $h_i \in H_0(Z)$ with $\tilde{\square}^\perp(h_i) \geq i$, $i = 1, 2, \dots$.

For instance, complete manifolds with non-positive sectional curvatures are enlargeable, since their universal coverings contain *arbitrarily large* compact domains $\tilde{V}$, i. e. with arbitrarily large $\square^\perp$-multispreads of their zero-dimensional homology.

It follows that if $X \underset{homeo}{=} Y \times Z$, where Y is an oriented m-manifold and where Z is a compact manifold, which admits a metric with non-positive sectional curvature, e.g. it is the $(n-m)$-torus, then the $\tilde{\square}^\perp$-spread (unlike the $\square^\perp$-spread) of the *fundamental homology class* $[Y] \in H_m(X, \partial_\infty X)$ of $Y = Y \times z_0 \subset X$, is *infinite.*

[26] This is explained in purely geometric terms in [Gro14b].

[27] This map *doesn't have* to send $\partial\tilde{V} \to \partial X$.

2.B. $\square^{\exists\exists}(n,m,N)$**-Inequality.** Let $X=(X,g)$ be an n-dimensional orientable Riemannian manifold with a boundary, let $h\in H_m(X,\partial_\infty X)$. If

$$n=dim(X)\leq 8,$$

then

$$[Sc^{\exists\exists}]\qquad Sc^{\exists\exists\rtimes}_{N+k}(h)\geq Sc^{\exists\rtimes}_N(X)-\frac{n+N-1}{n+N}\cdot\frac{4k\pi^2}{\tilde{\square}^{\perp}(h)^2},\ k=n-m,$$

that signifies the implication

$$Sc^{\exists\rtimes}_N(X)>\psi\implies Sc^{\exists\exists\rtimes}_{N+k}(h)>\psi$$

for all continuous functions $\psi=\psi(x)$ on X.

Furthermore,

if X is spin, then

$$[Sc^{\exists\exists}_{sp}]\qquad Sc^{\exists\exists\rtimes}_{N+k,sp}(h)\geq Sc^{\exists\rtimes}_N(X)-\frac{n+N-1}{n+N}\cdot\frac{4k\pi^2}{\tilde{\square}^{\perp}(h)^2},$$

and if the universal covering $\tilde{X}$ of X is spin, then

$$[Sc^{\exists\exists}_{\widetilde{sp}}]\qquad Sc^{\exists\exists\rtimes}_{N+k,\widetilde{sp}}(h)\geq Sc^{\exists\rtimes}_N(X)-\frac{n+N-1}{n+N}\cdot\frac{4k\pi^2}{\tilde{\square}^{\perp}(h)^2}.$$

Proof. If $k=1$ this follows from the equivariant separation theorem applied to $X\times\mathbb{T}^N$ with a metric $g^{\rtimes}$ and the general case follows by induction on k.[28]

(See the following section for immediate corollaries of this and for open questions.)

It is commonly believed that a presence of singularities in minimal hypersurfaces and/or in stable μ-bubbles for $n\geq 8$ is *non-consequential for scalar curvature*, which is confirmed by Natan Smale's generic regularity for $n=8$ [Sma93].

In the present case, this belief turns into the following.

2.C. $\square^{\exists\exists}(n>8)$**-Conjecture.** The inequalities $[Sc^{\exists\exists}]$, $[Sc^{\exists\exists}_{sp}]$ and $[Sc^{\exists\exists}_{\widetilde{sp}}]$ hold true for all m, n, and N.[29]

$\square^{\exists\exists}(n,m,N)$-inequality and thus $\rtimes^{\downarrow}$-extremality of solids for $n\leq 10$.

(It is conceivable that the inequality $[Sc^{\exists\exists}_{sp}]$ holds for non-zero multiples $i\cdot h\in H_m(X)$ in *non-spin* manifolds X.)

2.D. $\exists\rtimes$**-Stabilization of Riemannian Foliations.** Given a Riemannian foliation $(Q,\mathcal{Y},g)$ with m-dimensional leaves Y (see 1.1.B), define its $\mathbb{T}^{\rtimes}$-extension

$$(Q^{\rtimes}=Q\times\mathbb{T}^N,\mathcal{Y}^{\rtimes},g^{\rtimes}),$$

[28] See 5.4 in [Gro21] also [GH23] and compare with [Ric20] and [GZ21], where similar arguments are used

[29] The generic regularity theorem from [CMS23] extended to μ-bubbles (I haven't checked this extension) allows the above proof of the inequalities $[Sc^{\exists\exists}]$, $[Sc^{\exists\exists}_{sp}]$ and $[Sc^{\exists\exists}_{\widetilde{sp}}]$ for dimensions $n=9$ and $n=10$.

by foliating $Q^{\rtimes}$ into $Y^{\rtimes} = Y \times \mathbb{T}^N$ and taking $g^{\rtimes} = g + \sum_{i=1}^{N} \varphi_i^2 dt_i^2$ for smooth positive functions $\varphi_i(q) \geq 0$ on Q.

Accordingly, we introduce $Sc^{\exists\rtimes}(\mathcal{Y})$, where the sentence

"$Sc^{\exists\rtimes}(\mathcal{Y})$ has property P"

means that *there exist functions φ_i, such that*

the corresponding $\mathcal{Y}^{\rtimes}$ has property P.

Conceivably, one may also define $Sc^{\exists\exists\rtimes}(h)$, $h \in H_M(R)$, for Riemannian manifolds R by considering oriented M-dimensional manifolds Q foliated by m-dimensional Riemannian Y and maps $f : Q \to R$, which are distance decreasing on all leaves Y with respect to the metrics $Sc^{\exists\rtimes} \cdot g$ on Y.

Then one may formulate a counterpart of $\square^{\exists\exists}$-inequality 2.B for manifolds R foliated into n-dimensional leaves for $dim(R) - n = M - m$.

But there is little evidence for such a generalization of 2.B, where a more realistic conjecture should allow foliations with singular leaves Y. (See section 7.)

3 Maps to Spheres and to Sphere Bundles

Let $X = (X, g)$ and $S = (S, g_S)$ be smooth manifolds, let $\mathcal{P} \subset C(X, S))$ be a subset in the space of continuous maps $X \to S$, let $\Upsilon : X \times \mathcal{P} \to S$ be the evaluation map $\Upsilon : (x, a) \mapsto a(x)$. Let B be an aspherical topological space and $\beta : X \times \mathcal{P} \to B$ a continuous map, e.g. β is the composition of the coordinate projection $X \times \mathcal{P} \to X$ with a map $\underline{\beta} : X \to B$ and the classifiyng map $\underline{\beta} : X \to B = B(\Gamma)$, where Γ is the fundamental group $\pi_1(X)$.

What we do in this section is motivated by the following.

3.A. Aspherical Parametric Mapping Conjecture. Let X be a complete n-manifold, and g_p, $p \in \mathcal{P}$, be a C^2-continuous family of smooth complete Riemannian metrics on X with positive scalar curvatures,

Let S be the unit K-sphere S^K, $K \geq 1$, let $[S^K]^* \in H^K(S^K) = \mathbb{Z}$ be the fundamental cohomology class and let $\alpha \in H^k(B)$.

If all maps $p : X \to S^K$, $p \in \mathcal{P}$, are smooth, constant at infinity[30]

and strictly area decreasing on all smooth surfaces in X with respect to the Riemannian metrics $Sc(g_p) \cdot g_p$ on X and $m(m-1) g_{S^K}$ on S^K for $m = n - k$, then the cup product

$$\Upsilon^*[S^K]^* \smile \beta^*(\alpha) \in H^{K+k}(X \times \mathcal{P}, \partial_\infty X \times P)$$

vanishes on

$$H_n(X) \otimes H_{K-m}(\mathcal{P}) \subset H_{K+k}(X \times \mathcal{P}).$$

(Recall that $m = n - k$ and that $\partial_\infty X$ stands for the complement of a large unspecified compact subset in X.)

Remarks (i_A) Theorem 3.E below confirms this conjecture, where α is the fundamental cohomology class of an *enlargeable k-manifold*, provided that $dim(X) = n \leq 8$ and the universal covering of X is spin.

(ii_A) This conjecture makes sense for singular spaces X, where one can define scalar curvature, e.g. for Alexandrov spaces with lower curvature bounds.

[30]A map from an X is constant at infinity if it is constant outside a compact subset in X.

(iii$_A$) An ultimate form of this conjecture must apply to the space/category of germs of n-submanifolds (or singular spaces) in S^K with induced metrics having $Sc \geq \sigma > 0$ (e.g. $\sigma = m(m-1)$).

The following results (corresponding to $m = n$) give an idea of what may be expected in this regard.

3.B. Su-Wang-Zhang Foliated Area Contraction Theorem. Let $\mathcal{X} = (\mathcal{X}, G)$ be a complete oriented Riemannian K-manifold smoothly foliated into m-manifolds $\tilde{X} \subset \mathcal{X}$ and let $f : \mathcal{X} \to S^K$ be a smooth map locally constant at infinity.

If the leaf-wise scalar curvature of $\mathcal{X}$, denoted $Sc^{\sim}(\mathcal{X})$ is bounded from below by $Sc^{\sim}(\mathcal{X}) > m(m-1)$ and if the map f is area decreasing on all smooth surfaces contained in the leaves, then, provided either the manifold $\mathcal{X}$ is spin or the tangent bundle to the leaves is spin, the map f has zero degree,

$$deg(f) = 0.$$

Remarks. (i$_B$) The inequality $Sc^{\sim}(X) > m(m-1)$ is required in [SWZ21] only on the support of the differential of f, while everywhere else $Sc^{\sim}$ must be non-negative.

(ii$_B$) The conclusion $deg(f) = 0$ holds (unless I misunderstand something in [SWZ21]) if the map f is leaf-wise area decreasing for the metrics $Sc^{\sim} \cdot G$ in $\mathcal{X}$ and $m(m-1)g_{S^K}$ in S^K.

(iii$_B$) Probably, (I haven't properly checked this) the spin condition can be relaxed to the spin of the lifts of the tangent bundles (of the manifold $\mathcal{X}$ and of the leaves) to the universal covering of $\mathcal{X}$.

Preparation for 3.C. Let P and Q be smooth manifolds possibly with boundaries and $\Psi : Q \to P$ be a smooth *submersion*,[31] the fibers of which, denoted

$$\tilde{X}_p = \Psi^{-1}(p) \subset Q,\ p \in P,$$

are thought of as a family of manifolds parametrized by P.

Let $\mathcal{X}$ be the foliations of Q into (the connected components of) the fibers $\tilde{X}_p$ and let g be a Riemannian metric on $\mathcal{X}$.

(In the present case, this is a smooth family of Riemannian metrics g_p in $\tilde{X}_p$.)

Let $S^\bullet \to P$ be a K-dimensional sphere bundle, let $\delta : P \to S^\bullet$ be a section of this bundle and let $P_\bullet$ be the image of the opposite section,

$$P_\bullet = -\delta(P) \subset S^\bullet.$$

Let $\theta^\bullet \in H^K(S^\bullet, P_\bullet)$ be the *Thom cohomology class* of the K-ball bundle $S^\bullet \setminus P_\bullet \to P$, that is the Poincaré dual to

$$\delta_*[P] \in H_d(S^\bullet, \partial S^\bullet),$$

where $d = dim(P)$ and $[P] \in H_d(P, \partial P)$ is the fundamental class of P.

Let

$$H_{i/j}(Q, Q \setminus Q_0) = H_{i/j\downarrow\Psi}(Q, Q \setminus Q_0),\ Q_0 \subset Q,$$

be the group of relative homology classes of possibly infinite relative i-cycles $C \subset Q$, such that the images $\Psi(C) \subset P$ are *precompact* and such that $dim(\Psi(C)) \leq j$ and $dim(\Psi(\partial C)) \leq j - 1$.

Let

$$\mathcal{F} : Q \to S^\bullet$$

[31] The differential $d\Psi : T(Q) \to T(P)$ has everywhere $rank = dim(P)$

be a smooth *fiber bundle* map,[32] such that

(i) the pullback $\Psi^{-1}(\partial P) \subset Q$ is sent by $\mathcal{F}$ to $P_\bullet \subset S^\bullet$,

$$\mathcal{F}(\Psi^{-1}(\partial P)) \subset P_\bullet;$$

(ii) the complements of "uniformly controllably large" compact subsets R_p in the fibers $X_p \subset Q$ are also sent to $P_\bullet$, i.e.

there is a closed subset, $R \subset Q$ such that the restriction $\Psi_{|R} : R \to P$ is a proper map and $\mathcal{F}(Q \setminus R) \subset P_\bullet$.

Let

$$\mathcal{F}^*(\theta^\bullet) \in H^K(Q, \mathcal{F}^{-1}(P_\bullet))$$

be the (relative) cohomology class induced by $\mathcal{F}$ from $\theta^\bullet \in H^K(S^\bullet, P_\bullet)$.

Spherical Metric γ. Let $\gamma = \{\gamma_p\}$, be a smooth family of Riemannian metrics in the K-spheres $S^\bullet_p \subset S^\bullet$ with constant sectional curvatures κ_p.

3.C. Prelimenary Area Contraction Inequality. Let the above map $\mathcal{F}$ be *fiberwise area decreasing* with respect to the metrics $Sc^{\exists\rtimes}(\mathcal{X}) \cdot g$ in Q[33] and $n(n-1)\kappa\gamma = \{\kappa_p \cdot n(n-1)\gamma_p\}$ in $S^\bullet$, i.e. the $\mathcal{F}$-images of all smooth surfaces $E \subset \tilde{X}_p$, $p \in P$, satisfy

$$[area <]_{\exists\rtimes} \qquad n(n-1)\kappa_p \cdot area_{\gamma_p}(\mathcal{F}(E)) < \int_E Sc^{\exists\rtimes}(\tilde{X}_p)(x_p) d_{g_p} x_p.$$

Let the fibers $\tilde{X}_p$ be manifolds *without boundaries* and the metrics g_p on $\tilde{X}_p$ be *complete*.

If the sphere bundle $S^\bullet \to P$ and the tangent bundle $T(\mathcal{X}) \to Q$ are *spin*, then the cohomology class $\mathcal{F}^*(\theta^\bullet)$ *vanishes on* $H_{K/K-n}(Q, \Psi^{-1}(\partial P))$.

About the Proof. If the bundle is trivial, then 3.C for area decreasing maps $(\mathcal{X}, Sc(\mathcal{X}) \cdot g) \to Q)$ (instead of $[area <]_{\exists\rtimes}$) follows from 3.B, which needs to be augmented with (ii$_B$), yields 3.C for $Sc^{\exists\rtimes}$ as well.

In fact, since the leaves $\tilde{X}_p$ are closed, the proof of 3.C is much simpler than that of 3.B in [SWZ21]) (where non-trivial holonomy creates the major problem) and it goes along the usual lines, roughly, as follows.

Let $\mathbb{S}^\mp(S^\bullet_p) \to P$ be the Clifford bundle of spinors in the fibers $S^\bullet_p = S^K$, let $\mathcal{T}^\rtimes \to \mathbb{T}^N$ be an *almost flat bundle with non zero top Chern class and all other classes zero* and let $\mathcal{D}^\rtimes_p$, $p \in P$, be the Dirac operators on the fibers $\tilde{X}^\rtimes_p = \tilde{X}_p \times \mathbb{T}^N$ with coefficients in $\mathbb{S}^\mp(S^\bullet_p) \otimes \mathcal{T}^\rtimes \to P$,

$$\mathcal{D}^\rtimes_p : C^\infty(\mathbb{S}^\mp(X^\rtimes_p) \otimes \mathbb{S}^\mp_{\mathcal{F}(p)}(S^\bullet_p) \otimes \mathcal{T}^\rtimes) \circlearrowleft .$$

Then, by Llarull's trace evaluation (4.6 in [Lla98]) of the curvature term in the twisted Lichnerowitz formula (see [GL80, GL83] and 4.16 and 8.17 in [LM89]), the inequality $[area <]_{\exists\rtimes}$ implies that the operators $\mathcal{D}^\rtimes_p$ are positive. (Zhang's Dirac deformation argument [Zha20, Zha22] allows $\inf_x Sc(X,x) = 0$ at infinity, but this is unneeded for our applications.)

[32] p-Fibers of Q go to p-fibers of $S^\bullet$ for all $p \in P$,

[33] Recall that $Sc(g^\rtimes)$ actually is a function on Q, see 2.D.

Hence, the $K(P)$-theoretic relative ∓index (see §13 in [GL83] and references therein) is zero. Then, if K, N and $dim(\tilde{X}_p)$ are even, the relative version of the Atiyah-Singer index theorem for families [GL83] shows that

$$\mathcal{F}^*(\theta^\bullet)(h) = 0 \text{ for all } h \in H_{K/K-n}(Q, \Psi^{-1}(\partial P)).$$

Finelly, a spherical suspension construction, (see [Lla98] and sections 3.4 and 6.3.4 in [Gro21]) reduces the case of odd dimensions to that of the above even ones. QED.

Remarks. **(i$_C$)** The above mentioned spherical suspension construction also allows a reduction of the theorem to the case, where sphere bundle $S^\bullet \to P$ is *trivial*, as follows (compare with the proof of theorem 13.8 in [GL83]).

Let $S^+ \supset S^\bullet$ be an extension of $S^\bullet \to P$ to a trivial sphere bundle, $S^+ = P \times S^{K^+}$, $K^+ > K + dim(P)$ and let $\mathbb{R}_p^{K^+ + 1} = \mathbb{R}^{K^+ + 1} \supset S_p^+ = p \times S^+$ be the Euclidean spaces containing the fibers of this trivial sphere bundle.

Let $R_p^\perp$, $p \in P$, be the $(K^+ - K^\bullet)$-dimensional family of unit vectors $\nu_r \in T_{-\delta(p)}(\mathbb{R}^{K^+ + 1})$ normal to S_p^+ and pointing inward the unit ball in $\subset \mathbb{R}_p^{K^+ + 1}$ bounded by S_p^+.

Let $S_r \subset S_p^+$, $r \in R_p^\perp$, be K-dimensional subspheres $S_r \subset S_p^+$ obtained by intersecting S_p^+ by affine subspaces of dimension $K^\bullet + 1$ in $\mathbb{R}_p^{K^+ +} \supset S_p^+$, spanned by tangent subspace $T_{\delta(p)}(S_p^\bullet) \subset T_{\delta(p)}(\mathbb{R}_p^{K^+ + 1}) \subset \mathbb{R}_p^{K^+ + 1}$ and vectors ν_r, where we allow vectors ν_r tangent to $S_p^+ \subset S_p^\bullet$, where the corresponding K-spheres S_r collapse to the point $-\delta(p) \in S_p^\bullet$.

Let $R^\perp \to P$ be the fibration with the fibers $R_p^\perp$ and let $Q^+ \to Q$ be the induced fibration over Q.

The map $\mathcal{F} : Q \to S^\bullet$ naturally suspends to a map $\mathcal{F}^+ : Q^+ \to S^{+*}$, where $S^{+*} \to R^\perp$ is the (trivial!) sphere bundle induced from $S^+ \to P$ by the map $R^\perp \to P$, and the inequality 3.A for $\mathcal{F}$ reduces (this is an exercise) to that for $\mathcal{F}^+$.

(ii$_C$) A similar suspension argument allows us to replace $[area <]_{\exists\rtimes}$ by the corresponding inequality with the plane Sc for the foliation of the Riemannian product $Q \times \mathbb{R}^N$ (regarded as the $\mathbb{Z}^N$-covering of $Q \times \mathbb{T}^N$ in the definition of $Sc^\rtimes$) by the leaves $\tilde{X}_p \times \mathbb{R}^N$ endowed with the metrics $g_p^\rtimes$ (lifted from $Q \times \mathbb{T}^N$ to $Q \times \mathbb{R}^N$.

This $Q \times \mathbb{R}^N$ is mapped to the suspension $S^\bullet \wedge S^N$ via the one point compactification maps $\mathbb{R}_q^N \to S^N$ composed with the ε-scaling map

$$Q \times \mathbb{R}^N \overset{\varepsilon}{\to} Q \times \mathbb{R}^N \text{ for } (q, r) \mapsto (q, \varepsilon r).$$

(Recall that the spherical suspension corresponds to adding the trivial vector bundle of rank N to the $(K+1)$-vector bundle associated with $S^\bullet$.)

What is essential is that

- the map ε preserves the splitting of the tangent bundle $T(Q \times \mathbb{R}^N) = T(Q) \times \mathbb{R}^N$;
- the map ε is $g^\rtimes$-isometric on the tangent spaces to $Q \times r$;
- the map ε is ε-contracting on all $q \times \mathbb{R}^N$ with respect to the metric $g^\rtimes$.

Therefore, Llarull's trace evaluation (4.6 in [Lla98]) of the curvature term in the twisted Lichnerowitz formula applied to a small positive ε yields positivity of leaf-wise Dirac operator twisted with $\mathbb{S}^\mp_{\mathcal{F}(p)}(S_p^\bullet \wedge S^N)$ and the proof follows.

Notice that this argument automatically takes care of odd dimensions and it needs no additional twist with almost flat bundles $\mathcal{T}^\rtimes$. However, when it comes to rigidity theorems, such a twist, or rather twisting with families of flat bundles, makes the proofs easier.

(iii$_C$) The spin condition on $T(\mathcal{X})$ can be relaxed to *spin of the lift of $T(\mathcal{X})$ to the universal covering of X* (as in §§$9\frac{1}{9}$, $9\frac{1}{8}$ in [Gro96] and section 10 in [Gro19] and footnote 78 in section 2.7 in [Gro21]) but one has no idea what to do in the fully non-spin situation.

(iv$_C$) The recent development in the Dirac index theory for manifolds with boundaries (see [CZ21, WXY21] and references therein) shows – I haven't truly checked this – that completeness of the fibers X_p can be replaced by a specific (finite!) lower bound on the distances from the supports of the differentials of the maps $\mathcal{F}_p : X_p \to S^\bullet_p$ to ∂X_p (and /or to $\partial_\infty X_p$).

Let us now turn to the case $m \le n$ and let us define the $\tilde{\square}^\perp$-multispread of an $h \in H_{j/k}(Q, Q \setminus \Psi^{-1}(\partial P))$ by exhausting P by compact domains P_i,

$$P_1 \subset \ldots \subset P_i \subset \ldots \subset P,$$

observing that h can be represented by infinite j-cycles $C_i \subset P_i$ for all sufficiently large i letting $h(i) = [C_i] \in H_j(\Psi^{-1}(P_i^\circ)$ for the interiors $P_i^\circ \subset P_i$, and setting

$$\tilde{\square}^\perp(h) = \limsup_{i\to\infty} \tilde{\square}^\perp(h(i)).$$

3.D. Area Contraction Conjecture. Let us allow the fibers $\tilde{X}_p = \Psi^{-1}(p) \subset Q$ of the submersion $\Psi : Q \to P$ to have boundaries and allow $m \le n = dim(\tilde{X}_p)$.

Let

$$h_K \in H_{K/K-m}(Q, \Psi^{-1}(\partial P)),$$

and

$$\sigma_N^{\rtimes} = \sigma_N^{\rtimes}(q) = Sc(g_p^{\rtimes}(q)) - \frac{n+N-1}{n+N} \cdot \frac{4(n-m)\pi^2}{\tilde{\square}^\perp(h_K)^2}.$$

Let the map $\mathcal{F}$ be *fiberwise area decreasing* with respect to the metrics $\sigma_N^{\rtimes} \cdot g$ in Q and γ_m in $S^\bullet$. *If the fibers $\tilde{X}_p$ are metrically complete, then*

$$\mathcal{F}^*(\theta^\bullet)(h_K) = 0.$$

(This conjecture points toward our ultimate goal, but it may need an adjustment to avoid a possible counterexample in its present formulation, e.g. obtained with a parametric thin surgery.)

We prove below a special case of this conjecture by applying 3.B to families of m-dimensional submanifolds $Y_p \subset \tilde{X}_p$ for $m < n = dim(\tilde{X}_p)$, where the scalar curvatures of Y_p are bounded from below according to the $\square_\star^{\exists\exists}(n,m)$-inequality 2.B.

PREPARATION FOR THEOREM 3.E.

Let P and Q be smooth manifolds, possibly with boundaries, and let $\Psi : Q \to P$ be a true fibration, with an oriented, possibly disconnected, n-dimensional Riemannian fiber $\tilde{X} = (\tilde{X}, g)$, where the structure group Π of the fibration is countable and its action on $\tilde{X}$ *discrete, free, orientation preserving* and where, since Q is connected, the quotient space $\underline{X} = \tilde{X}/\Pi$ is connected.

Here, the fundamental group $\pi_1(P)$ acts on $\tilde{X}$ via a surjective homomorphism $\pi_1(P) \to \Pi$ and

$$Q = (\tilde{P} \times \tilde{X})/\pi_1(P)$$

for the Galois action of $\pi_1(P)$ on the universal covering $\tilde{P} \to P$ and the diagonal action of $\pi_1(P)$ on the product $\tilde{X} \times \tilde{P}$.

Thus, there is a natural covering map, say

$$\tilde{\Phi} : Q \to P \times \underline{X} = (\tilde{P} \times \tilde{X})/\pi_1(P) \times \pi_1(P).$$

Given homology classes $\underline{h}_m \in H_m(\underline{X}, \partial_\infty \underline{X})$, where $m \le n = dim(\underline{X}) = dim(\tilde{X})$, and $h_{K-m} \in H_{K-m}(P, \partial P)$, let

$$h_K \in H_{K/K-m}(Q, \Psi^{-1}(\partial P))$$

be the pullback of the product $\underline{h}_{K-m} \otimes h_m \in H_K(P \times \underline{X}, \partial P \times \underline{X} \cup P \times \partial_\infty X)$ to Q under the map $\tilde{\Phi} : Q \to P \times \underline{X}$.

Let $\underline{g}$ be a Riemannian metric in $\underline{X}$ for which $(\underline{X}, \underline{g})$ is metrically complete (bounded subsets are precompact), and let g_p be the the Riemannian metrics in the fibers $\tilde{X}_p$ induced by the covering maps $\tilde{X}_p = \tilde{X} \to \underline{X}$.

Let $\varphi_i(x)$, $i = 1, \ldots N$, be smooth positive functions on $\underline{X}$ and

$$\underline{g}^{\times} = \underline{g} + \sum_{i=1}^{N} \varphi(x)^2 dt_i^2,$$

be the metric on $\underline{X} \times \mathbb{T}^N$.

Recall the homology class $\underline{h}_m \in H_m(\underline{X}, \partial_\infty \underline{X})$ and let

$$\underline{\sigma}_N^{\rtimes} = \underline{\sigma}_N^{\rtimes}(x) = Sc(g^{\rtimes}) - \frac{n+N-1}{n+N} \cdot \frac{4(n-m)\pi^2}{\tilde{\Box}^{\perp}_{\underline{g}^{\rtimes}}(\underline{h}_m)^2}.$$

3.E. Area Contraction Theorem. Let the above map $\mathcal{F} : Q \to S^{\bullet}$ be *fiberwise area decreasing* with respect to the metrics $\underline{\sigma}_N^{\rtimes} \cdot g_p$ in the fibers $\tilde{X}_p = \tilde{X}$ of the fibration $\Psi : Q \to P$ and with the above normalized spherical metrics γ_m in the fiberes of $S^{\bullet} \to P$.

Let the following three conditions be satisfied:

$\bullet_{spin}$ *either the manifold $\tilde{X}$ is spin or $m \le 3$,*

$\bullet_{cmplt}$ *$\tilde{X} = (\tilde{X}, g)$ is metrically complete,*

$\bullet_{\le 8}$ *$n = dim(\tilde{X}) \le 8$.*

Then

$$\mathcal{F}^*(\theta^{\bullet})(h_K) = 0.$$

In fact, as we have already stated, this is a direct corollary of 2.B+3.B.

3.F. Representative Example of 3.D. Let Y and P be smooth compact oriented manifolds and let X be a Riemannian manifold homeomorphic to $Y \times Z$, where Z is a compact enlargeable k-manifold, e.g. $Z = \mathbb{T}^k$.

Let $Sc(X) \ge m(m-1)$ for $m = dim(Y)$.

Let $dim(P) = K$, let $S^{m+K} \subset \mathbb{R}^{m+K+1}$ be the unit sphere and let

$$f : X \times P \to S^{m+K}$$

be a smooth map, which is *area decreasing* on all $X = X_p = X \times p \subset X \times P$, $p \in P$.

If the universal covering of Y is *spin* and if $dim(X) \leq 8$, then

$$f_*([Y \times P]) = 0 \in H_{m+K}(S^{m+K}) = \mathbb{Z},$$

where $[Y \times P] \in H_{m+K}(X \times P)$ denotes the homology class of the submanifold $Y \times P = Y \times \mathbf{0} \times P \subset X = Y \times \mathbb{T}^k \times P$, $\mathbf{0} \in \mathbb{T}^k$.

Remarks. Granted validity of $\square^{\exists\exists}(n > 8)$-conjecture 2.C (irrelevance of singularities) the proof of 3.D applies to all $n = m + k$.

But this proof breaks down for all dimensions if we allow mutually non-isometric fibers $X_p = (X, g_p)$, (even for 3.E) , since the minimal hypersurfaces and stable μ-bubbles are not continuous in g_p.[34]

This begs for a *purely Dirac theoretic* proof of 3.D in the spirit of [CZ21] and [WXY21] that could yield more general inequalities, e.g. formulated in terms of the *K-area*,[35].

3.G. $\mathbf{Rad_{area}(h_*/S^K)}$-Invariant. Much (but not all) of the above can be expressed in the following terms.

Given a Riemannian manifold X and $h_m \in H_m(X, \partial_\infty X)$, let $Rad_{area}(h_m/S^K)$ be the supremum of the numbers $R > 0$ with the following property.

There exits a cellular topological space P, a bundle $S^\bullet \to P$ of K-spheres of radii R with a distinguished section $P_\bullet \hookrightarrow S^\bullet$ and a continuous map $F : X \times P \to S^\bullet$, such that

$\bullet_\bullet$ the map F sends $\partial_\infty X \to P_\bullet \subset S^\bullet$;

$\bullet_p$ the p-fiber maps $F_p = F_{|X \times p} : X \to S^\bullet_p = S^K$ of S, are smooth and C^1-continuous in $p \in P$; logycl $\bullet_{area}$ the maps F_p are area decreasing for all $p \in P$;

$\bullet_{top}$ there exists a rational homology class $h_{K-m} \in H_{K-m}(P)$ such that the F-pullback of the Thom class $\theta^\bullet \in H^K(S^\bullet, P_\bullet)$ doesn't vanish on $h_m \otimes h_{K-m} \in H_K(X \times P, \partial_X \times P)$,

$$F^*(\theta^\bullet)(h_m \otimes h_{K-m}) \neq 0.$$

The proof of 3.C shows that if the universal covering of X is *spin*, then

$$Sc^{\exists\exists\rtimes}(h_m) \geq \underline{\sigma} > 0 \implies Rad_{area}(h_m/S^K) \leq \sqrt{\frac{m(m-1)}{\underline{\sigma}}}.$$

This, together with the $\square^{\exists\exists}(n, m, N)$-inequality 2.B,

$$Sc^{\exists\exists\rtimes}(h_m) \geq Sc^{\rtimes}(X)) - \frac{n+N-1}{n+N} \cdot \frac{4(n-m)\pi^2}{\tilde{\square}^\perp(\underline{h}_m)^2},$$

proved for $n = dim(X) \leq 8$ yields the corresponding lower bound on $Rad_{area}(h_m/S^K)$ in terms of $Sc(X)$ and the $\square^\perp$-spread of h_m.

Remark on $\mathbf{Rad_{vol_n}(h_*/S^K)}$. The above definition obviously generalizes from $area = vol_2$ to all vol_i, where, for instance, Almgren's max-min argument (details need checking) yields the inequality

$$Rad_{vol_n}([X]_*/S^K) \leq \sqrt[n]{vol(X)/vol(S^n)},\ n = dim(X),$$

[34]We go around this discontinuity for $m = 3$ and $K = 1$ in the 4d-example 7.C.

[35]The only feasible approach to such inequalities lies in the Dirac theoretic realm but the scalar curvature geometry of minimal hypersurface and μ-bubbles depending on parameters is interesting in its own right; we say a few words about it in section 7.

and the convex partition argument yields the corresponding ε-waist inequality under the $\mathbb{Z}_2$-version of $\bullet_{top}$. (See [Gut14] and references therein.)

4 Injectivity Radii and Maps to Spheres

Notation and Definitions. Let P be a Riemannian manifold, $\tilde{P} \to P$ be the universal covering over P regarded as a principal fibration with the fiber equal to the fundamental group $\pi(P)$.

Let

$$\Psi = \Psi_P : \tilde{P}^\Delta \to P$$

be the fibration over P which has $\tilde{P}$ as a fiber, and which is associated with the universal covering $\tilde{P}$ over P for the deck (Galois) action of the fundamental group $\pi(P)$ in this fiber.

Thus, the space $\tilde{P}^\Delta$ is equal to $(\tilde{P} \times \tilde{P})/\pi_1(P)$ for the diagonal $\pi_1(P)$-action, where the two $\tilde{P}$-factors of $(\tilde{P} \times \tilde{P})$ play here different roles.

Let $\delta : P \to \tilde{P}^\Delta$ be the section of this fibration defined by the diagonal embedding $\tilde{P} \to (\tilde{P} \times \tilde{P})/\pi_1(P)$.

Let $r = r(p) \geq 0$ be a continuous function on P,

let $BT(r) = BT(P, r) \subset T(P)$ be the "subbundle" of $r(P)$-balls in the tangent spaces $T_p(P)$[36]

let $\tilde{B}^\Delta(r) \subset \tilde{P}^\Delta$ be the "subbundle" of the r-balls $\tilde{B}_{\delta(p)}(r(p))$ in the fibers $\tilde{P}^\Delta_p \subset \tilde{P}^\Delta$, where these balls degenerate to points for r(p)=0.

Let $r(p) \leq inj.rad_{\delta(p)}(\tilde{P}_p)$[37] and let $r(p)$ be *strictly positive inside* P, i.e. $r(p) > 0$ for $p \in P \setminus \partial P$, and observe that these "subbundles" are true ball bundles over $P \setminus \partial P$.

In fact, the inverse exponential maps in the fibers $\tilde{P}^\Delta_{\delta(p)} = \tilde{P}$ isomorphically send $\tilde{B}^\Delta_\delta(r)$ to the bundle $BT(r)$,

$$\widetilde{\exp}^{-1} : \tilde{B}^\Delta_\delta(r) \to BT(r) = BT(P, r),$$

where the tangent r-balls $BT_p(r(p))$ are identified with the tangent balls to the fibers $\tilde{P}^\Delta_p$ at $\delta(p) \in \tilde{P}^\Delta_p$.

Let $S^\bullet = S^\bullet(r) \to P$ be the K-sphere "bundle", $K = dim(P)$ that is obtained by shrinking the complements of the open balls $BT_x(r(x)) \subset T_p(P)$ to points and let $P_\bullet \subset S^\bullet$ be the image of the section $\delta_\bullet : P \to S^\bullet$ which sends $p \in P$ to these points.

If $p \in \partial P$ and $r(p) = 0$, we agree that the "fiber" $S^\bullet_p \subset S^\bullet$ consists of a single point $\bullet_p = \delta_\bullet(p)$.

Let

$$\tilde{\mathcal{B}} : \tilde{P}^\Delta \to S^\bullet(r)$$

be the *composition* of $\widetilde{\exp}^{-1} : \tilde{B}^\Delta_\delta(r) \to BT(r) \subset T(P)$ with the (tautological) quotient map $T(X) \to S^\bullet(r)$.

Let $E : \tilde{P}^\Delta \to P$ be the map, where each fiber $\tilde{P}^\Delta_p = \tilde{P} \subset \tilde{P}^\Delta$ goes to P by the covering map $E_p = \tilde{P} \to P$, which sends $\delta(p) \mapsto p$.

Let X be a smooth m-dimensional manifold and let $f : X \to P$ be a proper map.

[36] This fails to be a true bundle at the points where $r(p) = 0$,

[37] If P is *geodesically incomplete*, i.e. the exponential map at an $p \in P$ is defined only on certain open tangent ball $BT_p(R_p) \subset T_p(P)$, then, by definition, the inequality $inj.rad_p \geq r$ signifies that $r \leq R_p$, i. e. the exponential map $\exp_p : BT_p(r) \to P$ *is defined* and it is diffeomorphism on its image.

Let $\tilde{Q}_X$ be the product of $\tilde{P}^\Delta \underset{E}{\to} P$ and $X \underset{f}{\to} P$ over P, i.e.

$$\tilde{Q}_X = \{\tilde{p}^\Delta, x) \in (\tilde{P}^\Delta, X\}_{E(\tilde{p}^\Delta)=f(x)}.$$

(If $X \subset P$, then $\tilde{Q}_X = E^{-1}(X)$.)

Let the map

$$\tilde{f}^\Delta : \tilde{Q}_X \to \tilde{P}^\Delta$$

be defined by the projection $(\tilde{p}^\Delta, x) \to \tilde{p}^\Delta \in \tilde{P}^\Delta$, thus $E \circ \tilde{f}^\Delta(\tilde{p}^\Delta, x) = f(x)$, and let

$$\Psi_X = \Psi_P \circ \tilde{f}^\Delta : \tilde{Q}_X \to P.$$

Observe that this Ψ_X is a fibration over P, where the fiber $\tilde{Q}_{X,p}$ is equal to the covering of X induced by the map $f : X \to P$ from the covering $E_p : \tilde{P}^\Delta_p \to P$, and that $\tilde{Q}_X$ covers the product of P and X, where this covering (map), say

$$\Psi^X_X : \tilde{Q}_X \to P \times X,$$

is induced from the covering $\tilde{P}^\Delta = (\tilde{P} \times \tilde{P})/\pi_1(P) \to P \times P$ by the map $P \times X \to P \times P$ for $(p,x) \mapsto (p, f(x))$.

Let

$$\tilde{\mathcal{F}}_{X,r} = \tilde{\mathcal{B}} \circ \tilde{f}^\Delta : \tilde{Q}_X \to S^\bullet(r).$$

4.A. Nonvanishing Lemma.[38] Let P and X be Riemannian manifolds, possibly with boundaries and $f : X \to P$ be a proper map with the image in the interior of P, i.e. $f(X) \subset P \setminus \partial P$. and let $h_m \in H_m(X, \partial_\infty X)$.

Let $r(x) \le inj.rad_x(P)$ be a continuous function on P, which is strictly positive inside P and which is *bounded by distance to* P_0 outside a *compact subset* $P_1 \subset P$

$$r(x) \le dist(x, P_0) \text{ for } x \in P \setminus P_1,$$

where necessarily $P_1 \supset P_0$.

If the $\mathbb{Q}$-tensorisation of the image of the class h_m in P doesn't vanish,

$$0 \neq f_*(h_m) \otimes \mathbb{Q} \in H_m(P, \partial_\infty P; \mathbb{Q}),$$

then the pullback of the Thom class $\theta^\bullet \in H^k(S^\bullet, P_\bullet;)$ *under the map* $\tilde{\mathcal{F}}_{X,r}$,

$$\tilde{\mathcal{F}}^*_{X,r}(\theta^\bullet) \in H^K(\tilde{Q}_X, \mathcal{F}^{-1}_{X,r}(P_\bullet)),$$

doesn't vanish on $H_{K/K-m}(\tilde{Q}_X, \Psi^{-1}_X(\partial P))$.

Proof. Let $h_{K-m} \in H_{K-m}(P, \partial P)$ for $K = dim(P)$ be a homology class, such that the intersection number $[h_m \cap h_{K-m}]_{ind} \in \mathbb{Z}$ doesn't vanish (we assume P is connected oriented) and let

$$\tilde{h}_K \in H_{K/(K-m)\downarrow\Psi_X}(\tilde{Q}_X, \Psi^{-1}_X(\partial P \times X \cup P \times \partial_\infty X)) \subset H_K(\tilde{Q}_X, \Psi^{-1}_X(\partial P \times X \cup P \times \partial_\infty X))$$

[38] Compare with Lemma 13.5 in [GL83].

(for the above $\tilde{Q}_X \subset \tilde{P}^{\Delta} \times X$ and $\Psi_X : \tilde{Q}_X \to P$) be the pullback of the product

$$h_{K-m} \otimes h_m \in H_K(P \times X, \partial P \times X \cup P \times \partial_\infty X)$$

under the natural (covering) map

$$\tilde{}\Phi : \tilde{Q}_X \to P \times X.$$

The class $\tilde{\mathcal{F}}^*_{X,r}(\theta^\bullet)$ doesn't change if we replace r by a smaller continuous positive function $r'(x) \leq r(x)$; thus our geometrical lemma reduces to a *purely topological one*, where r is taken arbitrarily small.

To simplify further, properly embed $P \subset \mathbb{R}^{2K}$ and replace P by a small regular neighbourhood $P^+ \subset \mathbb{R}^{2K}$.

By the Thom isomorphism, this reduces the lemma to the case where the K-sphere bundle bundle $S^\bullet$ is trivial

$$S^\bullet = P \times S^K \text{ (with 2K instead of } K).$$

Here the Thom class $\theta^\bullet$ is induced from the fundamental class of the sphere S^K by the map $S^\bullet \to S^K$ and *non-vanishing* of $\tilde{\mathcal{F}}^*_{X,r}(\theta^\bullet)$ follows by the obvious degrees comparison argument as in [GL83]. QED.

4.B. Let us give geometric conditions that make the class $\tilde{\mathcal{F}}^*_{X,r}(\theta^\bullet)$, hence homology image class $f_*(X) \in H_m(P)$, *vanish.*

Definition of $Rad^\bullet_p(P)$. The radius

$$Rad^\bullet_p(P,r) = Rad^\bullet_p(P,r)_{area} \text{ for } r \leq inj.rad_p(P)$$

is the supremum of the numbers $R > 0$, such that the open ball $BT_p(r) \subset T_p(P) = \mathbb{R}^n$ for $n = dim(P)$ and $r = inj.rad_p(P)$ admits a smooth $O(n-1)$-*equivariant* map to the n-sphere of radius R, say

$$\alpha_{x,r} : BT_p(r) \to S^n(R),$$

where $0 \in BT_p(r)$ goes to the south pole and the boundary sphere $S^{n-1}_p(r) = \partial BT_p(r)$ to the north pole, and such that the composition of this α with the inverse exponential map that sends the open ball $B_p(r) \subset P$ to the n-sphere,

$$\alpha_{x,r} \circ (\exp_p)^{-1} : B_p(r) \to S^n(R),$$

is *area decreasing*.

Example. Let $a(r,\kappa)$ denote the area of the disk of radius r in the complete simply connected surface with constant curvature κ, e.g.

$$a(r,1) = 2\pi(1-\cos r),\ a(\pi,0) = r^2,\ a(r,-1) = 2\pi(\cosh r - 1),$$

and let

$$R^\bullet(r,\kappa) = \sqrt{\frac{a(r,\kappa)}{4\pi}}$$

be the radius of the 2-sphere with area $a(r,\kappa)$.

Then, by the textbook comparison theorem, the inequalities

$$sect.curv(P) \leq \kappa \text{ and } inj.rad_p(P) \geq r, \text{ where } \kappa \leq \tfrac{1}{r^2},$$

imply that

$$rad^{\bullet}_{p}(P)\geq R^{\bullet}(r,\kappa).$$

for all complete Riemannian n-manifold P.

PREPARATION FOR THEOREM 4.C.

Let, as earlier, P and $X=(X,g)$ be complete Riemannian manifolds, possibly with boundaries, and let

$$f:X\to P\smallsetminus\partial P$$

be a smooth proper map. Let $U=U_f\subset P$ be the set of points $p\in P$, which are closer to the the image of f, than to the boundary of P,

$$U=\{p\in P\}_{dist(p,f(X))\leq dist(p,\partial P)}$$

and let

$$\tilde{R}^{\bullet}=\tilde{R}^{\bullet}_{U}=\tilde{R}^{\bullet}_{f}=\inf_{p\in U}Rad^{\bullet}_{\tilde{p}}(\tilde{P},r=inj.rad_{\tilde{p}}(\tilde{P})).$$

for the points $\tilde{p}$ over $p\in P$ in the universal covering $\tilde{P}$ of P.

Now we are able to state the ***main result*** of the present paper.

4.C. Injectivity Radius Theorem. Let the map $f:X\to P\smallsetminus\partial P$ be *area decreasing* with respect to the metric $Sc^{\rtimes}_{N}(X)\cdot g$ in X and the original metric in P (used in the definition of $\tilde{R}^{\bullet}_{f}$) and let

$$h\in H_m(X,\partial_\infty X).$$

If the $\mathbb{Q}$-tensorisation of the f-image of h in the rational homology group $H_k(P,\partial_\infty P;\mathbb{Q})$ *doesn't vanish,*

$$f_*(h)\otimes\mathbb{Q}\neq 0,$$

then, in the following $\bullet_1$-$\bullet_3$ cases, the $\tilde{\square}^{\perp}$-spread of h is related to the above $\tilde{R}^{\bullet}$ by the inequality

$$[\tilde{\square}^{\perp}\&\tilde{R}^{\bullet}]\qquad \frac{n+N-1}{n+N}\frac{4(n-m)\pi^2}{(\tilde{\square}^{\perp}(h))^2}\geq Sc^{\rtimes}_{N}(X)-\frac{m(m-1)}{(\tilde{R}^{\bullet})^2}.$$

- $\bullet_1$ *The universal covering of X is spin and $m=n$,*
- $\bullet_2$ *the universal covering of X is spin and $n\leq 8$,*
- $\bullet_3$ *$n\leq 8$ and $m\leq 3$.*

Proof. The above example 4.B shows that the map $\tilde{\mathcal{F}}_{X,r}=\tilde{\mathcal{B}}\circ\tilde{f}^{\Delta}:\tilde{Q}_X\to S^{\bullet}(r)$ is area decreasing and the proof follows by using the $\square^{\exists\exists}(n,m,N)$-inequality 2.B (this is where $n\leq 8$ comes from) and confronting non-vanishing lemma 4.A with the area contraction theorem 3.C (where the spin condition resides).

EXAMPLES, REMARKS, COROLLARIES.

4.D. $(\mathbf{m}=\mathbf{n})$-Example. The inequality $[\tilde{\square}^{\perp}\&\tilde{R}^{\bullet}]$ for $m=n$ translates as follows.

$\bigstar_{m=n}$ Let $X = (X, g)$ be a complete oriented Riemannian n-manifold, $n \geq 2$ with *spin* universal covering $\tilde{X}$ and let P be a complete Riemannian manifold of dimension $K \geq n$, such that that $sec.curv(P) \leq \kappa \geq 0$ and the injectivity radius of the universal covering of P satisfies

$$inj.rad(\tilde{P}) \geq \frac{\pi}{\sqrt{\kappa}}.$$[39]

Let $f : X \to P$ be a smooth quasi-proper map[40] (e.g. f is proper or it is locally constant at infinity).

Let $Sc^{\exists\rtimes}(X, x) \geq \sigma(x) > 0$ for a positive continuous function σ on X, (e.g. $Sc(X) > \sigma(X)$).

If f is *area decreasing* with respect to the metric $\sigma \cdot g$, then the rational fundamental homology class of X *is sent by* f *to zero* in $H_n(P, \partial_\infty P; \mathbb{Q})$,

$$f_*[X]_{\mathbb{Q}} = 0.$$

(If $\kappa = 0$, this generalizes theorem 13.8 from [GL83] and if P is the unit n-sphere, this is the $\rtimes$-stabilized Llarull-Listing-Zhang theorem.[41])

4.E. Generalization and Proof of 1.B. Let $X = (X, g)$ be a Riemannian manifold of dimension $n = m + k$, such that $Sc^{\exists\rtimes}(X) \geq \sigma = \sigma(x) > 0$ (e.g. $Sc(X) \geq \sigma$), let Z be a compact connected enlargeable manifold.

Let

$$\phi : X \to Z$$

be a smooth map, let $\tilde{X}$ be a covering of X induced by ϕ from the universal covering $\tilde{Z} \to Z$ and let $\hat{\phi} : \hat{X} \to \tilde{Z}$ be the lift of ϕ to $\hat{X}$.

Let $Y = \hat{\phi}^{-1}(\tilde{z}) \subset \hat{X}$ be the $\hat{\phi}$-pullback of a generic point $\tilde{x} \in \tilde{Z}$,

Let P be a Riemannian manifold with $sect.curv(P) \leq \kappa \geq 0$ and let $f : \hat{X} \to P$ be a smooth map, which is strictly area decreasing with respect to the metric

$$\frac{\kappa}{m(m-1)} \cdot \sigma \cdot g \text{ in } X.$$

.

Let

$\bullet_{spin}$ *either* $m \leq 3$ *or the universal covering of* X *is spin,*

$\bullet_{\leq 8}$ $n = dim(X) \leq 8$.

4.E(i). Compact Case. Let X be compact without boundary, let $dim(Z) = k$ and suppose the f-image of the fundamental class $h_m = [Y] \in H_m(\hat{X})$ doesn't vanish in the rational homology $H_m(P; \mathbb{Q})$,

$$\hat{f}_*(h_m) \otimes \mathbb{Q} \neq 0.$$

[39]Conjecturally the inequality $injrad(\tilde{P}) \geq \pi/\sqrt{\kappa}$ can be relaxed to $conj.rad(\tilde{P}) \geq \pi/\sqrt{\kappa}$ and, possibly, the inequality $sect.curv(P) \leq \kappa$ can be dropped at the same time.

[40]A map f is quasi-proper if it extends to a continuous map between the compactified spaces, from $X^{+ends} \supset X$ to $P^{+ends} \supset P$.

[41]This argument, which relies on the trace inequality 4.6 in [Lla98], automatically delivers, as it was observed in [Lis10], a sharper result, where the area decreasing condition on f is replaced by the corresponding bound on *trace norm* of $\wedge^2 df$ as it is explained in detail in section 3.4.1 in [Gro21].

Similarly, $m < n$, one may use the supremum of the $\wedge^2$-trace norms over the restrictions of the differential df to all m-dimensional subspaces in the tangent bundle $T(X)$.

Then the universal covering of P satisfies

$$inj.rad(\tilde{P}) \le \frac{\pi}{\sqrt{\kappa}}.$$

4.E(ii). Non-compact Case. Let $dim(Z) = k-1$, let $\hat{Z} \subset \hat{X}$ be a closed subset, and let $h_m \in H_m((\hat{X} \setminus \hat{Z}) \cap Y)$ be a homology class, such that the f-image of h_m doesn't vanish in the rational homology $H_m(P;\mathbb{Q})$.

Let $\hat{R} = dist_{metr}(\hat{Z}, \partial X)$, (where, by definition, $\hat{R} = \infty$ for geodesically complete X), let $\hat{U} = U_r(\hat{Z}) \subset X$ be the closed (but non-compact) r-neighbourhood of $\hat{Z} \subset X$, and let

$$\hat{\sigma} = \sigma - \frac{36(n-1)\pi^2}{n\hat{R}^2} \ge m(m-1)\kappa.$$

Then the injectivity radius of the universal covering $\tilde{P}$ at the pullbacks of the points $f(x) \in P$ to $\tilde{P}$ satisfies

$$\inf_{x \in U} inj.rad_{\tilde{f}(x)}(\tilde{P}) \le \pi\sqrt{\frac{1}{\kappa}}.$$

Proof. These propositions are obvious corollaries of 4.C, while the examples 2B(i) and 2B(ii) for a manifold X follow from 4.E(i) and 4.E(ii) applied to the covering map $f : \hat{X} \to P = X$.

4.F. About Rigidity. The μ-bubbles inequalities present in our argument, as well as area inequalities for maps to spheres are accompanied by the corresponding rigidity phenomena in the extremal cases (see [Lla98, Lis10, Bre11, Zhu19, GZ21] and section 5.7 in [Gro21].). Then bringing these rigidity properties together shows that the inequalities

$$sect.curv(\underline{g}) \le \frac{\sigma}{m(m-1)}$$

and

$$inj.rad(\tilde{(}X, \tilde{\underline{g}}) \ge \pi\sqrt{\frac{m(m-1)}{\sigma}}$$

together with non-vanishing of $[Y] \otimes \mathbb{Q} \in H_m(X, \partial_\infty X; \mathbb{Q})$ imply that the $g = \underline{g}$ and the universal covering $\tilde{(}X, \tilde{g})$ is isometric to $S^m(R) \times \mathbb{R}^k$ for some $R > 0$.

4.E. $\tilde{\mathcal{R}}^\bullet_{area}$*-Invariant.* The proof of the injectivity radius theorem 4.C remains valid if $R^\bullet$ is replaced by $\tilde{\mathcal{R}}^\bullet_{area}(P) \ge R^\bullet$, which quantifies the λ^2-enlargeability from [GL83] and which is defined as follows.

Recall the map $\tilde{\mathcal{B}} : \tilde{P}^\Delta \to S^\bullet(r)$ and let us agree that the inequality

$$\tilde{\mathcal{R}}^\bullet_{area}(P) \ge r = r(p)$$

signify that there exists a smooth map

$$\tilde{B} : \tilde{P}^\Delta \to S^\bullet(r)$$

with the following two properties.

$\bullet_{hom}$ the map $\tilde{B}$ sends p-fibers to p-fibers

$$\tilde{P}_p \xrightarrow{\tilde{B}} S^\bullet_p(r(p)),\ p \in P,$$

such that $\partial\tilde{P}_p \cup \partial_\infty P_p$ is sent to the point $\bullet_p \in S^\bullet_p(r(p))$ for all $p \in P$, and such that $\tilde{B}$ is homotopic to $\tilde{\mathcal{B}}$ by such fiber preserving continuous maps $\tilde{P}^\Delta \to S^\bullet(r)$

$\bullet_{area}$ The map $\tilde{B}$ is fiber-wise smooth, it is C^1-continuous in $p \in P$[42] and area decreasing on all fibers $\tilde{P}_p$.

Example. It is shown in (different terms) in [GL83] that locally homogeneous spaces P with contractible universal coverings have $\tilde{\mathcal{R}}^\bullet_{length}(P) = \infty$,[43] i.e. $\tilde{\mathcal{R}}^\bullet_{lengh}(P) \geq r$ for all r.

It follows, for instance, that the Riemannian products of these P with unit spheres satisfy

$$\tilde{\mathcal{R}}^\bullet_{area}(P \times S^l) \geq \pi - \varepsilon$$

for all $\varepsilon > 0$.

5 Focal Radii and Normal Curvatures

Let $X = (X, g)$ be a complete Riemannian n-manifold, possibly with a boundary, e.g. $X = B^{n_1}(1) \times \mathbb{R}^{n_2}$.

Let Z be a $(k+l)$-dimensional manifold, e.g. homeomorphic to $\mathbb{T}^k \times S^l$ or to $\mathbb{R}^k \times S^l$ and let $f : Z \to X$ be a smooth immersion.

Let (compare with the proof of 1.C)

$$BT^\perp(R) \subset T^\perp(Z)$$

be the R-ball subbundle in the normal bundle of Z for $R < foc.rad(Z)$, remove arbitrarily small open ε-neighbourhood of the zero section $Z \hookrightarrow T^\perp(R)$ from this ball bundle, let

$$P = BT^\perp(R) \setminus U_\varepsilon(Z)$$

and endow this P with the same Riemannian metric g as X, that is induced by the normal exponential map $\exp^\perp : BT^\perp(R) \to X$ from the Riemannian metric g in X.

Let $S(r/2) \subset T = BT^\perp(R)$ be the $(R/2)$-sphere subbundle, and let

$$X^\perp = X^\perp_{r,R-r} \subset P,\ r < R/2,$$

be the bundle of annuli pinched between of the sphere subbundles

$$S(r), S(R-r) \subset P.$$

Let $inj.rad_f(z)(X) \geq \frac{R}{2} - r$, $z \in Z$, and observe that

$$dist_P(S(r), S(R-r)) = R - 2r$$

in this case.

[42] I am not certain if this is truly necessary for the validity of 4.C.

[43] "Length" instead of "area" means that the corresponding maps $B : \tilde{P}_p \overset{\tilde{B}}{\to} S^\bullet_p(r(p))$ are length decreasing.

As in the proof of 1.C, we derive a bound on $R = foc.rad(Z) - \varepsilon$, $\varepsilon \to 0$, hence on $foc.rad(Z)$,[44] by applying 4.C to the imbedding $X^\perp \hookrightarrow P$ as follows.

Let

$$h_l \subset H_l(Z, \partial_\infty Z; \mathbb{Q})$$

be a *non-zero rational homology class*, which has *infinite* $\tilde{\square}^\perp$*-spread* with respect to the induced metric in Z.

$$\tilde{\square}^\perp(h_l) = \infty.$$

Let $m = l + (n-k-l-1) = n-k-1$ and let

$$h_m \in H_m(S(R/2), \partial_\infty S(R/2) = H_m(X^\perp, \partial_\infty X^\perp) = H_m(P, \partial_\infty P)$$

be the image of $h_l \in H_l(\tilde{Z}, \partial_\infty \tilde{Z}; \mathbb{Q})$ under the ratioanal *Gysin's homomorphism*

$$H_l(Z; \partial_\infty Z; \mathbb{Q}) \to H_m(P; \partial_\infty P; \mathbb{Q})$$

for the $(n-k-l-1)$-sphere bundle $S(R/2) \to Z$.[45]

Observe that if the Euler class of the normal bundle $T^\perp(Z) \to Z$,

$$\chi^\perp_{n-k-l} \in H_{n-k-l}(Z^\perp),$$

vanishes on some $h_k \in H_k(Z)$, which has *non-zero intersection number with* h_l, then h_m *doesn't vanish*. This happens, for instance, if k is odd or if $n-k-l-1$ is even, or if the map $f : Z \to X$ is an *embedding* and $H_{k+l}(X) = 0$.

Also there often exists a covering $\tilde{Z} \to Z$, such that the lift of h_l to $\tilde{h}_l \in H_l(\tilde{Z}, \partial_\infty Z; \mathbb{Q})$ doesn't vanish, while the Euler class of the bundle $\tilde{S}(R/2) \to \tilde{Z}$ vanishes on some $\tilde{h}_k$.

For instance if h_l is the homology class of the pullback of a generic point under a smooth proper map $Z \to \underline{Z}$, where $\underline{Z}$ is a compact enlargeable aspherical manifold (e.g. admitting a metric with non-positive curvature) and $\tilde{Z} \to Z$ is the covering induced from the universal covering of $\underline{Z}$, then the corresponding $\tilde{h}_m \in H_m(\tilde{P}, \partial_\infty \tilde{P}; \mathbb{Q})$ *doesn't vanish*.

Finally, observe that the $\square^\perp$-spread of the corresponding $\tilde{h}_m \in H_m(\tilde{X}^\perp, \partial_\infty \tilde{X}^\perp) = H_m(\tilde{S}(R/2), \partial_\infty \tilde{S}(R/2)$ satisfies

$$\square^\perp(\tilde{h}_m) \geq dist_P(S(r), S(R-r)) = R - 2r$$

for all coverings $\tilde{Z}$ of Z and that $U \subset P$ defined in 4.C as

$$U = \{p \in P\}_{dist(p, X^\perp) \leq dist(p, \partial P)}$$

is equal to the $(\rho, R-\rho)$ annulus bundle for $\rho = r/2$.

---

[44]Since we prefer the *closed ball* bundle for P we can't use $R = foc.radZ$.

[45]The Gysin homomorphism is defined for orientable sphere bundles; if the bundle $S(R/2) \to Z$ is non-orientable, we pass to an orientable double covering of it.

5.A. Conclusion. If the universal covering of $T^{\perp}(Z)$ is spin and $n = dim(X) = n \leq 8$,[46] then theorem 4.C applied with this U, with

$$r = \frac{1}{3}R, \rho = \frac{1}{6}R$$

and with

$$\tilde{R}^{\bullet} = \inf_{p \in U} Rad^{\bullet}_{\tilde{p}}(\tilde{P}, r = inj.rad_{\tilde{p}}(\tilde{P})).$$

for the points $\tilde{p}$ over $p \in P$ in the universal covering $\tilde{P}$ of P shows that

$$[\square^{\perp}\& foc.rad] \qquad \frac{n+N-1}{n+N}\frac{4(n-m)\pi^2}{9foc.rad(Z)^2} \geq Sc^{\rtimes}_N(X) + \frac{m(m-1)}{(\tilde{R}^{\bullet})^2}.$$

DISCUSSION. Focal radius is closely related to the *maximal normal curvature* of an immersion $f : Z \to X$, that is *the supremum of the X-curvatures of the geodesic lines in Z,* i.e. of the curvatures measured in the Riemannian geometry of X of geodesics in Z for the induced Riemannian metric in Z.

However, the inequality $[\square^{\perp}\& foc.rad]$ yields a significant lower bound on the curvatures of immersions to "simple" manifolds X, e.g. to the unit ball $B^n \subset \mathbb{R}^n$ (where $foc.rad(Z) = \min(dist(Z, \partial B^n, 1/curv(Z)))$ only for large $n/m >> 8$, where this inequality remains conjectural.

In contrast, Anton Petrunin [Pet23] recently proved the following.

5.B. ***Asymptotically Optimal $\sqrt{3}$-Inequality.*** If a compact k-manifold Z *admits no metric with* $Sc > 0$, e.g. Z is enlargeable, then immersions $Z \hookrightarrow B^n$ have $curv(Z) \geq \sqrt{3k/(k+2)}$ for all k and n.

(Optimality of this result is proved in [Gro22], where we construct immersions $Z^k \to B^{8k^2}$ with $curv(Z) = \sqrt{3k/(k+2)} + \varepsilon$ and where we also present upper and lower bounds on the curvatures of immersions $Z^k \to X^n$ for $n \sim k$.)

6 Mean Convex Domains in $\mathbb{R}^n$

Let X be an infinite tunnel in the 3d space, that is a closed subset $X \subset \mathbb{R}^3$ diffeomorphic to the cylinder $B^2 \times \mathbb{R}$.

6.A. Example: ***Fat Mouse in a Narrow Tunnel.*** If the mean curvature of the boundary of X is everywhere $\geq 1 = mean.curv(S^2(2))$, then a "mouse", which contains an $(1+\varepsilon)$ ball $B^3(1+\varepsilon)$ inside its body won't be able to crawl trough this tunnel:

If a connected subset $Z \subset X$ infinitely stretches out to the both ends of X, i.e. it is not contained in $B^2 \times S$ for a proper subset $S \subsetneq \mathbb{R}$, *then the 1-neighbourhood $U_1(Z) \subset \mathbb{R}^3$ of Z intersects the boundary of X.*

Remarks. (a) The model X is the infinite round cylinder $B^2(1) \times \mathbb{R} \subset \mathbb{R}^3$, where $mean.curv(\partial X) = 1$. and where the optimal Z is the central line $0 \times \mathbb{R} \subset X$ with $U_1(Z) = X$.

(b) Instructive examples of tunnels with $mean.curv(\partial X) \geq 1$ are made by interconnecting infinite chains of disjoint balls $B^3_{x_i}(2-\varepsilon) \subset \mathbb{R}^3$ of radii $2-\varepsilon$, $\varepsilon > 0$, by narrow tubes, such that X contains all

[46] Both "spin" and "≤ 8" conditions are, probably, redundant. One has no idea what to do with spin, but, as we mentioned earlier, the argument in [CMS23] extended to μ-bubbles, (which seems not difficult) would allow one to relax the dimension bound to $n \leq 10$.

these balls, the mean curvature of X remains $\geq 2-2\varepsilon$ close to the balls and is arbitrarily large everywhere else.[47]

Proof of 6.A. This follows from the following proposition together with the *Gehring-Bombieri-Simon* linking/filling theorem (see section 8.1 in [Gro83] and references therein.)

6.B. Lemma. Let $X \subset \mathbb{R}^3$ be a smooth closed connected, possibly infinite, domain with smooth *non-simply connected* boundary. If $mean.curv(\partial X) \geq 1$, Then the boundary of X contains *non-contractible* closed curves $\Theta \subset \partial X$ of *lengths* $\leq 2\pi + \varepsilon$ for all $\varepsilon > 0$.

In fact, since $Sc(X) \geq 0$, theorem 1.1 in [GZ21] applied to large compact parts of X, shows that the manifold X contains connected simply connected surfaces Σ_ε, for all $\varepsilon >$ with $\partial\Sigma = \Theta \subset \partial X$, which represent non-trivial homology classes in $H_2(X, \partial X) = \mathbb{Z}$ and such that

$$\int_\Theta mean.curv(\partial X, \theta) d\theta \leq 2\pi + \varepsilon.$$

Since $mean.curv(\partial X) \geq 1$, the lemma follows.

Remarks. (a) The proof in [GZ21] elaborates on the following observation, which applies to all Riemannian 3-manifols X (compare with [SY79a] and 5.4 in [Gro14a]).

Let $\Sigma \subset X$ be a *connected stable minimal* surface with boundary $\Theta \subset \partial X$ and observe that the stability condition implies that the intrinsic curvature of $\partial\Sigma \subset \Sigma$ is bounded from below by the mean curvature of ∂X at all points $\theta \in \partial\Sigma \subset X$. Then, as in [SY79a], the stability inequality to the unit normal field on Σ one shows that

$$\int_\Sigma Sc(X, \sigma) d\sigma + \int_\Theta mean.curv(\partial X, \theta) d\theta \leq 2\pi,$$

which, for $Sc(X) \geq 0$ and $mean.curv(\partial X) \geq 1$, yields the inequality $length(\Theta) \leq 2\pi$.

(b) Somewhat paradoxically, *locally length minimizing* ("length-stable") geodesics $\Theta \subset \partial X$ may have arbitrarily large lengths and the minimal surfaces $\Sigma_\Theta \subset X$ with boundaries Θ can't be deformed to locally minimizing ones with free boundaries by area decreasing homotopies.

In fact, the minimization process of areas of surfaces bounded by long $\Theta_{min} \subset \partial X$ divides Θ_{min} into several shorter curves in ∂X.

(c) If X is a *compact non-simply connected* domain in $\mathbb{R}^3$ with $mean.curv(\partial X) \geq 1$, then, by the rigidity theorem 1.3 in [GZ21], the shortest non-contractible curve $\Theta_{min} \subset \partial X$ has length$< 2\pi$.

Probably, if $diam(X) \leq 1$, then $length(\Theta_{min}) \leq 2\pi - 0.01$. (Round tori seem good candidates for extremal X.)

Let us generalize 6.A. to Riemannian manifolds X, $Sc(X) \geq \sigma$ of higher dimensions n, where the boundary of our X is divided into two parts, $\partial X = \partial_\square \cup \partial_{side}$, where these are smooth domains in ∂X, which meet along their common boundary, $\partial\partial_\square = \partial\partial_{side} \subset \partial X$.[48]

[47] Similarly to the thin codimension 2 surgery of manifolds with $Sc > \sigma$ (1.3 in [Gro21]), hypersurfaces with $mean.curv > \mu$ admit codimension 1 surgery, (24 in [Gro17]).

Namely, let V be a domain with a smooth boundary in a Riemannian n-manifold U and $Y \subset U$ be a smooth submanifold of dimension $dim(Y) \leq n-2$ with a boundary, such that $V \cap Y = \partial Y \subset \partial V$, where the intersection between Y and ∂V is transversal.

Let $\mu(u)$ be a continuous function on U, such that $\mu(u) > mean.curv(\partial V)$ at all boundary points $u \in \partial V$. Then the union $V \cup Y$ admits an arbitrarily small regular neighbourhood $V_+ \supset V \cup Y$ with smooth bounary, such that $mean.curv\partial(V_+, u) > \mu(u)$ for all $u \in \partial V_+$.

[48] Our $\partial\square$ corresponds to ∂_{eff} in [GZ21].

Example. If X is the product of a smooth manifold B by the k-cube $\square^k = [-1,1]^k$ then $\partial_{side}X = \partial \times \square^k$ and $\partial_\square = B \times \partial\square^k$.

(For the above cylindrical $X = B^2 \times \mathbb{R}$, the side boundary is $\partial B^2 \times \mathbb{R}$ and $\partial_\square$ corresponds to the ideal top and bottom of the cylinder.)

Let $\Psi : X \to \square^k = [-1,1]^k$ be a continuous map which sends $\partial_\square X \to \partial\square^k$ and let, as in section 2, d_i, $i = 1,\dots,k$, be the distances between the pullbacks of the opposite faces of the cube $\square^k$ and

$$d_\square = d_\square(\Psi) = \left(\frac{1}{k}\sum_{i=1}^{k}\frac{1}{d_i^2}\right)^{-\frac{1}{2}}.$$

Let $Z \subset X$ be a closed subset and $h_k \in H_k(Z, Z \cap \partial_\square X)$ be a homology class which doesn't vanish under the homology homomorphism induced by Ψ,

$$0 \neq \Psi_*(h_k) \in H_k(\square^k, \partial\square^k) = \mathbb{Z}.$$

6.C. (n,k)**-Mouse Conjecture.** Let $sect.curv(X) \leq \kappa$, $\kappa \geq 0$, let

$$mean.curv(\partial_{side}X) \geq \mu.$$

and let

$[Sc_\square(n,k)]$ $$Sc(X) \geq \frac{n-1}{n}\cdot\frac{4k\pi^2}{d_\square^2}.$$

Then

$$inj.rad_Z(X) = \inf inj.rad_{z\in Z}(X) \leq r_\kappa,$$

where r_k is equal to the radius of the ball B in the sphere $S^{n-k}(1/\sqrt{\kappa})$, such that $mean.curv(\partial B) = \mu$. and where we prove this if the universal covering $\tilde{X}$ is spin and $n \leq 8$.

6.D. Corollary/Example. Let $X \subset \mathbb{R}^n$ be a closed domain homeomorphic to $B^2 \times Z_0 \times \mathbb{R}$, where Z_0 is a compact $(k-1)$-dimensional manifold, such that $\tilde{\square}^\perp(Z_0) = \infty$, e.g. Z_0 is the $(k-1)$-torus $\mathbb{T}^{k-1}$.

Let $f : Z_0 \times \mathbb{R} \to X$ be a proper map properly homotopic to the imbedding $Z_0 \times \mathbb{R} = 0 \times Z_0 \times \mathbb{R} \hookrightarrow X$. If $mean.curv(\partial X) \geq n-k-1$, then, provided $n \leq 8$,

$$dist(f(Z_0 \times \mathbb{R}), \partial X) \leq 1.$$

In fact, 6.C applies to compact domains $X_1 \subset \dots \subset X_i \subset \dots \subset X$ which exhaust X and where the Euclidean metrics are slightly perturbed to make $Sc(X_i) > \sigma_i$ for $0 \leq \sigma_i \to 0$.[49]

6.E. Theorem. *If $n \leq 8$, then conjecture 6.C and corollary 6.D hold true.*

Proof. Our proof of the injectivity radius $\square_*^{\exists\exists}(n,m)$-theorem 4.C shows that 6.E reduces to a generalization of the area contraction theorem 3.C to families of maps from manifolds with mean convex boundaries to balls in the spheres S^N.

This generalization reduces to two other propositions 6.F and 6.H below, where 6.F generalizes 2.C to manifolds with mean convex boundaries and which is resolved for $n \leq 8$, while 6.H generalizes

[49]One may also use $g^{\rtimes} = g_{Eucl} + \phi^2 dt^2$ on $X \times \mathbb{T}^1$, for $\phi(x) = 1 + \varepsilon \cdot dist(x, \partial X)$.

codimension zero parametric area contraction inequality 3.A, where an intended proof needs an extension of Llarull's (algebraic) inequality and where this inequality for maps from surfaces follows from [GZ21]. (This is sufficient for the proof of 6.C for $n-k=2$.[50])

PREPARATIONS FOR 6.F.

Given a Riemannian manifold $(Y, g=g_Y)$ with a distinguished domain $\partial_* \subset \partial Y$. e.g. $\partial_* = \partial Y$, let us incorporate the mean curvature of ∂ in the definition of $Sc^{\exists\rtimes}$ from section 2 as follows.

Let $Sc(Y\&\partial_*)$ denote the pair $(Sc(Y), mean.curv_g(\partial_*))$, where $Sc(Y)$ is a function on Y and $mean.curv_g(\partial_*))$ is a function on ∂_* and if

$$Y^{\rtimes} = Y \times \mathbb{T}^N, g^{\rtimes} = g_Y + \sum_{i=1}^{N} \varphi_i^2 dt_i^2,$$

then $Sc(Y^{\rtimes}\&\partial_*^{\rtimes})$ is the pair $(Sc(g^{\rtimes}), mean.curv_{g^{\rtimes}}(\partial_* \times \mathbb{T}^N))$.

Accordingly, we introduce the "inequality"

$$Sc^{\exists\rtimes}(Y\&\partial_*) > (\sigma, \mu)$$

for functions σ on Y and μ on ∂_* as the existence of $g^{\rtimes}$, such that $Sc(g^{\rtimes}) > \sigma$ and $mean.curv_{g^{\rtimes}}(\partial_* \times \mathbb{T}^N) \geq \mu$, where this is used below for $\partial_* = \partial Y$.

Then, we define $Sc^{\exists\exists\partial\rtimes}$ on homology classes of Riemannian manifolds $X = (X, g = g_X)$ relative to a given $\partial_* \subset \partial X$, where we represent such classes $h_m \in H_m(X, \partial_*)$ by 1-Lipschitz maps from m-manifolds Y to X, such that $\partial(Y) \to \partial_*$, and where, similarly to section 2, we use notation

$$Sc^{\exists\exists\partial\rtimes}(h_m) > (\psi, \nu) \text{ for functions } \psi \text{ on } X \text{ and } \nu \text{ on } \partial_*$$

as the existence of Y and f, such that

$$Sc^{\exists\rtimes}(Y\&\partial Y) > (\sigma \circ f, \mu \circ f).$$

Next, let the boundary of X be decomposed as above, $\partial X = \partial_\square \cup \partial_{side}$, define $\tilde{\square}^{\perp}(h_m)$ for $h_m \in H_m(X, \partial_{side} \cup \partial_\infty X)$ with maps Ψ from X to the cube $[-1,1]^k$, $k = n-m$, such that $\partial_\square(X)$ goes to the boundary of the cube and theΨ-pullback of a generic point is homologous to h_m.

If $n \leq 8$, then the proof of 2.C extended to μ-bubbles with boundaries in $\partial_{side}X$ (see section in [Gro21] and [GZ21]) yields the following proposition, which remains conjectural for $n \geq 8$.

6.F. $\square_{\partial}^{\exists\exists}(n,m)$**-Theorem**. Let X be a Riemannian n-manifold with decomposed boundary $\partial X = \partial_\square \cup \partial_{side}$, let $X^{\rtimes} = (X \times \mathbb{T}^N, g^{\rtimes})$ be a $\mathbb{T}^{\rtimes}$-extension of X (for $g^{\rtimes} = g_X^{\rtimes} = g_X + \sum_{i=1}^N \phi_i^2 dt_i^2$), denote $\nu^{\rtimes} = mean.curv_{g^{\rtimes}}(\partial_{side} \times \mathbb{T}^N)$ and let $h \in H_m(X, \partial_{side} \cup \partial_\infty X)$, $m = n-k$.

If $n \leq 8$, then

$$[Sc_{\partial}^{\exists\exists\partial\rtimes}] \qquad Sc^{\exists\exists\partial\rtimes}(h) \geq \left(Sc(X^{\rtimes}) - \frac{N+n-1}{N+n} \cdot \frac{4k\pi^2}{\tilde{\square}^{\perp}(h)^2},\ \nu^{\rtimes}\right),$$

[50] Conjecture 6.C for $n-k=2$ and all n follows from 6.F below, which itself, as in similar cases mentioned earlier, would follows for all n granted a μ-bubble version of theorem 4.6 from [SY17].

$[Sc^{\exists\exists\partial\rtimes}_{\partial,sp}]$ $$Sc^{\exists\exists\partial\rtimes}_{sp}(h) \geq \left(Sc(X^{\rtimes}) - \frac{N+n-1}{N+n}\cdot\frac{4k\pi^2}{\tilde{\square}^{\perp}(h)^2},\ \nu^{\rtimes}\right), \text{ if } X \text{ is spin,}$$

$[Sc^{\exists\exists\partial\rtimes}_{\partial,\widetilde{sp}}]$ $$Sc^{\exists\exists\partial\rtimes}_{\widetilde{sp}}(h) \geq \left(Sc(X^{\rtimes}) - \frac{N+n-1}{N+n}\cdot\frac{4k\pi^2}{\tilde{\square}^{\perp}(h)^2},\ \nu^{\rtimes}\right), \text{ if } \tilde{X} \text{ is spin,}$$

PREPARATIONS FOR 6.G AND 6.H

Let $Q = (Q,G)$ be an orientable n-dimensional Riemannian manifold with a boundary and let $\mathcal{X}$ be a smooth orientable m-dimensional foliation on Q where all leaves are transversal to the boundary ∂X.

Let $\varphi_i(q)$, $i = 1,...N$, be smooth positive functions on Q, let

$$G^{\times} = G + \sum_{i=1}^{N} \varphi_i^2 dt_i^2$$

and let $Q^{\rtimes} = Q \rtimes \mathbb{T}^N = (Q \times \mathbb{T}^N, G^{\rtimes})$.

Let $\sigma^{\rtimes}(q) = Sc(X_q^{\rtimes})$, $q \in Q$, where $X_q^{\rtimes} = X_q \rtimes \mathbb{T}^N \subset Q^{\rtimes}$ for the leaf $X_q \subset Q$ passing through $q \in Q$ and where $X_q^{\rtimes}$ is endowed with the Riemannian metric $g_q^{\rtimes}$ induced from $G^{\rtimes}$ on $Q^{\rtimes} \supset X_q^{\rtimes}$.

Let $\mu^{\rtimes}(q) = mean.curv_{g_q^{\rtimes}}(\partial X_q^{\rtimes})$, $q \in \partial Q$.

Let $\underline{S}^n\left(\sqrt{1/\underline{\kappa}}\right)$ denote the complete simply connected n-space with constant curvature $\underline{\kappa}$: spherical for $\underline{\kappa} > 0$, Euclidean for $\underline{\kappa} = 0$ and hyperbolic for $\underline{\kappa} < 0$.[51]

Let $\underline{B} = B(\underline{\kappa}, \underline{r}) \subset \underline{S}^n\left(\sqrt{1/\underline{\kappa}}\right)$, be the $\underline{r}$-ball in this "sphere" and let $\underline{\mu} = \underline{\mu}(\underline{\kappa}, \underline{r})$ be the mean curvature of the boundary $\partial\underline{B}$.

Let $F : Q \to \underline{B}$ be a smooth map, which sends $\partial Q \to \partial\underline{B}$ and which is *locally constant at infinity*, i.e. the complement of a compact subset in Q goes to a *finite subset* in $\underline{B}$.

6.G. Conjecture: Maps from Foliations to Balls. Let the manifold $Q = (Q,G)$ be metrically complete and let $\sigma^{\rtimes} > 0$ and $\mu^{\rtimes} > 0$.

Let the differential $dF(q)$ restricted to the leaf X_q and its exterior square $\wedge^2 dF(q)$ satisfy the following inequalities:

$\bullet_\mu$ $||dF|\partial X_q|| < \frac{\underline{\mu}}{\mu^{\rtimes}(q)}$ *for all $q \in \partial Q$,*

$\bullet_\sigma$ $||\wedge^2 dF|X_q|| \leq \frac{m(m-1)\underline{\kappa}}{\sigma^{\rtimes}(q)}$ *for all $q \in Q$.*

Then the map F has zero degree.

6.H. Theorem. Let, besides the above $\bullet_\mu$ and $\bullet_\sigma$, the following three conditions be satisfied.

(i) $\mathcal{X}$ is a fibration.

(ii) The scalar curvature of $\mathcal{X}^{\rtimes}$ is uniformly positive, $\sigma^{\rtimes} \geq \varepsilon > 0$.

(iii) The tangent bundle of the lift of the foliation $\mathcal{X}$ to the universal covering $\tilde{Q}$ of Q is spin.

Then the conclusion of 6.G holds true: the map F does have *zero degree.*

About the Proof. If $m = 2$ and $N = 0$, this follows from the Gauss-Bonnet theorem and if $N \geq 1$ a similar proof seems plausible (but I didn't check this).

[51] This notation is different from that in section 2.1.B_4.

But if $m \geq 3$, except maybe for $m = 3$ and $N = 0$, the only available approach to the proof is via Dirac operators where either $T(Q)$ or $T(\mathcal{X})$ lifted to the universal covering of Q must be spin; in fact if X has no boundary, then 6.F is proved [SWZ21], provided either $T(Q)$ or $T(\mathcal{X})$ is spin.

For an individual manifold $X = Q$ with a boundary, where the universal covering $\tilde{X}$ is spin, 6.G and 6.H follow from *Goette-Semllman's theorem* [GS02] applied to suitably smoothed double $\mathbb{D}X = X \sqcup_{\partial X} X$ (see section 4.3 in [Gro21]) mapped to a double of the receiving ball, and where a more satisfactory proof, which relies on a relative index theorem and which has an advantage of yielding rigidity of convex domains in symmetric spaces, is due to John Lott [Lot21].

To extend either of these arguments to foliations one needs a version of inequality 4.6 from [Lla98], which allows replacement of the sup-norms in the inequalities $\bullet_\mu$ and $\bullet_\sigma$ by the *normalized trace norms*

$||dF|\partial X|| \rightsquigarrow ||dF|\partial X||_{trace}/(m-1)$ and $|| \wedge^d F|| \rightsquigarrow || \wedge^d F||_{trace}/m(m-1)$

(see section 3.4 in [Gro21], where this is explained for individual manifolds) and to prove 6.G, one needs only the following.

6.I. Local Comparison Lemma. Let a point $s_\circ \in S^n$ be taken for a *pole* in the sphere a let us call an equatorial subsphere $S^m \subset S^n$ *radial* if it is made by geodesic segments radiating from the pole $s_\circ \in S^n$, which is equivalent to inclusion $S^m \ni s_\circ$.

Let $X_\rtimes$ be a Riemannian manifold, let $x_\rtimes \in X_\rtimes$ and let $f : X_\rtimes \to S^n$ be a smooth map, such that the differential $df : T(X_\rtimes) \to T(S^n)$ at a point $x_\rtimes \in X_\rtimes$ sends the tangent space $T_{x_\rtimes}(X_\rtimes)$ to the tangent space of a *radial equatorial* (sub)sphere $S^m \subset S^n$,

$$df(T_{x_\rtimes}(X_\circ)) \subset T_{f(x_\rtimes)}(S^m).$$

Let $f' : X_\rtimes \to S^m$ be a smooth map, differential of which at $x_\rtimes$, is equal to that of f, i.e. $f'(x_\rtimes) = f(x_\rtimes) \in S^m \subset S^n$ and the linear map $d_{x_\rtimes} f' : T_{x_\rtimes}(X_\rtimes 0 \to T_{f(x_\rtimes)}(S^m)$ is equal to $d_{x_\rtimes} f$.

Let

$$g_\circ = \phi_1(r)^2 dr^2 + \phi_2(r)^2 ds^2, \ r(s) = dist(s, s_\circ),$$

be a *radial* Riemannian metric on S^n, which is equivalent in the present case to *invariance of* $g_\circ$ *under the orthogonal subgroup* $O(n-1) \subset O(n)$*, which fixes the pole* $s_\circ \subset S^n$.

Let $\mathcal{D}$ be the Dirac operator on X with the coefficients in the pullback $f^*(\mathbb{S}^\pm(S^N))$ for the Clifford (spinor) bundle $\mathbb{S}^\mp$ over S^n,

$$\mathcal{D} : C^\infty((\mathbb{S}^\mp(X_\rtimes) \otimes f^*(\mathbb{S}^\mp(S^n)) \circlearrowleft$$

and

$$\mathcal{D}' : C^\infty((\mathbb{S}^\mp(X_\rtimes) \otimes (f')^*(\mathbb{S}^\mp(S^m)) \circlearrowleft,$$

be Dirac operator with the coefficients in the f'-pullback $(f')^*(\mathbb{S}^\pm(S^m))$,

Let

$$B = \nabla\nabla^* : C^\infty((\mathbb{S}^\mp(X_\rtimes) \otimes (f')^*(\mathbb{S}^\mp(S^m)) \circlearrowleft$$

and

$$B' = \nabla'(\nabla')^* : C^\infty((\mathbb{S}^\mp(X_\rtimes) \otimes (f')^*(\mathbb{S}^\mp(S^m)) \circlearrowleft$$

be the corresponding (positive) *Bochner Laplacians* and let us represent the (zero order) difference operators $\mathcal{D} - B$ and $\mathcal{D}' - B'$ by endomorphisms of the corresponding vector bundles, while keeping the notation

$$\mathcal{D} - B : \mathbb{S}^\mp(X_\rtimes) \otimes (f)^*(\mathbb{S}^\mp(S^n)$$

and
$$\mathcal{D}' - B' : \mathbb{S}^{\mp}(X_{\rtimes}) \otimes (f')^*(\mathbb{S}^{\mp}(S^m)).$$

Then *the lowest eigenvalue of the (linear selfadjoint operator) action of $\mathcal{D} - B$ on the fiber*
$$[\mathbb{S}^{\mp}(X_{\rtimes}) \otimes (f)^*(\mathbb{S}^{\mp}(S^n)]_{x_{\rtimes}} = \mathbb{S}^{\mp}(X_{\rtimes})_{x_{\rtimes}} \otimes \mathbb{S}^{\mp}_{f(x_{\rtimes})}(S^n)$$
is equal to lowest eigenvalue of the operator $\mathcal{D}' - B'$ on
$$[\mathbb{S}^{\mp}(X_{\rtimes}) \otimes (f)^*(\mathbb{S}^{\mp}(S^n)]_{x_{\rtimes}} = \mathbb{S}^{\mp}(X_{\rtimes})_{x_{\rtimes}} \otimes \mathbb{S}^{\mp}_{f(x_{\rtimes})}(S^m).$$

Proof. Since the $g_\circ$-Riemannian (Levi-Cevita) connection on the normal bundle $T^\perp(S^m) = T(S^n)_{|S^m} \ominus T(S^m)$ is (obviously) parallel, the $(S^n, g_\circ)$-spin bundle restricted to S^m decomposes into a sum of 2^{n-m} copies of the $(S^m, g_\circ)$-spin bundle.

Then $\mathbb{S}^{\mp}(X_{\rtimes})_{x_{\rtimes}} \otimes \mathbb{S}^{\mp}_{f(x_{\rtimes})}(S^n)$ decomposes into the corresponding sum of copies of $\mathbb{S}^{\mp}(X_{\rtimes})_{x_{\rtimes}} \otimes \mathbb{S}^{\mp}_{f'(x_{\rtimes})}(S^m)$, for $f(x_{\rtimes}) = f'(x_{\rtimes})$ and $d_{x_{\rtimes}} f = d_{x_{\rtimes}} f'$ and, by theorems II.4.16 and II.8.17 in [LM89] (compare with section 4 in [Lla98]), the operator $\mathcal{D} - B$ on $\mathbb{S}^{\mp}(X_{\rtimes})_{x_{\rtimes}} \otimes \mathbb{S}^{\mp}_{f(x_{\rtimes})}(S^n)$ decomposes accordingly.

Thus, the two operators have the same eigenvalues but with different multiplicities.

6.J. Corollary. Let $X_{\rtimes}$ be a Riemannian manifold, let $x_{\rtimes} \in X_{\rtimes}$, let $(S^n, g_\circ)$ be the n-sphere with a radial metric with respect to a given pole in S^n and let $f : X_{\rtimes} \to \underline{S}^n$ be a smooth map.

If $sect.curv(g_\circ, x_{\rtimes}) \geq 0$ and if
$$\|\wedge^2 d_{x_{\rtimes}} f\| < \frac{m(m-1)\underline{\kappa}(x_{\rtimes})}{Sc(X_{\rtimes}, x_{\rtimes})}$$
then, in the cases $\bullet_{cnst}$ and $\bullet_\circ$ below,

the operator
$$(\mathcal{D} - B)_{\rtimes} : [\mathbb{S}^{\mp}(X_{\rtimes}) \otimes (f)^*(\mathbb{S}^{\mp}(S^n)]_{x_{\rtimes}} \circlearrowleft$$
is positive, i.e. the lowest eigenvalue of $(\mathcal{D} - B)_{\rtimes}$ is >0.

$\bullet_{cnst}$ The sectional curvature of the metric $g_\circ$ at the point $f(x_{\rtimes}) \in S^n$ is *constant* on the set of 2-planes in $T_{f(x_{\rtimes})}(S^n)$.

$\bullet_\circ$ The image $d_{x_{\rtimes}} f(T_{x_{\rtimes}}) \subset T_{f(x_{\rtimes})}(S^n)$ is *contained in the tangent subbundle* of a radial sphere $S^m \subset S^n$.

Proof. The case $\bullet_{cnst}$ follows from the trace inequality 4.6 from [Lla98], (also see proposition 2 in [Lis10]).

The above lemma reduces the general case of $\bullet_\circ$, where $n \geq m$, to $n = m$, where inequality 1.11 in [GS02], applies. (Also see proposition 1 in [Lis10].)

Remark. In both cases $\bullet_{cnst}$ and $\bullet_\circ$, the f-pullback of the curvature operator of $g_\circ$ to $X_{\rtimes}$ is diagonalizable at $x_{\rtimes}$ and 6.G follows by the argument from [Lla98].

Conclusion of the Proof of 6.H. The doubling & smoothing argument in section 3.5 of [Gro21][52] reduces 6.H to the corresponding property of maps from foliations on manifolds *without boundaries* to

[52] See [GL80, Mia02, BMN11, Gro14a, BH21] for various versions of this argument.

spheres with radial metrics, where, in the spin case, the argument in the proof of the mapping theorem 1.2 from [SWZ21] applies.

However, we need an extension of [SWZ21]-arguments to a $\mathbb{T}^{\rtimes}$-stabilized scalar curvature $Sc^{\rtimes}(\mathcal{X}) = Sc(\mathcal{X} \rtimes \mathbb{T}^N)$, instead of the plain $Sc(\mathcal{X})$ and also to where the universal covering $\tilde{\mathcal{X}}$, rather than $\mathcal{X}$ is spin.

For this reason, we need the extra conditions (i), (ii), (iii) but this is sufficient for the proof of of 6.E.

7 A Few Words on Foliations

Let Q be a compact smooth orientable manifold and $\mathcal{X}$ a smooth n-dimensional Riemannian foliation[53] Q with leaves $X = X_q \subset Q$, the scalar curvatures of which are bounded from below by $Sc(X_q) > 0$, which is also written as $Sc(\mathcal{X}) > \sigma$.

Let $h \in H_K(Q)$ be a homology class and let $A \to S^K$ be a smooth leaf-wise area decreasing map, which doesn't vanish at h under the homology homomorphism induced by A,

$$A_*(h) \neq 0 \in H_K(S^K) = \mathbb{Z}.$$

7.A. Problem. Given $\sigma_- \leq m(m-1)$ for $m = n - k$ and $k = dim(Q) - K$, find topological and/or geometric conditions on h that would imply an inequality

$$\sigma \leq \sigma_-.$$

Remark. One could derive some (non-sharp) inequalities of this kind, from the corresponding (mainly conjectural) ones for individual manifolds with $Sc^{\rtimes} \geq \sigma$, if one could proof the following.

7.B. Conjecture. The inequality $Sc(\mathcal{X}) \geq \sigma$, and even $Sc^{\exists\rtimes}(\mathcal{X}) \geq \sigma$, imply that $Sc^{\exists\exists\rtimes}(Q) \geq \sigma$.

(This is proven in [GH23] for codimension 1 foliations $\mathcal{X}$ on manifolds Q with $dim(Q) \leq 7$.)

7.C. 4d-Example. Let Q be a compact orientable 4-manifold and $\mathcal{X}$ an orientable 3-dimensinal Riemannian foliations with $Sc(\mathcal{X}) > 2$ and such that

$\bullet_{<\infty}$ the leafs $X \subset Q$ of $\mathcal{X}$ have at most finitely many ends,[54]

$\bullet_{\neq 0}$ there exists a leaf $X_0 \subset Q$ of $\mathcal{X}$, and a homology class $h \in H_2(X_0)$ such that the image of h in $H_2(Q)$, the image of which in $H_2(Q)$ under the homology inclusion homomorphism $H_2(X_0) \to H_2(Q)$ is *is non-torsion.*

Then there exists a *non-torsion homology class* $h_\circ \in H_3(Q)$, such that all smooth *leaf wise area decreasing* maps $f : Q \to S^3$ send h_3 to *zero*.

$$f_*(h_3) = 0 \in H_3(S^3) = \mathbb{Z}.$$

Moreover, this conclusion holds if the inequality $Sc(\mathcal{X}) > 2$ is *relaxed to* $Sc^{\rtimes}(\mathcal{X}) > 2$.

Proof. Start by recalling the following.

[53] *"Riemannian"* refers to a smooth positive quadratic form on the tangent bundle $T(\mathcal{X}) \to Q$.

[54] It is unclear if this condition is needed.

7.D. Lemma. Let X be a complete orientable Riemannian 3-manifold with $Sc(X) > \sigma + 2\varepsilon > 0$, possibly, with a uniformly mean convex boundary (i.e. $mean.curv(\partial X) \geq \mu > 0$), where $\sigma, \varepsilon > 0$, and let $Y_0 \subset X$ be a compact connected oriented surface with $area(Y_0) \geq \frac{8\pi}{\sigma}$.

Then there exists a one parameter deformation Y_t of Y_0, $0 \leq t \leq 1$, where all Y_t are piecewise smooth 2-cycles, which *continuously depend on t with respect to the flat topology* and such that

(i) $area(Y_t) \leq area(Y_0)$ for all t;

(ii) $Y_1 \subset X$ is a smooth surface with $area(Y_1) \leq \frac{8\pi}{\sigma+\varepsilon}$.

In fact, Y_0 admits an area decreasing deformation to a stable minimal surface Y_{min}, where the the second variation formula and the Gauss-Bonnet theorem, imply that

$$\int_{Y_{min}} Sc(X,y)dy \leq \int_{Y_{min}} Sc(Y_{min},y)dy \leq 8\pi.\text{[55]}$$

Furthermore, the same argument applies to $Sc^{\rtimes}(X) > \sigma + 2\varepsilon > 0$ instead of $Sc(X) > \sigma + 2\varepsilon > 0$ with a use of the Zhu area inequality (see [Zhu19] and section 2.8 in [Gro21]).

Sketch of the Proof of 7.C. Let ν be a nonvanishing vector field on Q nowhere tangent to the leaves of $\mathcal{X}$.

Let $Y_0 \subset X_0$ be a compact smooth connected surface, with *non-torsion* fundamental homology class $[Y] \in H_2(Q)$, where, by the Lemma, one may assume that $area(Y_0) \leq 4\pi - 2\varepsilon$.

Continuously move Y_0 in the direction of ν by small distance $\delta > 0$ locally away from X_0 to another leave $X_1 \subset Q$ locally "downstream" of X_0 relative to ν while keeping $area \leq 4\pi - \varepsilon$ all along. Then move the resulting surface within X_1 to get $Y_1 \subset X_1$ with $area(Y_1) \leq 2 - 2\varepsilon$.

Keep doing this and simultaneously regularize the surfaces, thus obtaining a sequence of surfaces $Y_2, \ldots, Y_i, \ldots$ in different leaves, where the diameters of Y_i with respect to the induced Riemannian metrics and the curvatures of Y_i in the leaves are uniformly bounded.

It follows that there are arbitrarily large i and j, such that Y_i and Y_{i+j} come arbitrarily close together in Q.

Join the two by a narrow cylinder, and obtain a 3-cycle $\mathcal{Y}_{ij}$ sliced by surfaces (some of which may be singular) of areas $\leq 4\pi - \varepsilon$.

Now, the condition $\bullet_{<\infty}$ and the area bounds on (almost minimal) surfaces ($\mu - bubbles$) for 3-manifolds X with $Sc^{\rtimes} > \sigma > 0$, shows that the leaves X can be exhausted by compact domains

$$B(1) \subset \ldots \subset B(2) \subset \ldots \subset X$$

with

$$area(\partial k) \leq const.$$

Hence, these B_k can be completed to 3-cycles $C_k \supset B_i$ with

$$vol_3(C_k \setminus B_k) \leq const'.$$

It is also clear that some leaf X transversally intersect C over a very large (of order of j) number l of our surfaces $Y_{i_1}, \ldots, Y_{i_l}$, which implies that the homology class of intersection $\mathcal{Y}_{ij} \cup C_k$ for large j and k represent a *non-zero* multiple of $h \in H_2(Q)$. Hence, the class $[\mathcal{Y}_{ij}] \in H_3(Q)$ is non-torsion as well.

[55] The second variation formula was brought to the needed form in [SY79a], see section 2.5 in [Gro21] for details.

Finally, since $\mathcal{Y}_{ij}$ is sliced by surfaces with areas $< \pi$, area decreasing maps $\mathcal{Y}_{ij} \to S^3$ have *degrees zero* by Almgren's waist inequality (See [Gro03, Gut14]) and the proof is achieved by taking $h_\circ = [\mathcal{Y}_{ij}]$ with the above (large) i and j.

Sub-Example. Let $\underline{\mathcal{X}}$ be a usual geodesic foliation on the 2-torus $\mathbb{T}^2$ and $\mathcal{X}_0$ be the corresponding foliation on $Q = \mathbb{T}^2 \times S^2$ with the leaves $X = \underline{X} \times S^2$, where these $\underline{X} \subset \mathbb{T}^2$ may be circles or lines.

Observe that the product metric $G_0 = dt^2 + ds^2$ in this foliation has $Sc = 2$, and that all non-zero $h \in H_3(Q) = \mathbb{Z} \oplus \mathbb{Z}$ allow area non-increasing maps $f = f_h : Q \to S^3$ with $f_*(h) \neq 0$.

Let $\underline{Y} \subset \mathbb{T}^2$ be a closed geodesic transversal to $\underline{X}$ and let $Y = \underline{Y} \times S^2 \subset Q$ and let

$$G_\varepsilon = dt^2 + (1 + \varepsilon(t))ds^2,$$

where $\varepsilon(t)$, $t \in \mathbb{T}^2$, is a smooth function which is negative away from a small δ-neighbourhood $U_\delta(\underline{Y}) \subset \mathbb{T}^2$ and is small positive in this neighbourhood.

Apparently, (I haven't checked this carefully) there are functions $\varepsilon(t)$ of this kind arbitrarily C^∞-close to zero, such that $Sc(\mathcal{X}, G_\varepsilon) > 2$ and where all leaf-wise G_ε-area decreasing maps $f : Q \to S^3$ send the class $[Y] \in H_3(Q)$ to zero.

Yet all $h' \in H_3(Q)$, which are *not multiples of h*, do allow leaf-wise G_ε-area decreasing maps $f' : Q \to S^3$ with $f'_*(h') \neq 0$.

7.C. Question. Is there a meaningful version of lemma 7.D for higher dimensions and codimensions?

One knows, for instance, that

if a compact Riemannian manifold X with $Sc(X) \geq 2$ is homeomorphic to $S^2 \times \mathbb{T}^k$, and if $k \leq 6$, then the homology class of the 2-sphere $S^2 = S^2 \times \{t\} \in X$ is representable by a smooth surface with *area* $\leq 4\pi$ (see [Zhu19]).

But if $n = dim(X) \geq 5$ (I am uncertain about $n = 4$) a manifold X with $Sc(X) \geq 2$ may contain locally minimal 2-spheres homologous to S^2 of *arbitrary large* areas.[56]

What is more promising in this regard is the geometry of the functional

$$Y \mapsto \frac{1}{Sc^{\rtimes}(Y)} \text{ on the space } \mathcal{Y} = \mathcal{Y}(X),$$

of closed cooriented smooth (singular?) *hypersurfaces* $Y \subset X$ endowed with induced Riemannian metrics.

Hopefully, the lower bound on $Sc^{\rtimes}(X)$, does have non-trivial influence on the *topologies*[57] of the *sublevels* $\mathcal{Y}_{\leq a} \subset \mathcal{Y}$, of this (or a similar) functional, where $\mathcal{Y}_{\leq a}$ is the space of hypersurfaces $Y \subset X$ with $Sc^{\rtimes}(Y) \geq a^{-1}$, and where we are concerned with *connectedness* of $\mathcal{Y}_{\leq a}$ for $a = Sc^{\rtimes}(X)$, as well as with the *images of the inclusion homology homomorphisms* $H_*(\mathcal{Y}_{\leq a}) \to H_*(\mathcal{Y}_{\leq a+b})$ where a and b/a tend to infinity.

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[56] Such an X can be obtained by *thin surgery* (section 1.3 in [Gro21]) applied to the product $S^2 \times \mathbb{T}^k$, $k \geq 3$, in a small neighbourhood of $S^2 \times \mathbf{0} \subset S^2 \times \mathbb{T}^k$.

[57] Defining an adequate topology in $\mathcal{Y}$ is quite an issue here.

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