JOURNAL OF THE ASSOCIATION FOR MATHEMATICAL RESEARCH

VOLUME 3 : ISSUE 1

2025

JOURNAL OF THE ASSOCIATION FOR MATHEMATICAL RESEARCH

PUBLISHED BY

Association for Mathematical Research, amathr.org,
Davis, CA; Jenkintown PA.

JOURNAL OF THE ASSOCIATION FOR MATHEMATICAL RESEARCH, 3(1):1–26, 2025
jamathr.org

Prime Hasse principles via Diophantine second moments

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Received 28 Nov 2023; Revised 11 Mar 2024; Accepted 25 Mar 2024

Abstract: We show that almost all primes $p \not\equiv \pm 4 \bmod 9$ are sums of three cubes, assuming a conjecture due to Hooley, Manin, et al. on cubic fourfolds. This conjecture is approachable under standard statistical hypotheses on geometric families of L-functions.

Key words and phrases:
Integral points, densities, Manin conjectures, representing primes, Selberg sieve

1 Introduction

Let $F_0(\boldsymbol{y}) = F_0(y_1, y_2, y_3) := y_1^3 + y_2^3 + y_3^3$ and $F_0(S) := \{F_0(\boldsymbol{y}) : \boldsymbol{y} \in S\}$. For well-known local reasons, $F_0(\mathbb{Z}^3) \subseteq \{a \in \mathbb{Z} : a \not\equiv \pm 4 \bmod 9\}$. The *Hasse principle* holds if the set

$$\mathcal{E} := \{a \in \mathbb{Z} : a \not\equiv \pm 4 \bmod 9\} \setminus F_0(\mathbb{Z}^3) \tag{1.1}$$

is empty. The analog of $\mathcal{E}$ for $5y_1^3 + 12y_2^3 + 9y_3^3$ contains a sparse sequence [GS22, p. 691, footnote 3],[1] produced by means that do not apply to F_0 [CTW12, p. 1304]. We confine ourselves to a statistical analysis of $\mathcal{E}$ relative to $\mathbb{Z}$ and the set of primes. For "critical" equations such as $F_0(\boldsymbol{y}) = a$, "subcritical" statistical frameworks (such as those of [Vau80, Brü91, Hoo16]) break down, and new features come into play [GS22, Dia19].

Here "critical" refers to the well-known fact that if, say, $X \geq 1$ and $A \in [X^2, X^3]$, then

$$\mathbb{E}_{|a| \leq A}[\#\{\boldsymbol{y} \in \mathbb{Z}^3 : \|\boldsymbol{y}\| \leq X,\ F_0(\boldsymbol{y}) = a\}] \ll \log(1 + X/A^{1/3}).$$

*This work was partially supported by the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie Grant Agreement No. 101034413.

[1] See [LMU23] for a general study of Brauer–Manin obstructions for ternary diagonal cubic forms.

DOI: 10.56994/JAMR.003.001.001

(Any solutions only "barely" exist!) To produce integral solutions to $F_0(\boldsymbol{y}) = a$ in general, one must take $X/A^{1/3} \to \infty$, not fixed. In fact, for any fixed $\lambda \geq 1$, the set

$$\{a \not\equiv \pm 4 \bmod 9 : a \notin F_0([-|a|^{1/3}\lambda, |a|^{1/3}\lambda]^3)\} \subseteq \mathbb{Z}$$

has lower density > 0 [Dia19, §1]. For further discussion, see [HB92, pp. 622–623].

The scarcity of solutions also forces us to pass from the *sparse* setting of F_0 to a *richer* setting in 6 variables. If $r_3(a) := \#\{\boldsymbol{y} \in \mathbb{Z}^3_{\geq 0} : F_0(\boldsymbol{y}) = a\}$, then by Cauchy,

$$|F_0(\mathbb{Z}^3_{\geq 0}) \cap [0, A]| \geq (\mathbb{E}_{0 \leq a \leq A}[r_3(a)])^2 / \mathbb{E}_{0 \leq a \leq A}[r_3(a)^2]. \tag{1.2}$$

Whereas $\mathbb{E}_{0 \leq a \leq A}[r_3(a)] \asymp 1$ by classical geometry of numbers, the mean square $\mathbb{E}_{0 \leq a \leq A}[r_3(a)^2]$ corresponds to a difficult point count in $3 + 3 = 6$ cubes, which we now introduce.

Write $\boldsymbol{y} = (y_1, y_2, y_3)$, $\boldsymbol{z} = (z_1, z_2, z_3)$, and let $\boldsymbol{x} = (x_1, \dots, x_6) := (y_1, y_2, y_3, z_1, z_2, z_3)$. Let

$$F(\boldsymbol{x}) = F(\boldsymbol{y}, \boldsymbol{z}) := F_0(\boldsymbol{y}) - F_0(\boldsymbol{z}). \tag{1.3}$$

Let Υ denote the set of 3-dimensional vector spaces $L \subseteq \mathbb{Q}^6$ over $\mathbb{Q}$ such that $F|_L = 0$. (The equation $\boldsymbol{y} = \boldsymbol{z}$ cuts out one such L. All other $L \in \Upsilon$ can be generated from $\boldsymbol{y} = \boldsymbol{z}$ by suitable permutations and negations of variables.) Call a tuple $\boldsymbol{x} \in \mathbb{Z}^6$ *special* if

$$\boldsymbol{x} \in \bigcup_{L \in \Upsilon} L. \tag{1.4}$$

We will concentrate on smoothly weighted point counts away from the origin $\mathbf{0}$; there are, however, standard methods to pass to unrestricted, unweighted point counts.

Given integers $X, d, q \geq 1$, and a function $w \in C_c^\infty(\mathbb{R}^6)$ supported away from $\mathbf{0}$, let

$$N_w(X; d) := \sum_{\boldsymbol{x} \in \mathbb{Z}^6 :\, F(\boldsymbol{x}) = 0,\, d \mid F_0(\boldsymbol{y}), F_0(\boldsymbol{z})} w(\boldsymbol{x}/X), \tag{1.5}$$

$$\rho(q; d) := q^{-5} \cdot \#\{\boldsymbol{x} \in (\mathbb{Z}/q\mathbb{Z})^6 : q \mid F(\boldsymbol{x}) \text{ and } \gcd(q, d) \mid F_0(\boldsymbol{y}), F_0(\boldsymbol{z})\}, \tag{1.6}$$

$$\sigma_p(d) := \lim_{l \to \infty} \rho(p^l; d), \quad \sigma_{\infty, w} := \lim_{\varepsilon \to 0} (2\varepsilon)^{-1} \int_{|F(\boldsymbol{x})| \leq \varepsilon} w(\boldsymbol{x})\, d\boldsymbol{x}, \tag{1.7}$$

$$E_w(X; d) := N_w(X; d) - \mathfrak{S}(d) \cdot \sigma_{\infty, w} \cdot X^3 - \sum_{\text{special } \boldsymbol{x} \in \mathbb{Z}^6 :\, d \mid F_0(\boldsymbol{y}), F_0(\boldsymbol{z})} w(\boldsymbol{x}/X), \tag{1.8}$$

where $\mathfrak{S}(d) := \prod_{p \text{ prime}} \sigma_p(d)$ is the *singular series* associated to the equation $F(\boldsymbol{x}) = 0$ with the congruence condition $F_0(\boldsymbol{y}) \equiv F_0(\boldsymbol{z}) \equiv 0 \bmod d$. By directly generalizing [Hoo86a, Conjecture 2 for $l = 3$], one conjectures

$$\lim_{X \to \infty} X^{-3} E_w(X; d) = 0. \tag{1.9}$$

This is a Manin-type randomness-structure dichotomy conjecture on the 6-variable cubic hypersurface $F = 0$ (based on *special subvarieties*, i.e. thin sets of type I), compatible with the general framework of [FMT89], [VW95, Appendix], [Pey95], et al.

By (1.2), $F_0(\mathbb{Z}^3_{\geq 0})$ has lower density > 0, assuming $E_w(X; 1) \ll X^3$ holds for a suitable w. There are conjectures on the density of $F_0(\mathbb{Z}^3_{\geq 0})$ [DHL06], but we focus on $F_0(\mathbb{Z}^3)$. Diaconu, building on [GS22], showed that (1.1) has density 0, assuming a version of (1.9) for $d = 1$ where $N_w(X;1)$ is replaced by a point count over a fairly skew region [Dia19, R_N^* on p. 24]. In fact, as we will see, (1.9) itself suffices. We can say even more about (1.1), assuming that there exists a triple $(\xi, \delta, k) \in \{0,1\} \times \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 1}$ for which the following holds:

$$E_w(X;d) \ll \|w\|_{k,\infty} B(w)^k X^{3-\delta}, \quad \text{uniformly over } d \leq X^{\xi\delta} \text{ and clean } w \in C_c^\infty(\mathbb{R}^6). \tag{1.10}$$

We define the quantities $\|w\|_{k,\infty}$, $B(w)$ in §1.1; they measure the complexity of w. We call a function $f\colon \mathbb{R}^s \to \mathbb{R}$ *clean* if it is supported away from the coordinate hyperplanes, i.e.

$$(\operatorname{Supp} f) \cap \{\boldsymbol{u} \in \mathbb{R}^s : u_1 \cdots u_s = 0\} = \emptyset. \tag{1.11}$$

Theorem 1.1. *Suppose that (1.9) for $d = 1$ holds for all clean functions $w \in C_c^\infty(\mathbb{R}^6)$. Then (1.1) has density 0 in $\mathbb{Z}$. Now fix $(\delta, k) \in \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 1}$, and assume (1.10) for $\xi = 0$. Then $|\mathcal{E} \cap [-A,A]| \ll_\varepsilon A/(\log A)^{1-\varepsilon}$ for all integers $A \geq 2$ (for all $\varepsilon > 0$).*

The qualitative density result in Theorem 1.1 was proven in [Wan22, Theorem 2.1.8] using the strategy of [Dia19, §§2–3]. But our present approach is stronger and more flexible.

Theorem 1.2. *Fix $(\delta, k) \in \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 1}$. Assume (1.10) for $\xi = 1$. Then (1.1) contains at most $O_\varepsilon(A/(\log A)^{2-\varepsilon})$ primes $p \leq A$, for any integer $A \geq 2$ (for any $\varepsilon > 0$).*

Before proceeding, some notes on (1.9), (1.10) are in order:

1. Unconditionally, $E_w(X;1) \ll_{w,\varepsilon} X^{7/2}(\log X)^{\varepsilon - 5/2}$ for $X \geq 2$ [Vau20]. Improving on this is difficult; classically, [Hua38] proved $E_w(X;1) \ll_{w,\varepsilon} X^{7/2+\varepsilon}$.

2. Conditionally, [Hoo86b, Hoo97, HB98] proved the near-optimal bound $E_w(X;1) \ll_{w,\varepsilon} X^{3+\varepsilon}$ assuming GRH for geometric L-functions.

3. Building thereon, [Wan23] proved (1.9) for $d = 1$ (and in fact $E_w(X;1) \ll_w X^{3-\delta}$ for some δ) for all clean functions $w \in C_c^\infty(\mathbb{R}^6)$, assuming mainly some statistical hypotheses of Random Matrix Theory type on geometric families of L-functions.

4. The world of L-functions is very rich. Over function fields, for instance, GRH is known and [BDPW23, MPPRW24, Wan24] give a promising route towards [Wan23]'s hypotheses. See [BGW24] for further details.

5. The hypothesis (1.10) is essentially a version of (1.9) with a specified power saving, uniform over a specified range of integers d and weights w. The conditional approach of [Wan23] should in principle lead to (1.10) for some $(\delta, k) \in \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 1}$.

6. If $E_w(X;1) \ll_w X^\theta$ for all w, where $\theta \geq 3$, then $E_w(X;d) \ll d^{O(1)} \|w\|_{0,\infty} (B(w)X)^\theta$. The uniformity in (1.10) is thus natural, but it requires extra work if $\theta < 3$.

How are Theorems 1.1–1.2 proven? Given a compactly supported function $\nu\colon \mathbb{R}^3 \to \mathbb{R}$, let

$$N_{a,\nu}(X) := \sum_{\boldsymbol{y}\in\mathbb{Z}^3:\, F_0(\boldsymbol{y})=a} \nu(\boldsymbol{y}/X). \tag{1.12}$$

To prove Theorem 1.1, we will choose weights ν (smooth as in [Hoo16], yet tentacled as in [GS22, Dia19]), and bound $N_{a,\nu}(X)$ in approximate variance over $a \ll X^3$. To prove Theorem 1.2, we will also need precise estimates (not just bounds) for such variances, and not just over $\mathbb{Z}$ but also over arithmetic progressions $a \equiv 0 \bmod d$ with $d \le X^\delta$.

In §2, we will define and estimate an approximate variance. In §3, we will show that certain truncated singular series are typically sizable. In §4, we will apply our estimates from §§2–3 to prove Theorems 1.1 and 1.2. Here it is important to allow ν to deform with X. For fixed ν, counting prime values of F_0 (with or without multiplicity) up to constant factors remains a challenge, even conditionally; see [DS19, Conjecture A.3] (Bateman–Horn in several variables) and the present §5 for some concrete open questions in this direction. (It might also be interesting to ask analogous questions for sequences other than the primes.)

How many solutions to $x^3+y^3+z^3=a$ of a given size can our methods produce, for typical a? By slightly modifying §4, one could provide several kinds of answers to this question. A comprehensive discussion would be tedious, so we limit ourselves to a murky remark:

Remark 1.3. In Theorems 1.1 and 1.2, there are three levels of hypotheses in total, say H1, H2, H3, from weakest to strongest. H1 implies that if $h(a)\to\infty$ as $|a|\to\infty$, then there exists $f\colon \mathbb{Z}\to\mathbb{R}$, with $\lim_{|a|\to\infty} f(a)=\infty$, such that for almost all integers $a \not\equiv \pm 4 \bmod 9$, the equation $x^3+y^3+z^3=a$ has $\ge f(a)$ solutions with $\max(|x|,|y|,|z|) \le h(a)\cdot|a|^{1/3}$.[2] Under H2 (resp. H3), one can show that if $g(a)\to 0$ as $|a|\to\infty$, then $x^3+y^3+z^3=a$ has $\ge g(a)\cdot\log(1+|a|)$ solutions for almost all integers (resp. primes) $a\not\equiv \pm 4 \bmod 9$.

We expect that Theorems 1.1 and 1.2 can be generalized from F_0 to arbitrary ternary cubic forms G_0 with nonzero discriminant, if one modifies (1.11) and (2.28). For (1.11), see [Wan22, Definition 1.4.3]. For (2.28), one needs that for any L in the (finite) set Υ, if $I(L)$ denotes the ideal of $\mathbb{Q}[\boldsymbol{y},\boldsymbol{z}]$ defining L, then $I(L)\cap\mathbb{Q}[\boldsymbol{y}]$ is a principal ideal of $\mathbb{Q}[\boldsymbol{y}]$.

The proof of Theorem 1.2 might also adapt to the Markoff cubic $x^2+y^2+z^2-xyz$ to *unconditionally* extend [GS22, Theorem 1.2(ii)] to primes, provided one can handle certain quadratic subtleties uncovered in [GS22]. For practical reasons, we focus on $x^3+y^3+z^3$.

1.1 Conventions

We let $\mathbb{Z}_{\ge 0} := \{n\in\mathbb{Z} : n\ge 0\}$, and define $\mathbb{Z}_{\ge 1}, \mathbb{R}_{>0}, \dots$ similarly. For a finite nonempty set S, we let $\mathbb{E}_{b\in S}[f(b)] := |S|^{-1}\sum_{b\in S} f(b)$. For an event E, we let $\mathbf{1}_E := 1$ if E holds, and $\mathbf{1}_E := 0$ otherwise. We let $e(t) := e^{2\pi i t}$, and $e_r(t) := e(t/r)$.

For a vector $\boldsymbol{u}\in\mathbb{R}^s$, we let $\|\boldsymbol{u}\| := \max_i(|u_i|)$ and $d\boldsymbol{u} := du_1\cdots du_s$. We let $C_c^\infty(\mathbb{R}^s)$ denote the set of smooth compactly supported functions $\mathbb{R}^s\to\mathbb{R}$. Given a function $f\colon \mathbb{R}^s\to\mathbb{R}$, we let $\operatorname{Supp} f$ denote the

[2] One may simultaneously satisfy the condition $\max(|x|,|y|,|z|) \ge f(a)\min(|x|,|y|,|z|)$, as the referee has suggested. This is a weak analog of the "minicube" problem discussed in [BW10]. Minicubes do sometimes, if rarely, occur in practice; see the lopsided solution of [BS21] to $x^3+y^3+z^3=3$, for instance.

closure of $\{\boldsymbol{u} \in \mathbb{R}^s : f(\boldsymbol{u}) \neq 0\}$ in $\mathbb{R}^s$. We let $\partial_x := \partial/\partial x$. We write $\int_X dx\, f(x)$ to mean $\int_X f(x)\, dx$; we write $\int_{X\times Y} dx\, dy\, f(x,y)$ to mean $\int_X dx\, (\int_Y dy\, f(x,y))$.

Given $f = f(u_1,\dots,u_s) \in C_c^\infty(\mathbb{R}^s)$ and $k \in \mathbb{Z}_{\geq 0}$, we define the Sobolev norm

$$\|f\|_{k,\infty} := \max_{\substack{\alpha_1,\dots,\alpha_s \in \mathbb{Z}_{\geq 0}:\\ \alpha_1+\cdots+\alpha_s \leq k}} \max_{\mathbb{R}^s} |\partial_{u_1}^{\alpha_1} \cdots \partial_{u_s}^{\alpha_s} f|. \tag{1.13}$$

Given a clean, compactly supported function $f\colon \mathbb{R}^s \to \mathbb{R}$, we let $B(f)$ denote the smallest integer $B \geq 1$ such that $B^{-1} \leq |u_1|,\dots,|u_s| \leq B$ for all $\boldsymbol{u} \in \operatorname{Supp} f$.

We write $f \ll_S g$, or $g \gg_S f$, to mean $|f| \leq Cg$ for some $C = C(S) > 0$. We let $O_S(g)$ denote a quantity that is $\ll_S g$. We write $f \asymp_S g$ if $f \ll_S g \ll_S f$.

We let $v_p(-)$ denote the usual p-adic valuation. For integers $n \geq 1$, we let $\phi(n)$ denote the totient function, $\tau(n)$ the divisor function, $\mu(n)$ the Möbius function, $\omega(n)$ the number of distinct prime factors of n, and $\operatorname{rad}(n)$ the radical of n.

2 General variance setup and estimation

Let $F_a := F_0 - a$ and $\nu \in C_c^\infty(\mathbb{R}^3)$. Suppose $\mathbf{0} \notin \operatorname{Supp} \nu$. Let us define local densities corresponding to the point count (1.12). For all $a \in \mathbb{Z} \setminus \{0\}$, let

$$\sigma_{p,a} := \lim_{l\to\infty} (p^{-2l} \cdot \#\{\boldsymbol{y} \in (\mathbb{Z}/p^l\mathbb{Z})^3 : F_a(\boldsymbol{y}) = 0\}); \tag{2.1}$$

this exists by [CLT10, §5.4] since $F_a = 0$ is smooth in $\mathbb{A}^3_{\mathbb{Q}}$. Similarly, but for all $a \in \mathbb{R}$, let

$$\sigma_{\infty,a,\nu}(X) := \lim_{\varepsilon\to 0} (2\varepsilon)^{-1} \int_{\boldsymbol{y}\in\mathbb{R}^3 : |F_a(\boldsymbol{y})| \leq \varepsilon} d\boldsymbol{y}\, \nu(\boldsymbol{y}/X); \tag{2.2}$$

this exists because the surface $F_a = 0$ in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ is smooth (even if $a = 0$). At least for cube-free $a \neq 0$, the density $\sigma_{p,a}$ and a real density are computed on [HB92, p. 622].

For $a \in \mathbb{Z} \setminus \{0\}$, informally write $\mathfrak{S}_a := \prod_{p \text{ prime}} \sigma_{p,a}$, and consider the *Hardy–Littlewood prediction* $\mathfrak{S}_a \cdot \sigma_{\infty,a,\nu}(X)$ for $N_{a,\nu}(X)$. Smaller moduli should have a greater effect in $\mathfrak{S}_a$; furthermore, $\mathfrak{S}_a$ itself—*as is*—can be subtle (see §3). So in (2.5) below, we use a "restricted" version of $\mathfrak{S}_a \cdot \sigma_{\infty,a,\nu}(X)$. For technical quantitative reasons, we use a series approximation (2.4) to $\mathfrak{S}_a$, rather than a product approximation as in [GS22, Dia19, Wan22].

Definition 2.1. For integers $m \geq 1$, let

$$T_a(m) := \sum_{u\in(\mathbb{Z}/m\mathbb{Z})^\times} \sum_{\boldsymbol{y}\in(\mathbb{Z}/m\mathbb{Z})^3} e_m(uF_a(\boldsymbol{y})) = \sum_{\substack{1\leq u\leq m:\\ \gcd(u,m)=1}} \sum_{1\leq y_1,y_2,y_3\leq m} e_m(uF_a(y_1,y_2,y_3)) \tag{2.3}$$

whenever a lies in $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$ (or maps canonically into $\mathbb{Z}/m\mathbb{Z}$). For $a \in \mathbb{Z}$ and $K \in \mathbb{Z}_{\geq 1}$, let

$$s_a(K) := \sum_{m\leq K} m^{-3} T_a(m). \tag{2.4}$$

For integers $d \geq 1$, let $d\mathbb{Z} := \{a \in \mathbb{Z} : d \mid a\}$ and define the *K-approximate variance*

$$\operatorname{Var}(X,K;d) := \sum_{a\in d\mathbb{Z}} [N_{a,\nu}(X) - s_a(K)\sigma_{\infty,a,\nu}(X)]^2. \tag{2.5}$$

Both ν and $T_a(m)$ are real-valued, so (2.5) is a reasonable definition. For later use, let

$$\nu^{\otimes 2}(\boldsymbol{x}) = \nu^{\otimes 2}(\boldsymbol{y},\boldsymbol{z}) := \nu(\boldsymbol{y})\nu(\boldsymbol{z}), \tag{2.6}$$

$$\|\nu\|^2_{L^2(\mathbb{R}^3)} := \int_{\boldsymbol{y}\in\mathbb{R}^3} d\boldsymbol{y}\,\nu(\boldsymbol{y})^2. \tag{2.7}$$

For clean ν, we will rewrite $\operatorname{Var}(X,K;d)$ after expanding the square (Theorem 2.13 below). Due to the nonnegativity of squares in (2.5), the Selberg sieve can then be used to bound a variant of (2.5) over primes, via a certain multiplicative structure over d, under additional conditions on ν; this will be Theorem 2.14. The proofs of Theorems 2.13 and 2.14 rest on delicate local calculations, and require particular care at primes $p \mid d$. On a first reading, one may wish to focus on the simplest case $d = 1$, which contains most of the key ideas.

2.1 Non-archimedean work

For integers $m,d \geq 1$, let

$$S^+_{\mathbf{0}}(m;d) := \sum_{n\geq 1:\,\operatorname{lcm}(n,d)=m}\ \sum_{u\in(\mathbb{Z}/n\mathbb{Z})^\times}\ \sum_{\boldsymbol{x}\in(\mathbb{Z}/m\mathbb{Z})^6:\,d\mid F_0(\boldsymbol{y}),F_0(\boldsymbol{z})} e_n(uF(\boldsymbol{x})). \tag{2.8}$$

Then $S^+_{\mathbf{0}}(m;d) = 0$ unless $d \mid m$. For every positive integer $q \in d\mathbb{Z}$, we have by (1.6)

$$\begin{aligned}
\rho(q;d) &= \sum_{a\in\mathbb{Z}/q\mathbb{Z}} q^{-6} \sum_{\boldsymbol{x}\in(\mathbb{Z}/q\mathbb{Z})^6:\,d\mid F_0(\boldsymbol{y}),F_0(\boldsymbol{z})} e_q(aF(\boldsymbol{x})) \\
&= \sum_{n\mid q}\ \sum_{u\in(\mathbb{Z}/n\mathbb{Z})^\times} q^{-6} \sum_{\boldsymbol{x}\in(\mathbb{Z}/q\mathbb{Z})^6:\,d\mid F_0(\boldsymbol{y}),F_0(\boldsymbol{z})} e_n(uF(\boldsymbol{x})) \\
&= \sum_{n\geq 1:\,\operatorname{lcm}(n,d)\mid q}\ \sum_{u\in(\mathbb{Z}/n\mathbb{Z})^\times} q^{-6} \sum_{\boldsymbol{x}\in(\mathbb{Z}/q\mathbb{Z})^6:\,d\mid F_0(\boldsymbol{y}),F_0(\boldsymbol{z})} e_n(uF(\boldsymbol{x})),
\end{aligned}$$

since $n \mid q \Leftrightarrow \operatorname{lcm}(n,d) \mid q$ when $q \in d\mathbb{Z}$. Reducing $\boldsymbol{x}$ modulo $\operatorname{lcm}(n,d)$, we get (for $q \in d\mathbb{Z}$)

$$\begin{aligned}
\rho(q;d) &= \sum_{n\geq 1:\,\operatorname{lcm}(n,d)\mid q}\ \sum_{u\in(\mathbb{Z}/n\mathbb{Z})^\times} \operatorname{lcm}(n,d)^{-6} \sum_{\boldsymbol{x}\in(\mathbb{Z}/\operatorname{lcm}(n,d)\mathbb{Z})^6:\,d\mid F_0(\boldsymbol{y}),F_0(\boldsymbol{z})} e_n(uF(\boldsymbol{x})) \\
&= \sum_{m\mid q} m^{-6} S^+_{\mathbf{0}}(m;d);
\end{aligned} \tag{2.9}$$

this is well known when $d = 1$ [Vau97, Lemma 2.12], but perhaps less so for $d > 1$.

Proposition 2.2. *Let $m_1,m_2,d_1,d_2 \geq 1$ be integers. If $\gcd(m_1,m_2) = 1$, then $T_a(m_1m_2) = T_a(m_1)T_a(m_2)$. If $\gcd(m_1d_1,m_2d_2) = 1$, then $S^+_{\mathbf{0}}(m_1m_2;d_1d_2) = S^+_{\mathbf{0}}(m_1;d_1)S^+_{\mathbf{0}}(m_2;d_2)$.*

Proof. It is well known that $T_a(m)$ and $S^+_{\mathbf{0}}(m;1)$ are multiplicative in m; see [Bro21, Lemma 2.13] for a general treatment. A similar proof should show that $S^+_{\mathbf{0}}(m;d)$ is multiplicative in pairs (m,d), but we instead use double Dirichlet series. By (2.9),

$$\sum_{q,d\geq 1:\,d\mid q} \rho(q;d)q^{-r}d^{-s} = \sum_{d,m\geq 1} m^{-6}S^+_{\mathbf{0}}(m;d)d^{-s} \sum_{q\geq 1:\,m\mid q} q^{-r} = \zeta(r) \sum_{d,m\geq 1} m^{-6-r}S^+_{\mathbf{0}}(m;d)d^{-s}.$$

The left-hand side has an Euler product (by (1.6) and the Chinese remainder theorem). Multiplying by $\zeta(r)^{-1} = \prod_p(1-p^{-r})$ gives the desired multiplicativity of $S^+_{\mathbf{0}}(m;d)$. □

Lemma 2.3. *Let $m,d \geq 1$ be integers. Suppose $v_p(m) > v_p(d)$ for all primes $p \mid m$. Then $|S^+_{\mathbf{0}}(m;d)| \leq |S^+_{\mathbf{0}}(m;1)|$.*

Proof. Here $\{n \geq 1 : \operatorname{lcm}(n,d) = m\} = \{m\}$. So by (2.8), the quantity $d^2 \cdot S^+_{\mathbf{0}}(m;d)$ equals

$$\begin{aligned}
&d^2 \sum_{u \in (\mathbb{Z}/m\mathbb{Z})^\times} \sum_{\boldsymbol{y}_1,\boldsymbol{y}_2 \in (\mathbb{Z}/m\mathbb{Z})^3 : d \mid F_0(\boldsymbol{y}_1), F_0(\boldsymbol{y}_2)} e_m(uF_0(\boldsymbol{y}_1) - uF_0(\boldsymbol{y}_2)) \\
&= \sum_{u \in (\mathbb{Z}/m\mathbb{Z})^\times} \sum_{\boldsymbol{y}_1,\boldsymbol{y}_2 \in (\mathbb{Z}/m\mathbb{Z})^3} e_m(uF_0(\boldsymbol{y}_1) - uF_0(\boldsymbol{y}_2)) \sum_{v_1,v_2 \in \mathbb{Z}/d\mathbb{Z}} e_d(v_1F_0(\boldsymbol{y}_1) + v_2F_0(\boldsymbol{y}_2)) \\
&= \sum_{v_1,v_2 \in \mathbb{Z}/d\mathbb{Z}} \sum_{u \in (\mathbb{Z}/m\mathbb{Z})^\times} \prod_{1 \leq i \leq 2} \left(\sum_{\boldsymbol{y}_i \in (\mathbb{Z}/m\mathbb{Z})^3} e_m((-1)^{i-1} uF_0(\boldsymbol{y}_i)) e_d(v_iF_0(\boldsymbol{y}_i)) \right).
\end{aligned}$$

But for any $\varepsilon \in \{-1,1\}$ and $v \in \mathbb{Z}/d\mathbb{Z}$, we have

$$\sum_{u \in (\mathbb{Z}/m\mathbb{Z})^\times} \left| \sum_{\boldsymbol{y} \in (\mathbb{Z}/m\mathbb{Z})^3} e_m(\varepsilon uF_0(\boldsymbol{y})) e_d(vF_0(\boldsymbol{y})) \right|^2 = S^+_{\mathbf{0}}(m;1),$$

since the formula $u \mapsto \varepsilon u + (m/d)v$ defines a bijection on $(\mathbb{Z}/m\mathbb{Z})^\times$. So Cauchy over $u \in (\mathbb{Z}/m\mathbb{Z})^\times$ implies $d^2 \cdot |S^+_{\mathbf{0}}(m;d)| \leq \sum_{v_1,v_2 \in \mathbb{Z}/d\mathbb{Z}} S^+_{\mathbf{0}}(m;1) = d^2 \cdot S^+_{\mathbf{0}}(m;1)$. This suffices. □

Lemma 2.4. *Let $N,d \geq 1$ be integers. Then the following holds:*

$$\sum_{m \in [N,2N)} m^{-6} |S^+_{\mathbf{0}}(m;d)| \leq d \sum_{ab \in [N,2N) : a \mid d} b^{-6} |S^+_{\mathbf{0}}(b;1)| \ll_\varepsilon d^{5/3} N^{-2/3+\varepsilon}.$$

Proof. Write $m = ef$, where $p \mid e \Rightarrow v_p(e) \leq v_p(d)$ and $p \mid f \Rightarrow v_p(f) > v_p(d)$. Trivially, $|S^+_{\mathbf{0}}(e;\gcd(d,e^\infty))| \leq \sum_{n \mid e} \phi(n) e^6 = e^7$. By Lemma 2.3, $|S^+_{\mathbf{0}}(f;\gcd(d,f^\infty))| \leq |S^+_{\mathbf{0}}(f;1)|$. Therefore, $|S^+_{\mathbf{0}}(m;d)| \leq e^7 |S^+_{\mathbf{0}}(f;1)|$ by Proposition 2.2. Since $e \mid d$, we get

$$\sum_{m \in [N,2N)} m^{-6} |S^+_{\mathbf{0}}(m;d)| \leq \sum_{e \mid d} e \sum_{f \in [N/e,2N/e)} f^{-6} |S^+_{\mathbf{0}}(f;1)| \leq d \sum_{e \mid d} \sum_{f \in [N/e,2N/e)} f^{-6} |S^+_{\mathbf{0}}(f;1)|.$$

By [Vau97, $(t,k,\lambda) = (4,3,0)$ case of Lemma 4.9] and [Vau97, Theorem 4.2], the inner sum over f is $\ll_\varepsilon (N/e)^{\varepsilon - 2/3}$. The lemma then follows from the bound $\sum_{e \mid d} e^{2/3-\varepsilon} \ll d^{2/3}$. □

Let us note some standard consequences of our work so far. First, the density $\sigma_p(d)$ from (1.7) exists by (2.9) and Lemma 2.4. Second, the singular series $\mathfrak{S}(d)$ converges, with

$$\mathfrak{S}(d) = \prod_{p \text{ prime}} \sigma_p(d) = \sum_{m \geq 1} m^{-6} S^+_{\mathbf{0}}(m;d) = \sum_{m \geq 1 : d \mid m} m^{-6} S^+_{\mathbf{0}}(m;d); \tag{2.10}$$

here we use (2.9), Proposition 2.2, and Lemma 2.4 to see that the infinite product and sums in (2.10) all converge absolutely, and equal one another.

We now prove several results relating $T_a(-)$, from (2.3), to $S^+_{\mathbf{0}}(-;d)$, from (2.8).

Lemma 2.5. *Let $d,m \geq 1$ be integers with $d \mid m$. Then*

$$\sum_{n_1,n_2\geq 1:\,\mathrm{lcm}(n_1,d)=m,\ \mathrm{lcm}(n_2,d)=m} \frac{1}{m} \sum_{b\in d\mathbb{Z}/m\mathbb{Z}} \frac{T_b(n_1)T_b(n_2)}{(n_1n_2)^3} = \frac{S^+_{\mathbf{0}}(m;d)}{m^6}, \tag{2.11}$$

$$\sum_{n_1,n_2\geq 1:\,\mathrm{lcm}(n_1,d)=m,\ \mathrm{lcm}(n_2,d)=m} \left| \frac{1}{m} \sum_{b\in d\mathbb{Z}/m\mathbb{Z}} \frac{T_b(n_1)T_b(n_2)}{(n_1n_2)^3} \right| \leq \tau(m) \sum_{rn=m:\,r\mid d} \frac{|S^+_{\mathbf{0}}(n;1)|}{n^6}. \tag{2.12}$$

Proof. By Proposition 2.2, it suffices to prove the lemma when m is a prime power. So suppose $(d,m) = (p^e, p^f)$, where p is prime and $0 \leq e \leq f$.

Case 1: $e < f$. Then $\{n \geq 1 : \mathrm{lcm}(n,d) = m\} = \{m\}$. In particular,

$$\begin{aligned} S^+_{\mathbf{0}}(m;d) &= \sum_{u\in(\mathbb{Z}/m\mathbb{Z})^\times} \sum_{\boldsymbol{x}\in(\mathbb{Z}/m\mathbb{Z})^6:\,d\mid F_0(\boldsymbol{y}),F_0(\boldsymbol{z})} e_m(uF(\boldsymbol{x})) \\ &= p^e \sum_{u\in(\mathbb{Z}/p^{f-e}\mathbb{Z})^\times} \sum_{\boldsymbol{x}\in(\mathbb{Z}/m\mathbb{Z})^6:\,d\mid F_0(\boldsymbol{y}),F_0(\boldsymbol{z})} e_m(uF(\boldsymbol{x})) \end{aligned} \tag{2.13}$$

(since $e_m(ud)$ depends only on u mod p^{f-e}). And if $b \in d\mathbb{Z}/m\mathbb{Z}$, then

$$\begin{aligned} T_b(m) = \sum_{u\in(\mathbb{Z}/m\mathbb{Z})^\times} \sum_{\boldsymbol{y}\in(\mathbb{Z}/m\mathbb{Z})^3} e_m(uF_b(\boldsymbol{y})) &= \sum_{u\in(\mathbb{Z}/m\mathbb{Z})^\times} \sum_{\boldsymbol{y}\in(\mathbb{Z}/m\mathbb{Z})^3:\,d\mid F_0(\boldsymbol{y})} e_m(uF_b(\boldsymbol{y})) \\ &= p^e \sum_{u\in(\mathbb{Z}/p^{f-e}\mathbb{Z})^\times} \sum_{\boldsymbol{y}\in(\mathbb{Z}/m\mathbb{Z})^3:\,d\mid F_0(\boldsymbol{y})} e_m(uF_b(\boldsymbol{y})) \end{aligned} \tag{2.14}$$

(since $\sum_{u\in(\mathbb{Z}/p^f\mathbb{Z})^\times} e_{p^f}(uF_b(\boldsymbol{y})) = 0$ if $p^{f-1} \nmid F_b(\boldsymbol{y})$), whence

$$T_b(m)^2 = p^{2e} \sum_{u,v\in(\mathbb{Z}/p^{f-e}\mathbb{Z})^\times} \sum_{\boldsymbol{y},\boldsymbol{z}\in(\mathbb{Z}/m\mathbb{Z})^3:\,d\mid F_0(\boldsymbol{y}),F_0(\boldsymbol{z})} e_m(uF_b(\boldsymbol{y}) + vF_b(\boldsymbol{z})).$$

But $\sum_{b\in d\mathbb{Z}/m\mathbb{Z}} e_m(-ub - vb) = p^{f-e} \cdot \mathbf{1}_{p^{f-e}\mid u+v}$. So summing the previous display over $b \in d\mathbb{Z}/m\mathbb{Z}$, and then using (2.13), we get $\sum_{b\in d\mathbb{Z}/m\mathbb{Z}} T_b(m)^2 = p^f S^+_{\mathbf{0}}(m;d)$. This suffices for both (2.11), (2.12). (For (2.12), note that $|S^+_{\mathbf{0}}(m;d)| \leq |S^+_{\mathbf{0}}(m;1)|$ by Lemma 2.3).

Case 2: $e = f$. Then $d = m$ and $\{n \geq 1 : \mathrm{lcm}(n,d) = m\} = \{n \geq 1 : n \mid m\}$. So

$$S^+_{\mathbf{0}}(m;d) = \sum_{n\mid m} \sum_{u\in(\mathbb{Z}/n\mathbb{Z})^\times} \sum_{\substack{\boldsymbol{x}\in(\mathbb{Z}/m\mathbb{Z})^6:\\ m\mid F_0(\boldsymbol{y}),F_0(\boldsymbol{z})}} e_n(uF(\boldsymbol{x})) = m \sum_{\substack{\boldsymbol{x}\in(\mathbb{Z}/m\mathbb{Z})^6:\\ m\mid F_0(\boldsymbol{y}),F_0(\boldsymbol{z})}} 1. \tag{2.15}$$

But $d\mathbb{Z}/m\mathbb{Z} = 0$ (in (2.11), (2.12)), and

$$\begin{aligned} \sum_{n\mid m} \frac{T_0(n)}{n^3} &= \sum_{n\mid m} \sum_{u\in(\mathbb{Z}/n\mathbb{Z})^\times} \mathbb{E}_{\boldsymbol{y}\in(\mathbb{Z}/n\mathbb{Z})^3}[e_n(uF_0(\boldsymbol{y}))] \\ &= \sum_{n\mid m} \sum_{u\in(\mathbb{Z}/n\mathbb{Z})^\times} \mathbb{E}_{\boldsymbol{y}\in(\mathbb{Z}/m\mathbb{Z})^3}[e_n(uF_0(\boldsymbol{y}))] \\ &= \sum_{u\in\mathbb{Z}/m\mathbb{Z}} \mathbb{E}_{\boldsymbol{y}\in(\mathbb{Z}/m\mathbb{Z})^3}[e_m(uF_0(\boldsymbol{y}))] = m^{-2} \sum_{\boldsymbol{y}\in(\mathbb{Z}/m\mathbb{Z})^3:\,m\mid F_0(\boldsymbol{y})} 1. \end{aligned} \tag{2.16}$$

Upon squaring (2.16), dividing by m, and using (2.15), we get (2.11). For (2.12), note that by Cauchy, $|T_0(n)|^2 \le \phi(n) \sum_{u \in (\mathbb{Z}/n\mathbb{Z})^\times} |\sum_{\boldsymbol{y} \in (\mathbb{Z}/n\mathbb{Z})^3} e_n(uF_0(\boldsymbol{y}))|^2 = \phi(n) S^+_{\boldsymbol{0}}(n;1)$, so

$$\frac{1}{m} \sum_{n_1,n_2 \mid m} \frac{|T_0(n_1)T_0(n_2)|}{(n_1n_2)^3} \le \frac{\tau(m)}{m} \sum_{n \mid m} \frac{|T_0(n)|^2}{n^6} \le \frac{\tau(m)}{m} \sum_{n \mid m} \frac{\phi(n) S^+_{\boldsymbol{0}}(n;1)}{n^6}. \tag{2.17}$$

This suffices for (2.12) (since $\phi(n) \le n \le m$, and $(m/n) \mid m = d$). $\square$

Lemma 2.6. *Let $d,m \ge 1$ be integers with $d \mid m$. Then*

$$\sum_{n \ge 1 : \operatorname{lcm}(n,d)=m} \frac{1}{m^3} \sum_{\boldsymbol{e} \in (\mathbb{Z}/m\mathbb{Z})^3 : F_0(\boldsymbol{e}) \equiv 0 \bmod d} \frac{T_{F_0(\boldsymbol{e})}(n)}{n^3} = \frac{S^+_{\boldsymbol{0}}(m;d)}{m^6}, \tag{2.18}$$

$$\sum_{n \ge 1 : \operatorname{lcm}(n,d)=m} \left| \frac{1}{m^3} \sum_{\boldsymbol{e} \in (\mathbb{Z}/m\mathbb{Z})^3 : F_0(\boldsymbol{e}) \equiv 0 \bmod d} \frac{T_{F_0(\boldsymbol{e})}(n)}{n^3} \right| \le d \cdot \tau(m) \sum_{rn=m : r \mid d} \frac{|S^+_{\boldsymbol{0}}(n;1)|}{n^6}. \tag{2.19}$$

Proof. Again, we may assume that $(d,m) = (p^e, p^f)$, where p is prime and $0 \le e \le f$.

Case 1: $e < f$. Then $\{n \ge 1 : \operatorname{lcm}(n,d) = m\} = \{m\}$. But by (2.14),

$$\sum_{\boldsymbol{e} \in (\mathbb{Z}/m\mathbb{Z})^3 : F_0(\boldsymbol{e}) \equiv 0 \bmod d} T_{F_0(\boldsymbol{e})}(m) = \sum_{u \in (\mathbb{Z}/m\mathbb{Z})^\times} \sum_{\boldsymbol{e},\boldsymbol{y} \in (\mathbb{Z}/m\mathbb{Z})^3 : d \mid F_0(\boldsymbol{e}), F_0(\boldsymbol{y})} e_m(u(F_0(\boldsymbol{y}) - F_0(\boldsymbol{e}))),$$

which equals $S^+_{\boldsymbol{0}}(m;d)$ by (2.13). Both (2.18), (2.19) follow (in the present case).

Case 2: $e = f$. Then $d = m$ and $\{n \ge 1 : \operatorname{lcm}(n,d) = m\} = \{n \ge 1 : n \mid m\}$. So (2.18) follows from (2.16) and (2.15). Also, (2.19) follows from (2.17) (since $T_0(n) = T_0(n)T_0(1)$ for all $n \ge 1$, and we have $\#\{\boldsymbol{e} \in (\mathbb{Z}/m\mathbb{Z})^3 : F_0(\boldsymbol{e}) \equiv 0 \bmod d\} \le m^3 = m^2 d$ trivially). $\square$

Lemma 2.7. *Suppose $n_1, n_2, d \ge 1$ are integers with $\operatorname{lcm}(n_1,d) \ne \operatorname{lcm}(n_2,d)$. Then*

$$\sum_{b \in d\mathbb{Z}/n_1n_2d\mathbb{Z}} T_b(n_1) T_b(n_2) = 0. \tag{2.20}$$

Proof. We may assume $\operatorname{lcm}(n_1,d) < \operatorname{lcm}(n_2,d)$. Let $r := \operatorname{lcm}(n_1,d)$; then $n_2 \nmid r$. So $\sum_{a \in \gcd(n_2,r)\mathbb{Z}/n_2\mathbb{Z}} e_{n_2}(-ua) = 0$ for all $u \in (\mathbb{Z}/n_2\mathbb{Z})^\times$. Thus $\sum_{b \in \mathbb{Z}/n_1n_2d\mathbb{Z} : b \equiv c \bmod r} T_b(n_2) = 0$ for all $c \in \mathbb{Z}/r\mathbb{Z}$. Multiplying by $T_c(n_1)$ and summing over $c \in d\mathbb{Z}/r\mathbb{Z}$, we get (2.20). $\square$

Proposition 2.8. *Let $d, K \ge 1$ be integers. Then*

$$\sum_{n_1,n_2 \le K} \frac{1}{n_1n_2d} \sum_{b \in d\mathbb{Z}/n_1n_2d\mathbb{Z}} \frac{T_b(n_1)T_b(n_2)}{(n_1n_2)^3} = \mathfrak{S}(d) + O_\varepsilon(d^{5/3}K^{-2/3+\varepsilon}), \tag{2.21}$$

$$\sum_{n \le K} \frac{1}{(nd)^3} \sum_{\boldsymbol{e} \in (\mathbb{Z}/nd\mathbb{Z})^3 : F_0(\boldsymbol{e}) \equiv 0 \bmod d} \frac{T_{F_0(\boldsymbol{e})}(n)}{n^3} = \mathfrak{S}(d) + O_\varepsilon(d^{5/3}K^{-2/3+\varepsilon}). \tag{2.22}$$

Proof. If $n_1, n_2 \geq 1$ are integers with $\operatorname{lcm}(n_1, d) = \operatorname{lcm}(n_2, d) = m$, say, then $d \mid m$ and

$$\frac{1}{n_1 n_2 d} \sum_{b \in d\mathbb{Z}/n_1 n_2 d\mathbb{Z}} \frac{T_b(n_1) T_b(n_2)}{(n_1 n_2)^3} = \frac{1}{m} \sum_{b \in d\mathbb{Z}/m\mathbb{Z}} \frac{T_b(n_1) T_b(n_2)}{(n_1 n_2)^3}.$$

By Lemmas 2.7 and 2.5, it follows that the left-hand side of (2.21) equals

$$\begin{aligned}
&\sum_{m \geq 1 : d \mid m} \sum_{\substack{n_1, n_2 \leq K: \\ \operatorname{lcm}(n_1,d) = m,\ \operatorname{lcm}(n_2,d) = m}} \frac{1}{m} \sum_{b \in d\mathbb{Z}/m\mathbb{Z}} \frac{T_b(n_1) T_b(n_2)}{(n_1 n_2)^3} \\
&= \sum_{m \leq K : d \mid m} \frac{S_{\mathbf{0}}^{+}(m; d)}{m^6} + \sum_{m > K : d \mid m} \tau(m) \sum_{rn = m : r \mid d} \frac{O(|S_{\mathbf{0}}^{+}(n; 1)|)}{n^6} \\
&= \mathfrak{S}(d) + O_\varepsilon(d^{5/3} K^{-2/3+\varepsilon}),
\end{aligned}$$

where in the final step we write $\sum_{m \leq K : d \mid m} = \sum_{m \geq 1 : d \mid m} - \sum_{m > K : d \mid m}$, use (2.10) to write $\sum_{m \geq 1 : d \mid m} S_{\mathbf{0}}^{+}(m; d)/m^6 = \mathfrak{S}(d)$, and use Lemma 2.4 to bound the $m > K$ contributions.

The second part, (2.22), is similar, but simpler. If $n \geq 1$ and $\operatorname{lcm}(n, d) = m$, say, then

$$\frac{1}{(nd)^3} \sum_{\boldsymbol{e} \in (\mathbb{Z}/nd\mathbb{Z})^3 : F_0(\boldsymbol{e}) \equiv 0 \bmod d} \frac{T_{F_0(\boldsymbol{e})}(n)}{n^3} = \frac{1}{m^3} \sum_{\boldsymbol{e} \in (\mathbb{Z}/m\mathbb{Z})^3 : F_0(\boldsymbol{e}) \equiv 0 \bmod d} \frac{T_{F_0(\boldsymbol{e})}(n)}{n^3}.$$

By Lemma 2.6, the left-hand side of (2.22) thus equals

$$\begin{aligned}
&\sum_{m \geq 1 : d \mid m} \sum_{\substack{n \leq K: \\ \operatorname{lcm}(n,d) = m}} \frac{1}{m^3} \sum_{\boldsymbol{e} \in (\mathbb{Z}/m\mathbb{Z})^3 : F_0(\boldsymbol{e}) \equiv 0 \bmod d} \frac{T_{F_0(\boldsymbol{e})}(n)}{n^3} \\
&= \sum_{m \leq K : d \mid m} \frac{S_{\mathbf{0}}^{+}(m; d)}{m^6} + \sum_{m > K : d \mid m} d \cdot \tau(m) \sum_{rn = m : r \mid d} \frac{O(|S_{\mathbf{0}}^{+}(n; 1)|)}{n^6} \\
&= \mathfrak{S}(d) + O_\varepsilon(d^{5/3} K^{-2/3+\varepsilon}),
\end{aligned}$$

where in the last step we again use (2.10) and Lemma 2.4. This completes the proof. □

2.2 Archimedean work

Let $\nu \in C_c^\infty(\mathbb{R}^3)$, and suppose $\mathbf{0} \notin \operatorname{Supp} \nu$. Given $X \in \mathbb{R}_{>0}$ and $(\boldsymbol{y}, a) \in \mathbb{R}^3 \times \mathbb{R}$, write $\tilde{\boldsymbol{y}} := \boldsymbol{y}/X$ and $\tilde{a} := a/X^3$. By (2.2) (after replacing ε with $X^3 \varepsilon$),

$$\sigma_{\infty,a,\nu}(X) = \lim_{\varepsilon \to 0} (2\varepsilon)^{-1} \int_{\tilde{\boldsymbol{y}} \in \mathbb{R}^3 : |F_0(\tilde{\boldsymbol{y}}) - \tilde{a}| \leq \varepsilon} d\tilde{\boldsymbol{y}}\, \nu(\tilde{\boldsymbol{y}}) = \sigma_{\infty,\tilde{a},\nu}(1). \tag{2.23}$$

For convenience (when working with $\sigma_{\infty,a,\nu}(X)$), we now observe the following:

Observation 2.9. Suppose $|y_1| \geq \delta > 0$ for all $\boldsymbol{y} \in \operatorname{Supp} \nu$. Let $(a, X) \in \mathbb{R} \times \mathbb{R}_{>0}$. Then a change of variables in (2.23) from $\tilde{y}_1$ to $F_0 := F_0(\tilde{\boldsymbol{y}})$ gives (by [CLT10, §5.4, par. 4])

$$\sigma_{\infty,a,\nu}(X) = \int_{\mathbb{R}^2} d\tilde{y}_2\, d\tilde{y}_3\, \nu(\tilde{\boldsymbol{y}}) \cdot |\partial F_0/\partial \tilde{y}_1|^{-1}, \tag{2.24}$$

where $F_0, \tilde{y}_1$ are constrained by the equation $F_0 = \tilde{a}$ (and thus determined by $\tilde{y}_1, \tilde{y}_2$), and where $\partial F_0/\partial \tilde{y}_1 = 3\tilde{y}_1^2 \geq 3\delta^2 > 0$ over the support of the integrand.

At least in the absence of better surface coordinates, the definition (2.2) (via "ε-thickening") still provides greater intuition, while the surface integral allows for effortless rigor. The area form $d\tilde{y}_2\, d\tilde{y}_3\, |\partial F_0/\partial \tilde{y}_1|^{-1}$ in (2.24) is often called a *Leray form*, as in [Pey95].

We now prove three results on real densities. For technical convenience, we assume for the rest of §2 that ν is clean (see (1.11)). Let $B(\nu)$ and $\|\nu\|_{k,\infty}$ be as in §1.1.

Proposition 2.10. *For integers $k \geq 0$, we have $\partial_{\tilde{a}}^k[\sigma_{\infty,a,\nu}(X)] \ll_k \|\nu\|_{k,\infty} B(\nu)^{A_0+A_1k}$ (for some constants $A_0, A_1 \in [1, 10]$), uniformly over $(a, X) \in \mathbb{R} \times \mathbb{R}_{>0}$.*

Proof. The integrand on the right in (2.24) vanishes unless $\tilde{a} \ll B(\nu)^3$ and $\tilde{y}_1^{-1}, \tilde{y}_2, \tilde{y}_3 \ll B(\nu)$. Fix $\tilde{y}_2, \tilde{y}_3$, and let $\tilde{y}_1$ vary with $\tilde{a}$ according to $F_0 = \tilde{a}$. Then $\partial_{\tilde{a}}[\tilde{y}_1] = (3\tilde{y}_1^2)^{-1} \ll B(\nu)^2$. Now repeatedly apply $\partial_{\tilde{a}}$ to the integrand (using Leibniz and the chain rule). This gives

$$\partial_{\tilde{a}}^k[\sigma_{\infty,a,\nu}(X)] \ll_k \int_{\tilde{y}_2,\tilde{y}_3 \ll B(\nu)} d\tilde{y}_2\, d\tilde{y}_3\, \|\nu\|_{k,\infty} B(\nu)^{2k}/|\tilde{y}_1|^{2+k} \ll_k \|\nu\|_{k,\infty} B(\nu)^{4+3k},$$

which is satisfactory. □

Proposition 2.11. *Let $X \in \mathbb{R}_{>0}$. The "pure L^2 moment" $\int_{a\in\mathbb{R}} d\tilde{a}\, \sigma_{\infty,a,\nu}(X)^2$ and the "mixed L^1 moment" $\int_{\tilde{\boldsymbol{z}}\in\mathbb{R}^3} d\tilde{\boldsymbol{z}}\, \nu(\tilde{\boldsymbol{z}}) \sigma_{\infty,F_0(\boldsymbol{z}),\nu}(X)$ both equal $\sigma_{\infty,\nu^{\otimes 2}}$ (defined via (1.7)).*

Proof. First, $\int_{a\in\mathbb{R}} d\tilde{a}\, \sigma_{\infty,a,\nu}(X)^2$ expands (via (2.24) and the relation $\tilde{a} = F_0(\tilde{\boldsymbol{y}})$) to

$$\int_{\mathbb{R}^4} d\tilde{y}_2\, d\tilde{y}_3\, d\tilde{z}_2\, d\tilde{z}_3 \int_{\tilde{y}_1\in\mathbb{R}} dF_0(\tilde{\boldsymbol{y}})\, \frac{\nu(\tilde{\boldsymbol{y}})\nu(\tilde{\boldsymbol{z}})}{\partial_{\tilde{y}_1} F_0|_{\tilde{\boldsymbol{y}}}} (\partial_{\tilde{z}_1} F_0|_{\tilde{\boldsymbol{z}}})^{-1}|_{F_0(\tilde{\boldsymbol{y}})=F_0(\tilde{\boldsymbol{z}})},$$

which simplifies to $\int_{\mathbb{R}^5} d\tilde{y}_2\, d\tilde{y}_3\, d\tilde{z}_2\, d\tilde{z}_3\, d\tilde{y}_1\, \nu^{\otimes 2}(\tilde{\boldsymbol{x}}) \cdot (\partial_{\tilde{z}_1} F_0|_{\tilde{\boldsymbol{z}}})^{-1}|_{F(\tilde{\boldsymbol{x}})=0}$ (by (2.6)), which equals $\sigma_{\infty,\nu^{\otimes 2}}$ by [CLT10, §5.4, par. 4]. Second, $F_0(\boldsymbol{z})/X^3 = F_0(\tilde{\boldsymbol{z}})$, so by (2.24),

$$\int_{\tilde{\boldsymbol{z}}\in\mathbb{R}^3} d\tilde{\boldsymbol{z}}\, \nu(\tilde{\boldsymbol{z}}) \sigma_{\infty,F_0(\boldsymbol{z}),\nu}(X) = \int_{\mathbb{R}^3\times\mathbb{R}^2} d\tilde{\boldsymbol{z}}\, d\tilde{y}_2\, d\tilde{y}_3\, \nu(\tilde{\boldsymbol{y}})\nu(\tilde{\boldsymbol{z}}) \cdot (\partial_{\tilde{y}_1} F_0|_{\tilde{\boldsymbol{y}}})^{-1}|_{F_0(\tilde{\boldsymbol{y}})=F_0(\tilde{\boldsymbol{z}})},$$

which again simplifies to $\sigma_{\infty,\nu^{\otimes 2}}$. □

Proposition 2.12. *Let $X, N \geq 1$ be integers. Let $b \in \mathbb{Z}/N\mathbb{Z}$ and $\boldsymbol{e} \in (\mathbb{Z}/N\mathbb{Z})^3$. Then for integers $j \geq 4$, we have (for some constants $A_2, \dots, A_5 \in [1, 30]$)*

$$\sum_{a \in \mathbb{Z}:\, a \equiv b \bmod N} \sigma_{\infty,a,\nu}(X)^2 = \frac{X^3 \sigma_{\infty,\nu^{\otimes 2}}}{N} + \frac{O_j(\|\nu\|_{j,\infty}^2 B(\nu)^{A_2+A_3 j})}{(X^3/N)^{j-1}}, \tag{2.25}$$

$$\sum_{\boldsymbol{z} \in \mathbb{Z}^3:\, \boldsymbol{z} \equiv \boldsymbol{e} \bmod N} \nu(\boldsymbol{z}/X) \sigma_{\infty,F_0(\boldsymbol{z}),\nu}(X) = \frac{X^3 \sigma_{\infty,\nu^{\otimes 2}}}{N^3} + \frac{O_j(\|\nu\|_{j,\infty}^2 B(\nu)^{A_4+A_5 j})}{(X/N)^{j-3}}, \tag{2.26}$$

$$\sum_{\boldsymbol{z} \in \mathbb{Z}^3:\, \boldsymbol{z} \equiv \boldsymbol{e} \bmod N} \nu(\boldsymbol{z}/X)^2 = \frac{X^3 \|\nu\|_{L^2(\mathbb{R}^3)}^2}{N^3} + \frac{O_j(\|\nu\|_{j,\infty}^2 B(\nu)^3)}{(X/N)^{j-3}}. \tag{2.27}$$

Proof. We want to replace certain sums with integrals. This is loosely analogous to [Dia19, proof of Lemma 3.1], but we can get better error terms using the smoothness of ν. The error will depend polynomially on the size and support of ν. We do not optimize the exponents of $B(\nu)$ above, since they are ultimately not so important.

The method is standard. Poisson summation, together with Proposition 2.11, gives

$$\sum_{a \equiv b \bmod N} \sigma_{\infty,a,\nu}(X)^2 = \frac{X^3 \sigma_{\infty,\nu^{\otimes 2}}}{N} + \sum_{c \neq 0} \frac{O(1)}{N} \left| \int_{a \in \mathbb{R}} da\, \sigma_{\infty,a,\nu}(X)^2 e(-c \cdot a/N) \right|.$$

We plug in $a = X^3 \tilde{a}$, integrate by parts $j \geq 2$ times in $\tilde{a}$, and invoke Proposition 2.10, to get

$$\frac{1}{N} \left| \int_{a \in \mathbb{R}} da\, \sigma_{\infty,a,\nu}(X)^2 e(-c \cdot a/N) \right| \ll_j \frac{\|\nu\|_{j,\infty}^2 B(\nu)^{A_2+A_3 j}}{|c|^j (X^3/N)^{j-1}} \quad (A_2 = 3 + 2A_0,\ A_3 = A_1).$$

(Note that $\sigma_{\infty,a,\nu}(X) = 0$ unless $\tilde{a} \ll B(\nu)^3$.) The estimate (2.25) follows.

The estimate (2.26) similar. Let $\boldsymbol{c} \in \mathbb{Z}^3 \setminus \{\boldsymbol{0}\}$. Choose $i \in \{1,2,3\}$ with $|c_i| = \|\boldsymbol{c}\|$. Integrating by parts $j \geq 4$ times in $\tilde{z}_i = z_i/X$ gives (via Proposition 2.10 with $a = F_0(\boldsymbol{z})$)

$$\frac{1}{N^3} \left| \int_{\boldsymbol{z} \in \mathbb{R}^3} d\boldsymbol{z}\, \nu(\boldsymbol{z}/X) \sigma_{\infty,F_0(\boldsymbol{z}),\nu}(X) e(-\boldsymbol{c} \cdot \boldsymbol{z}/N) \right| \ll_j \frac{\|\nu\|_{j,\infty}^2 B(\nu)^{A_4+A_5 j}}{\|\boldsymbol{c}\|^j (X/N)^{j-3}},$$

with $A_4 = 3 + A_0$, $A_5 = A_1 + 2$; this is because a factor of $\partial_{\tilde{z}_i} F_0(\tilde{\boldsymbol{z}}) = 3\tilde{z}_i^2 \ll B(\nu)^2$ appears each time we differentiate $\sigma_{\infty,F_0(\boldsymbol{z}),\nu}(X)$, by the chain rule. (Note that $\nu(\boldsymbol{z}/X) = 0$ unless $\tilde{\boldsymbol{z}} \ll B(\nu)$.) Poisson summation and Proposition 2.11 then give (2.26).

Finally, (2.27) is similar to (2.26), but easier; integrating by parts in $\tilde{z}_i = z_i/X$ gives

$$\frac{1}{N^3} \left| \int_{\boldsymbol{z} \in \mathbb{R}^3} d\boldsymbol{z}\, \nu(\boldsymbol{z}/X)^2 e(-\boldsymbol{c} \cdot \boldsymbol{z}/N) \right| \ll_j \frac{\|\nu\|_{j,\infty}^2 B(\nu)^3}{\|\boldsymbol{c}\|^j (X/N)^{j-3}},$$

for $j \geq 4$. Poisson summation and (2.7) then give (2.27). □

2.3 Final calculations

We are finally ready to establish the main results of §2, which concerns the variance defined in (2.5).

Theorem 2.13. *Let $\nu \in C_c^\infty(\mathbb{R}^3)$ satisfy (1.11). Let $X, K, d \geq 1$ be integers with $Kd \leq X^{9/10}$. Then*

$$\operatorname{Var}(X,K;d) = [N_{\nu^{\otimes 2}}(X;d) - \mathfrak{S}(d) \cdot \sigma_{\infty,\nu^{\otimes 2}} X^3] + O_\varepsilon(d^{5/3} K^{-2/3+\varepsilon} X^3 \|\nu\|_{100,\infty}^2 B(\nu)^{5000}).$$

Proof. Squaring out (2.5) yields $\operatorname{Var}(X,K;d) = \Sigma_1 - 2\Sigma_2 + \Sigma_3$, where

$$\Sigma_1 := \sum_{a \in d\mathbb{Z}} N_{a,\nu}(X)^2, \quad \Sigma_2 := \sum_{a \in d\mathbb{Z}} N_{a,\nu}(X) s_a(K) \sigma_{\infty,a,\nu}(X), \quad \Sigma_3 := \sum_{a \in d\mathbb{Z}} [s_a(K) \sigma_{\infty,a,\nu}(X)]^2.$$

Plugging in (1.12) gives $\Sigma_1 = N_{\nu^{\otimes 2}}(X;d)$ (by (1.3), (1.5), and (2.6)). Next, write

$$\Sigma_3 = \sum_{n_1,n_2 \leq K} \sum_{b \in d\mathbb{Z}/n_1 n_2 d\mathbb{Z}} (n_1 n_2)^{-3} T_b(n_1) T_b(n_2) \sum_{a \in \mathbb{Z}:\, a \equiv b \bmod n_1 n_2 d} \sigma_{\infty,a,\nu}(X)^2$$

by plugging in (2.4) and then grouping terms by $a \bmod n_1 n_2 d$; and write

$$\begin{aligned}
\Sigma_2 &= \sum_{\boldsymbol{z} \in \mathbb{Z}^3:\, F_0(\boldsymbol{z}) \equiv 0 \bmod d} \nu(\boldsymbol{z}/X) s_{F_0(\boldsymbol{z})}(K) \sigma_{\infty,F_0(\boldsymbol{z}),\nu}(X) \\
&= \sum_{n \leq K} \sum_{\boldsymbol{e} \in (\mathbb{Z}/nd\mathbb{Z})^3:\, F_0(\boldsymbol{e}) \equiv 0 \bmod d} n^{-3} T_{F_0(\boldsymbol{e})}(n) \sum_{\boldsymbol{z} \in \mathbb{Z}^3:\, \boldsymbol{z} \equiv \boldsymbol{e} \bmod nd} \nu(\boldsymbol{z}/X) \sigma_{\infty,F_0(\boldsymbol{z}),\nu}(X)
\end{aligned}$$

by expanding $N_{a,\nu}(X)$, plugging in (2.4), and grouping terms by $\boldsymbol{z} \bmod nd$. Then by Proposition 2.12 and the trivial bound $|T_b(n)| \leq n^4$, we have (for $j \geq 4$)

$$\Sigma_3 - \sum_{n_1,n_2 \leq K} \sum_{b \in d\mathbb{Z}/n_1 n_2 d\mathbb{Z}} (n_1 n_2)^{-3} T_b(n_1) T_b(n_2) \cdot \frac{X^3 \sigma_{\infty,\nu^{\otimes 2}}}{n_1 n_2 d} \ll K^6 \cdot \frac{O_j(\|\nu\|_{j,\infty}^2 B(\nu)^{A_2 + A_3 j})}{(X^3/K^2 d)^{j-1}}$$

by (2.25) (with $N = n_1 n_2 d$), and

$$\Sigma_2 - \sum_{n \leq K} \sum_{\boldsymbol{e} \in (\mathbb{Z}/nd\mathbb{Z})^3:\, F_0(\boldsymbol{e}) \equiv 0 \bmod d} n^{-3} T_{F_0(\boldsymbol{e})}(n) \cdot \frac{X^3 \sigma_{\infty,\nu^{\otimes 2}}}{(nd)^3} \ll K^2 (Kd)^3 \cdot \frac{O_j(\|\nu\|_{j,\infty}^2 B(\nu)^{A_4 + A_5 j})}{(X/Kd)^{j-3}}$$

by (2.26) (with $N = nd$), where $A_2, \dots, A_5 \leq 30$. Upon taking $j = 100$ above, and plugging in (2.21) and (2.22) from Proposition 2.8, we find that $\operatorname{Var}(X,K;d)$ equals

$$[N_{\nu^{\otimes 2}}(X;d) - \mathfrak{S}(d) \cdot \sigma_{\infty,\nu^{\otimes 2}} X^3] + O_\varepsilon(d^{5/3} K^{-2/3+\varepsilon} \cdot \sigma_{\infty,\nu^{\otimes 2}} X^3) + O(\|\nu\|_{100,\infty}^2 B(\nu)^{5000}),$$

since $X, K, d, B(\nu) \geq 1$ and $Kd \leq X^{9/10}$ (and thus K^6, $K^2 d$, $K^2 (Kd)^3$ are at most X^6, X^2, X^5, respectively). To finish, note that $\sigma_{\infty,\nu^{\otimes 2}} \ll B(\nu)^3 \cdot \|\nu\|_{0,\infty}^2 B(\nu)^{2A_0}$ by Propositions 2.11 and 2.10, and that $\|\nu\|_{100,\infty}^2 B(\nu)^{5000} \leq d^{5/3} K^{-2/3+\varepsilon} X^3 \|\nu\|_{100,\infty}^2 B(\nu)^{5000}$ trivially. □

Call a function $\nu\colon \mathbb{R}^3 \to \mathbb{R}$ *very clean* if

$$(\operatorname{Supp}\nu) \cap \{\boldsymbol{y} \in \mathbb{R}^3 : y_1y_2y_3(y_1+y_2)(y_1+y_3)(y_2+y_3) = 0\} = \emptyset, \tag{2.28}$$

and $\operatorname{LinAut}(F_0)$-*symmetric* if

$$\nu(y_1,y_2,y_3) = \nu(y_{\sigma(1)},y_{\sigma(2)},y_{\sigma(3)}) \quad \text{for all } \sigma \in S_3. \tag{2.29}$$

Theorem 2.14. *Let $\xi \in \{0,1\}$. Fix $\delta \in (0,9/10)$ and an integer $k \geq 5000$, and assume (1.10) for ξ. Let $\nu \in C_c^\infty(\mathbb{R}^3)$ satisfy (2.28) and (2.29). Let $X,K \geq 2$ be integers with $K \leq X^{9/10-\delta}$. Fix $\kappa \in (0,9\delta/20]$, and let P be the product of all primes $p < X^{\xi\kappa}$. Then*

$$\sum_{a\in\mathbb{Z}:\, \gcd(a,P)=1} [N_{a,\nu}(X) - s_a(K)\sigma_{\infty,a,\nu}(X)]^2 \ll \frac{X^3\|\nu\|^2_{L^2(\mathbb{R}^3)}}{(\log X)^\xi} + \frac{X^3\|\nu\|^2_{k,\infty}B(\nu)^k}{\min(X^{\delta/11},K^{2/3}X^{-3\delta})}.$$

Proof. We apply (1.10) with $w = \nu^{\otimes 2}$. Note that $\|\nu^{\otimes 2}\|_{k,\infty} \leq \|\nu\|^2_{k,\infty}$ by (1.13), and $B(\nu^{\otimes 2}) \leq B(\nu)$ since $\operatorname{Supp}(\nu^{\otimes 2}) \subseteq (\operatorname{Supp}\nu)^2$. Now let $d \leq X^{\xi\delta}$ (so that $Kd \leq X^{9/10}$). By Theorem 2.13, (1.10), and (1.8), we conclude that (since $k \geq 5000$, and $d^{5/3}K^\varepsilon \leq X^{2\delta}$ for $\varepsilon = \delta/3$, say)

$$\operatorname{Var}(X,K;d) = \sum_{\text{special } \boldsymbol{x}\in\mathbb{Z}^6:\, d\mid F_0(\boldsymbol{y}),F_0(\boldsymbol{z})} \nu^{\otimes 2}(\boldsymbol{x}/X) + O(X^{3-\delta} + X^{3+2\delta}K^{-2/3}) \cdot \|\nu\|^2_{k,\infty}B(\nu)^k.$$

By (2.28) and (1.4), any special $\boldsymbol{x} \in \mathbb{Z}^6$ with $\nu^{\otimes 2}(\boldsymbol{x}/X) \neq 0$ must satisfy $(y_1,y_2,y_3) = (z_{\sigma(1)},z_{\sigma(2)},z_{\sigma(3)})$ for some $\sigma \in S_3$, since $\prod_{1\leq i<j\leq 3}(y_i+y_j)(z_i+z_j) \neq 0$. So, by (2.29),

$$\sum_{\text{special } \boldsymbol{x}\in\mathbb{Z}^6:\, d\mid F_0(\boldsymbol{y}),F_0(\boldsymbol{z})} \nu^{\otimes 2}(\boldsymbol{x}/X) = 3! \sum_{\boldsymbol{y}\in\mathbb{Z}^3:\, d\mid F_0(\boldsymbol{y})} \nu(\boldsymbol{y}/X)^2 + O(\|\nu\|^2_{0,\infty} \cdot [B(\nu)X]^2). \tag{2.30}$$

But $\nu \in C_c^\infty(\mathbb{R}^3)$, so by (2.27) (with $N = d$),

$$\sum_{\boldsymbol{y}\in\mathbb{Z}^3:\, d\mid F_0(\boldsymbol{y})} \nu(\boldsymbol{y}/X)^2 = \sum_{\boldsymbol{e}\in(\mathbb{Z}/d\mathbb{Z})^3:\, F_0(\boldsymbol{e})\equiv 0 \bmod d} \left(\frac{X^3\|\nu\|^2_{L^2(\mathbb{R}^3)}}{d^3} + \frac{O_k(\|\nu\|^2_{k,\infty}B(\nu)^3)}{(X/d)^{k-3}} \right), \tag{2.31}$$

since $k \geq 4$. For integers $n \geq 1$, let

$$g(n) := n^{-3} \cdot |\{\boldsymbol{e} \in (\mathbb{Z}/n\mathbb{Z})^3 : F_0(\boldsymbol{e}) \equiv 0 \bmod n\}|.$$

Then our work above (in the the last several displays) implies

$$\operatorname{Var}(X,K;d) = 3!g(d)X^3\|\nu\|^2_{L^2(\mathbb{R}^3)} + r_d, \tag{2.32}$$

where $r_d \ll (X^{3-\delta} + X^{3+2\delta}K^{-2/3} + X^2 + d^3/(X/d)^{k-3}) \cdot \|\nu\|^2_{k,\infty}B(\nu)^k$. Since $d \leq X^{9/10}$ and $k \geq 100$, we in fact have

$$r_d \ll (X^{3-\delta} + X^{3+2\delta}K^{-2/3}) \cdot \|\nu\|^2_{k,\infty}B(\nu)^k. \tag{2.33}$$

But g is multiplicative in n, and we have $g(n) \in (0,1)$ for all $n \geq 2$ (because, for instance, $F_0(\mathbf{0}) = 0$ and $F_0(1,0,0) = 1$). And by definition, $P = \prod_{p<X^{\xi\kappa}} p$. Directly if $\xi = 0$, and by the Selberg sieve [IK04, Theorem 6.4 and (6.80)] if $\xi = 1$, we conclude from (2.32) that

$$\sum_{a\in\mathbb{Z}:\, \gcd(a,P)=1} [N_{a,\nu}(X) - s_a(K)\sigma_{\infty,a,\nu}(X)]^2 \leq \frac{3!X^3\|\nu\|^2_{L^2(\mathbb{R}^3)}}{H^\xi} + R(P), \tag{2.34}$$

where

$$H := \sum_{d<X^\kappa} \mu(d)^2 \prod_{p\mid d} \frac{g(p)}{1-g(p)} \geq 1,$$
$$R(P) \ll_\varepsilon \sum_{d\leq X^{2\xi\kappa}} d^\varepsilon |r_d| \ll X^\varepsilon (X^{3-\delta/10} + X^{3+2.9\delta}K^{-2/3}) \cdot \|\nu\|^2_{k,\infty} B(\nu)^k.$$

(We use (2.33) to bound r_d.) The Hasse bound for elliptic curves gives $g(p) = p^{-1} + O(p^{-3/2})$; so $\sum_{p\leq x} g(p)\log p = \log x + O(1)$ (for $x \geq 1$) and $\sum_{p>1} g(p)^2 \log p < \infty$. By [IK04, §6.6, derivation of (6.85) using Theorem 1.1], then, $H \asymp \log(X^\kappa)$. Theorem 2.14 follows. □

3 Statistics of truncated singular series

For integers $K \geq 1$, recall the definition of $s_a(K)$ from (2.4). In this section, we will prove that $s_a(K)$ is typically sizable (see Theorem 3.9 below). The proof makes use of an "approximate inverse" of $s_a(K) = \sum_{n\leq K} n^{-3} T_a(n)$. There is some flexibility here; we let

$$M_a(K) := \sum_{n\leq K:\, \gcd(n,30)=1} \mu(n) n^{-3} T_a(n). \tag{3.1}$$

The strategy for Theorem 3.9 is to show that $s_a(K)M_a(K)$ is typically sizable, and that $M_a(K)$ is typically bounded. For analytic convenience, let $T^\natural_a(n) := n^{-2} T_a(n)$.

For integers a and primes $p \nmid 3a$, Propositions 3.2 and 3.3 below imply $T^\natural_a(p) \ll 1$ and $T_a(p^{\geq 2}) = 0$. Indeed, if $a \neq 0$, then $s_a(K)$ resembles the value of an L-function at 1, and in this sense Theorem 3.9 is related to work such as [GS03]. But there are also significant issues with this comparison, because $|T^\natural_a(n)|$ can be as large as n or so. (Trivially $|T^\natural_a(n)| \leq n^2$.) The most serious issues arise when a has a large cube divisor.

For an integer $n \geq 1$, let $\operatorname{sq}(n) := \prod_{p^2\mid n} p^{v_p(n)}$ denote the *square-full part* of n, and $\operatorname{cub}(n) := \prod_{p^3\mid n} p^{v_p(n)}$ the *cube-full part* of n. Let $\mathcal{S}(D)$ be the set of nonzero integers a with $\operatorname{sq}(|a|) \leq D$. Let $\mathcal{C}(D)$ be the set of nonzero integers a with $\operatorname{cub}(|a|) \leq D$. We call an integer $n \geq 1$ *square-full* if $n = \operatorname{sq}(n)$, and *cube-full* if $n = \operatorname{cub}(n)$.

We now collect some basic facts about $T_a(n)$.

Lemma 3.1. *Let $a, n \in \mathbb{Z}$ with $n \geq 1$. Let $n_3 := \operatorname{cub}(n)$. Then $|T^\natural_a(n)| \leq O(1)^{\omega(n)} (n_3 n)^{1/2}$.*

Proof. By Proposition 2.2, it suffices to prove the lemma when $n = p^l$ for some prime p and integer $l \geq 1$. But by (2.3) and [Vau97, (4.24) and Lemma 4.7], we have

$$p^{3\lfloor (l-1)/3 \rfloor} \cdot p^{-3l} T_a(p^l) \ll p^{l-3/2} \mathbf{1}_{3|l-1} + p^{l-3} \mathbf{1}_{3\nmid l-1}.$$

Thus $T_a^\natural(p^l) \ll p^{l-1/2}\mathbf{1}_{3|l-1} + p^{l-1}\mathbf{1}_{3|l-2} + p^l \mathbf{1}_{3|l} \ll p^{l/2} \cdot \mathbf{1}_{l \in \{1,2\}} + p^l \cdot \mathbf{1}_{l \geq 3}$. (There should be more robust proofs, but since F_0 is diagonal, it is convenient to call on [Vau97].) □

For the proofs of the next two results, let $N_a(m)$ denote the number of solutions $\boldsymbol{y} \in (\mathbb{Z}/m\mathbb{Z})^3$ to $F_0(\boldsymbol{y}) \equiv a \bmod m$.

Proposition 3.2. *Let $a \in \mathbb{Z}$. Let p be a prime. Then the following hold:*

1. *If $p \nmid 3a$, then $|T_a^\natural(p)| \leq 6 + 2p^{-1/2}$.*
2. *If $p \mid 3a$, then $|T_a^\natural(p)| \leq 6p^{1/2}$.*
3. *If $p \geq 7$, then $|T_a^\natural(p)| < 0.99p$.*

Proof. By (2.3), we have

$$T_a(p) = pN_a(p) - p^3. \tag{3.2}$$

Case 1: $p \nmid 3a$. Then the equation $F_0(\boldsymbol{y}) = aw^3$ cuts out a smooth cubic hypersurface in $\mathbb{P}^3$ over $\mathbb{F}_p$. The "curve at infinity" (cut out by $w = 0$), namely $F_0(\boldsymbol{y}) = 0$ in $\mathbb{P}^2$, is also smooth. So by the Weil conjectures, $N_a(p) = (p^2 + p + 1 + \lambda_1) - (p + 1 - \lambda_2)$, where $\lambda_1 = \lambda_1(a;p)$ and $\lambda_2 = \lambda_2(p)$ are integers with $|\lambda_1| \leq 6p$ and $|\lambda_2| \leq 2\sqrt{p}$. Thus $|T_a(p)| \leq 6p^2 + 2p^{3/2}$, and $|T_a^\natural(p)| \leq 6 + 2p^{-1/2}$. So (1)–(3) hold (with (2) being vacuous).

Case 2: $p = 3$. Then $|T_a^\natural(p)| \leq p^2$ trivially. (In fact, $T_a(p) = 0$, but we do not need this special fact.) So (1)–(3) hold (with (1) and (3) being vacuous).

Case 2: $p \mid 3a$ and $p \neq 3$. Then $p \mid a$. So $T_a(p) = T_0(p)$. But $F_0(\boldsymbol{y}) = 0$ in $\mathbb{A}^3$ is an affine cone over a curve. This curve, $F_0(\boldsymbol{y}) = 0$ in $\mathbb{P}^2$, is smooth. So by the Weil conjectures, $N_0(p) = 1 + (p-1)(p+1-\lambda_2(p))$, where $|\lambda_2(p)| \leq 2\sqrt{p}$. Thus $|T_a(p)| \leq 2p(p-1)\sqrt{p}$, and $|T_a^\natural(p)| \leq 2(p^{1/2} - p^{-1/2})$. So (1)–(3) hold (with (1) being vacuous). □

We now strengthen the vanishing result in [Vau97, Lemma 4.7].

Proposition 3.3. *Let p be a prime. Let $a, l \in \mathbb{Z}$ with $l \geq 2$. If $p^{l-1} \nmid 3a$, then $T_a(p^l) = 0$.*

Proof when $p \neq 3$. By (2.3), we have

$$T_a(p^l) = p^l N_a(p^l) - p^{l-1} p^3 N_a(p^{l-1}). \tag{3.3}$$

Let $N_a^0(p^\nu)$ (for $\nu \geq 1$) denote the number of solutions $\boldsymbol{y} \in (\mathbb{Z}/p^\nu\mathbb{Z})^3$ to $F_0(\boldsymbol{y}) \equiv a \bmod p^\nu$ with $p \mid \nabla F_0(\boldsymbol{y})$. (Here $\nabla F_0(\boldsymbol{y}) = (3y_1^2, 3y_2^2, 3y_3^2)$.) Then by (3.3) and Hensel's lemma,

$$T_a(p^l) = p^l N_a^0(p^l) - p^{l-1} p^3 N_a^0(p^{l-1}). \tag{3.4}$$

Case 1: $p^3 \nmid a$. Suppose $p^{l-1} \nmid 3a$. Then $p^{\min(3,l-1)} \nmid a$, so all solutions $\boldsymbol{y}$ mod p^{l-1} to $F_0(\boldsymbol{y}) \equiv a \bmod p^{l-1}$ must be *primitive* (i.e. satisfy $p \nmid \boldsymbol{y}$), and thus satisfy $p \nmid \nabla F_0(\boldsymbol{y})$. Thus $N_a^0(p^v) = 0$ for $v \geq l-1$, whence $T_a(p^l) = 0$ by (3.4).

Case 2: $p^3 \mid a$. Suppose $p^{l-1} \nmid 3a$. Then $l \geq 5$. But $p \mid \nabla F_0(\boldsymbol{y}) \Rightarrow p \mid \boldsymbol{y}$, so $N_a^0(p^v) = p^6 N_{a/p^3}(p^{v-3})$ for $v \geq 4$, whence $T_a(p^l) = p^9 T_{a/p^3}(p^{l-3})$ by (3.4). Noting that $l-3 \geq 2$ and $p^{l-4} \nmid 3a/p^3$, a recursive argument now yields $T_a(p^l) = 0$. □

Proof when $p = 3$. Both (3.3) and (3.4) hold here, but (3.4) is no longer simpler than (3.3). However, let $N_a^1(p^v)$ (for $v \geq 2$) denote the number of solutions $\boldsymbol{y} \in (\mathbb{Z}/p^v\mathbb{Z})^3$ to $F_0(\boldsymbol{y}) \equiv a \bmod p^v$ with $p^2 \mid \nabla F_0(\boldsymbol{y})$. A standard Hensel argument (based on writing $\boldsymbol{y} = \boldsymbol{y}_0 + p^{l-2}\boldsymbol{z}$, rather than $\boldsymbol{y} = \boldsymbol{y}_0 + p^{l-1}\boldsymbol{z}$) shows that if $l \geq 3$, then

$$T_a(p^l) = p^l N_a^1(p^l) - p^{l-1} p^3 N_a^1(p^{l-1}). \tag{3.5}$$

Case 1: $p^3 \nmid a$. Suppose $p^{l-1} \nmid 3a$. Then $l \geq 3$, and $p^{\min(3,l-2)} \nmid a$. So all solutions $\boldsymbol{y}$ mod p^{l-1} to $F_0(\boldsymbol{y}) \equiv a \bmod p^{l-1}$ are *primitive*, and thus (since $l-1 \geq 2$) satisfy $p^2 \nmid \nabla F_0(\boldsymbol{y})$. Thus $N_a^1(p^v) = 0$ for $v \geq l-1$, whence $T_a(p^l) = 0$ by (3.5).

Case 2: $p^3 \mid a$. Suppose $p^{l-1} \nmid 3a$. Then $l \geq 6$. But $p^2 \mid \nabla F_0(\boldsymbol{y}) \Rightarrow p \mid \boldsymbol{y}$, so $N_a^1(p^v) = p^6 N_{a/p^3}(p^{v-3})$ for $v \geq 5$, whence $T_a(p^l) = p^9 T_{a/p^3}(p^{l-3})$ by (3.5). Noting that $l-3 \geq 3$ and $p^{l-4} \nmid 3a/p^3$, a recursive argument yields $T_a(p^l) = 0$. □

Now recall (2.4) and (3.1). For every $a \in \mathbb{Z}$, we have

$$s_a(K)M_a(K) = \sum_{n \leq K} c_a(n) + \sum_{\substack{n_1,n_2 \leq K: \\ n_1n_2 > K,\ \gcd(n_2,30)=1}} n_1^{-3} T_a(n_1) \cdot \mu(n_2) n_2^{-3} T_a(n_2), \tag{3.6}$$

where for integers $n \geq 1$ we let

$$c_a(n) := n^{-3} \sum_{n_1n_2 = n:\, \gcd(n_2,30)=1} T_a(n_1) \cdot \mu(n_2) T_a(n_2).$$

By Proposition 2.2, $c_a(n)$ is a Dirichlet convolution of two multiplicative functions, so $c_a(n)$ is multiplicative in n. For any prime p and integer $l \geq 1$, we have

$$c_a(p^l) = p^{-3l}[T_a(p^l) - T_a(p^{l-1})T_a(p)\mathbf{1}_{p \geq 7}] = p^{-l}[T_a^\natural(p^l) - T_a^\natural(p^{l-1})T_a^\natural(p)\mathbf{1}_{p \geq 7}]. \tag{3.7}$$

We now bound $c_a(n)$ using our work above (for convenience; [Vau97, Lemma 4.7], a "stratification" result for $p^{-l}T_a^\natural(p^l)$ based on $\gcd(p^l, a)$, might also suffice).

Lemma 3.4. *Let $a, n \in \mathbb{Z}$ with $a \neq 0$ and $n \geq 1$. Let $a_2 := \operatorname{sq}(|a|)$ and $n_3 := \operatorname{cub}(n)$. Let $n_2 := \operatorname{sq}(n/n_3)$ and $n_1 := n/n_2n_3$. Then n_2 is a square, and*

$$|c_a(n)| \leq O(1)^{\omega(n)} n_1^{-2} \cdot n_2^{-1} \gcd(n_2^{1/2}, 3a) \cdot n_3^{-1/2} \prod_{p \mid n_3:\, v_p(n_3) \leq 2v_p(9a_2)} p^{v_p(n_3)/2}. \tag{3.8}$$

Proof. Since n/n_3 is cube-free, $n_2 = \operatorname{rad}(n_2)^2$. As for (3.8), note that both sides of (3.8) are multiplicative in n, so we may assume $n = p^l$, where p is prime and $l \ge 1$ is an integer.

Case 1: $l = 1$. By (3.7), $c_a(p) = 0$ if $p \ge 7$, and $c_a(p) \ll p^{-2}$ trivially if $p \le 5$. So $c_a(n) \ll n^{-2}$. This proves (3.8), since $(n_1, n_2, n_3) = (n, 1, 1)$.

Case 2: $l = 2$. Again, use (3.7). If $p \nmid 3a$, then Propositions 3.3 and 3.2 give $c_a(p^2) = p^{-2}[0 + O(1)] \ll p^{-2}$. If $p \mid 3a$, then Lemma 3.1 and Proposition 3.2 give $c_a(p^2) \ll p^{-2}[p + O(p)] \ll p^{-1}$. Either way, $c_a(p^2) \ll n^{-1} \gcd(n^{1/2}, 3a)$. This proves (3.8), since $(n_1, n_2, n_3) = (1, n, 1)$.

Case 3: $l \ge 3$. If $p^{\max(2,l-2)} \nmid 3a$, then (3.7) and Proposition 3.3 (plus Lemma 3.1 if $l = 3$) give $c_a(p^l) \ll p^{-l/2}$. In general, (3.7) and Lemma 3.1 give $c_a(p^l) \ll 1$. Either way,

$$c_a(n) \ll n^{-1/2} (p^{v_p(n)/2})^{\mathbf{1}_{v_p(3a) \ge \max(2,l-2)}} \ll n^{-1/2} (p^{v_p(n)/2})^{\mathbf{1}_{v_p(9a_2) \ge l/2}},$$

where in the final step we have used the fact that $p^{\max(2,l-2)}$ is square-full, $\operatorname{sq}(3a) \mid \operatorname{sq}(9a_2)$, and $\max(2, l-2) \ge l/2$. This proves (3.8), since $(n_1, n_2, n_3) = (1, 1, n)$. $\square$

In general, the series $\sum_{n \le K} c_a(n)$ behaves better than $s_a(K)$.

Proposition 3.5. *Let $a, D, K \in \mathbb{Z}$ with $a \ne 0$ and $D, K \ge 1$. Suppose $a \not\equiv \pm 4 \bmod 9$.*

1. *For p prime, $1 + c_a(p) + c_a(p^2) + \cdots$ converges absolutely to a real number $\gamma_p(a) > 0$.*
2. *If $p \nmid 3a$, then $\gamma_p(a) \ge (1 - p^{-2})^{O(1)}$. If $p \mid 3a$, then $\gamma_p(a) \ge (1 - p^{-1})^{O(1)}$.*
3. *$\prod_{p \text{ prime}} \gamma_p(a)$ converges absolutely to a real number $\gamma(a) \gg \prod_{p \mid a} (1 - p^{-1})^{O(1)}$.*
4. *If $a \in \mathcal{S}(D)$, then $\sum_{n \le K} c_a(n) = \gamma(a) + O_\varepsilon(|a|^\varepsilon K^{\varepsilon - 1/6} D)$.*

Proof. By (3.7), Proposition 3.3, and (2.1), the sum $\sum_{j \ge 0} c_a(p^j)$ converges *absolutely* to

$$\gamma_p(a) := \sigma_{p,a} \cdot (1 - p^{-3} T_a(p) \mathbf{1}_{p \ge 7}), \tag{3.9}$$

since $a \ne 0$. Here $\sigma_{p,a} > 0$ by [Bro21, Lemma 2.16], since $a \not\equiv \pm 4 \bmod 9$. And

$$1 - p^{-3} T_a(p) \mathbf{1}_{p \ge 7} \in (0.01, 1.99) \tag{3.10}$$

by Proposition 3.2. So $\gamma_p(a) > 0$ by (3.9). This completes the proof of (1).

Now we bound $\gamma_p(a)$ from below. First suppose $p \le 5$. By [Bro21, Lemma 2.16], $\sigma_{p,a} \ge p^{-6} \gg 1$. So $\gamma_p(a) \gg 1$ by (3.9) and (3.10). Now suppose $p \ge 7$. Let $N_a^*(p)$ denote the number of solutions $\boldsymbol{y} \in \mathbb{F}_p^3 \setminus \{\boldsymbol{0}\}$ to $F_0(\boldsymbol{y}) = a$. By (2.1) and Hensel's lemma, $\sigma_{p,a} \ge p^{-2} N_a^*(p)$. But $N_a^*(p) \ge N_a(p) - 1$, and $N_a(p) = p^2 + p^{-1} T_a(p)$ by (3.2), so $\sigma_{p,a} \ge 1 + p^{-3} T_a(p) - p^{-2}$. Since $p \ge 7$, it follows from (3.9), (3.10), and Proposition 3.2 that $\gamma_p(a) \ge 1 - p^{-6} T_a(p)^2 - 1.99 p^{-2} = 1 - O(p^{-2} \mathbf{1}_{p \nmid 3a} + p^{-1} \mathbf{1}_{p \mid 3a})$. Since $\gamma_p(a) > 0$ by (1), we conclude that (2) holds for $p \ge 7$. By enlarging the $O(1)$'s in (2) if necessary, we can then ensure that (2) holds for all primes p.

For (3), first note that if $p \ge 7$ and $p \nmid 3a$, then $N_a^*(p) = N_a(p)$, so $\sigma_{p,a} = p^{-2} N_a(p)$ by (2.1) and Hensel's lemma; and hence the arguments in the previous paragraph imply $\gamma_p(a) = 1 + O(p^{-2})$ (uniformly over $p \nmid 30a$). Thus $\prod_{p \text{ prime}} \gamma_p(a)$ converges absolutely to some $\gamma(a) \in \mathbb{R}_{>0}$. By (2), then, $\gamma(a) \gg \prod_{p \mid 3a} (1-p^{-1})^{O(1)}$. So (3) holds.

For (4), note that by Lemma 3.4 (and the definition of $\mathcal{S}(D)$),

$$\sum_{n>K} |c_a(n)| \ll_\varepsilon \sum_{\substack{n_1,m_2,n_3 \geq 1: \\ n_1 m_2^2 n_3 \geq K,\ n_3 = \mathrm{cub}(n_3)}} (n_1 m_2^2 n_3)^\varepsilon n_1^{-2} m_2^{-2} \gcd(m_2, 3a) n_3^{-1/2} (9D). \tag{3.11}$$

The contribution to the right-hand side of (3.11) from $(n_1, m_2, n_3) \in [N_1, 2N_1) \times [M_2, 2M_2) \times [N_3, 2N_3)$ (where $(2N_1)(2M_2)^2(2N_3) \geq K$) is at most $O_\varepsilon((N_1 M_2 N_3)^{2\varepsilon} D)$ times

$$\frac{N_1 N_3^{1/3}}{N_1^2 M_2^2 N_3^{1/2}} \sum_{d \mid 3a} \sum_{m_2 \in [M_2, 2M_2):\, d \mid m_2} d \ll \sum_{d \mid 3a} \frac{N_1 M_2 N_3^{1/3}}{N_1^2 M_2^2 N_3^{1/2}} \ll_\varepsilon \frac{|a|^\varepsilon}{(N_1 M_2^2 N_3)^{1/6}} \ll_\varepsilon \frac{|a|^\varepsilon K^{3\varepsilon - 1/6}}{(N_1 M_2^2 N_3)^{3\varepsilon}}.$$

Summing over $N_1, M_2, N_3 \in \{1, 2, 4, 8, \ldots\}$, we get $\sum_{n>K} |c_a(n)| \ll_\varepsilon |a|^\varepsilon K^{3\varepsilon - 1/6} D$. Thus $\sum_{n \geq 1} c_a(n)$ converges absolutely to $\gamma(a)$; and furthermore, (4) holds. □

To bound $M_a(K)$ on average, and to handle the sum over $n_1 n_2 > K$ in (3.6), we can use Lemma 3.8 below, whose proof requires the following two lemmas.

Lemma 3.6. *Let $D, n \geq 1$ be integers. Let $a \in \mathcal{S}(D)$. If $T_a(n) \neq 0$, then $n \in \mathcal{C}(27D^{3/2})$.*

Proof. Suppose $T_a(n) \neq 0$. Let $a_2 \leq D$ be the square-full part of $|a|$. Suppose p is a prime dividing the cube-full part n_3 of n. Since $T_a(n) \neq 0$, Proposition 2.2 implies $T_a(p^{v_p(n)}) \neq 0$. So by Proposition 3.3, $p^{v_p(n)-1} \mid 3a$. Here $v_p(n) \geq 3$; so $p^{v_p(n)-1} \mid \mathrm{sq}(3a) \mid \mathrm{sq}(9a_2)$. Therefore, $9a_2$ is divisible by $\prod_{p \mid n_3} p^{v_p(n)-1} \geq n_3^{2/3}$, whence $n_3 \leq (9a_2)^{3/2} \leq 27D^{3/2}$. □

The next lemma controls the large values of $|T_a^\natural(n)|$ on average over a. Let

$$T_b^\natural(\boldsymbol{m}) := T_b^\natural(m_1) \cdots T_b^\natural(m_r), \quad T_b^\natural(\boldsymbol{n}) := T_b^\natural(n_1) \cdots T_b^\natural(n_r),$$

whenever the variables b, r, $m_1, \ldots, m_r$, $n_1, \ldots, n_r$ are clear from context.

Lemma 3.7. *Let $r, m_1, n_1, \ldots, m_r, n_r \geq 1$ be integers. Let $m_{i,3}$, $n_{i,3}$ denote the cube-full parts of m_i, n_i, respectively. Suppose $m_1 \cdots m_r n_1 \cdots n_r$ is square-full. Then*

$$\mathbb{E}_{b \in \mathbb{Z}/m_1 \cdots m_r n_1 \cdots n_r \mathbb{Z}} [|T_b^\natural(\boldsymbol{m})| \cdot |T_b^\natural(\boldsymbol{n})|] \ll_{r,\varepsilon} \frac{\prod_{i=1}^r [(m_{i,3} n_{i,3})^{1/2} (m_i n_i)^{1/2+\varepsilon}]}{\mathrm{rad}(m_1 \cdots m_r n_1 \cdots n_r)}.$$

Proof. By Proposition 2.2, we may assume that $m_1 \cdots m_r n_1 \cdots n_r$ is a prime power; or equivalently, that $\mathrm{rad}(m_1 \cdots m_r n_1 \cdots n_r)$ equals some prime p. Applying Lemma 3.1 to residues b with $p \mid 3b$, and Propositions 3.2 and 3.3 to residues b with $p \nmid 3b$, gives

$$\mathbb{E}_{b \in \mathbb{Z}/m_1 \cdots m_r n_1 \cdots n_r \mathbb{Z}} [|T_b^\natural(\boldsymbol{m})| \cdot |T_b^\natural(\boldsymbol{n})|] \ll_r \frac{\prod_{i=1}^r [(m_{i,3} n_{i,3})^{1/2} (m_i n_i)^{1/2}]}{p / \gcd(3, p)} + 1.$$

But $m_1 \cdots m_r n_1 \cdots n_r$ is square-full, so $\prod_{i=1}^r (m_i n_i)^{1/2} \geq p$, and the lemma follows. □

The following result can loosely be thought of as an algebro-geometric analog of [DK00, Theorem 4], emphasizing a different aspect. It only applies over relatively small moduli.

Lemma 3.8. *Let $A, K_1, K_2, D, j \geq 1$ be integers. Let $\beta\colon \mathbb{Z}^2 \to \mathbb{R}$ be a function supported on $[K_1, 2K_1) \times [K_2, 2K_2)$. For $a \in \mathbb{Z}$, let*

$$P_a := \sum_{m,n\geq 1:\, n\in\mathcal{S}(1)} \beta(m,n)\cdot T_a^\natural(m)T_a^\natural(n). \tag{3.12}$$

Suppose $A \geq (K_1K_2)^{3j}$. Then

$$\sum_{a\in[-A,A]\cap\mathcal{S}(D)} |P_a|^{2j} \ll_{j,\varepsilon} A\cdot \min(K_1, D^{3/2})^j \cdot (K_1K_2)^{j+\varepsilon}\cdot (\max|\beta|)^{2j}. \tag{3.13}$$

Proof. Let $K := K_1K_2$ and $r := 2j$. For $a \in \mathbb{Z}$, let

$$Q_a := \sum_{m,n\geq 1:\, m\in\mathcal{C}(27D^{3/2}),\, n\in\mathcal{S}(1)} \beta(m,n)\cdot T_a^\natural(m)T_a^\natural(n).$$

By (3.12) and Lemma 3.6, we have $P_a = Q_a$ for all $a \in \mathcal{S}(D)$. Thus we may replace each P_a in (3.13) with Q_a; and by positivity, we may then extend $[-A,A]\cap\mathcal{S}(D)$ to the full interval $[-A,A]$. It follows that the left-hand side of (3.13) is at most

$$\sum_{a\in[-A,A]} |Q_a|^r \ll \sum_{\substack{m_1,\dots,m_r\in\mathbb{Z}_{\geq 1}\cap\mathcal{C}(27D^{3/2}),\\ n_1,\dots,n_r\in\mathbb{Z}_{\geq 1}\cap\mathcal{S}(1),\\ b\in\mathbb{Z}/m_1\cdots m_rn_1\cdots n_r\mathbb{Z}}} \beta(\boldsymbol{m},\boldsymbol{n})T_b^\natural(\boldsymbol{m})T_b^\natural(\boldsymbol{n})\cdot \frac{A\cdot(1+O_k(K^{-k}))}{m_1\cdots m_rn_1\cdots n_r}, \tag{3.14}$$

where $\beta(\boldsymbol{m},\boldsymbol{n}) := \beta(m_1,n_1)\cdots\beta(m_r,n_r)$; the inequality (3.14) follows from Poisson summation after replacing (by positivity again) the sum over $a \in [-A,A]$ with a smoothed sum over $a \in \mathbb{Z}$. The total contribution from $O_k(K^{-k})$ to the right-hand side of (3.14) is trivially $\ll K^{2r}\cdot(\max|\beta|)^r\cdot K^r\cdot A\cdot O_k(K^{-k})$, which is satisfactory if $k := 3r$, say.

It remains to handle the "main" term in (3.14). Given $m_1, n_1, \dots, m_r, n_r \geq 1$, we have

$$\sum_{b\in\mathbb{Z}/m_1\cdots m_rn_1\cdots n_r\mathbb{Z}} T_b^\natural(\boldsymbol{m})T_b^\natural(\boldsymbol{n}) = 0 \quad \text{if } m_1\cdots m_rn_1\cdots n_r \text{ is not square-full.}$$

(This follows from Proposition 2.2 and the fact that $\sum_{a\in\mathbb{Z}/p\mathbb{Z}} T_a(p) = 0$ for every prime p. Note that we used a similar idea to prove Lemma 2.7.) On the other hand, if $m_1\cdots m_rn_1\cdots n_r$ is square-full, then Lemma 3.7 delivers the bound

$$\mathbb{E}_{b\in\mathbb{Z}/m_1\cdots m_rn_1\cdots n_r\mathbb{Z}}[|T_b^\natural(\boldsymbol{m})|\cdot|T_b^\natural(\boldsymbol{n})|] \ll_{r,\varepsilon} \frac{\min(K_1, D^{3/2})^{r/2}\cdot K^{r/2+\varepsilon}}{\operatorname{rad}(m_1\cdots m_rn_1\cdots n_r)},$$

provided that we have $m_1,\dots,m_r \in \mathcal{C}(27D^{3/2})$ and $n_1,\dots,n_r \in \mathcal{S}(1)$. It follows that

$$\sum_{\substack{m_1,\dots,m_r\in\mathbb{Z}_{\geq 1}\cap\mathcal{C}(27D^{3/2}),\\ n_1,\dots,n_r\in\mathbb{Z}_{\geq 1}\cap\mathcal{S}(1),\\ b\in\mathbb{Z}/m_1\cdots m_rn_1\cdots n_r\mathbb{Z}}} \frac{\beta(\boldsymbol{m},\boldsymbol{n})T_b^\natural(\boldsymbol{m})T_b^\natural(\boldsymbol{n})}{m_1\cdots m_rn_1\cdots n_r} \ll \sum_{R\leq(4K)^{r/2}} \sum_{\substack{m_1,\dots,m_r\mid R^\infty,\\ n_1,\dots,n_r\mid R^\infty}} \frac{C(\boldsymbol{m},\boldsymbol{n})}{R}, \tag{3.15}$$

where $C(\boldsymbol{m},\boldsymbol{n}) := |\beta(\boldsymbol{m},\boldsymbol{n})| \cdot \min(K_1, D^{3/2})^{r/2} \cdot K^{r/2+\varepsilon}$. (For $u, v \in \mathbb{Z}$, we write $u \mid v^\infty$ if there exists a positive integer k with $u \mid v^k$.) But for any integers $R, N \geq 1$, we have $\sum_{n \mid R^\infty} \mathbf{1}_{n \in [N,2N)} \ll_\varepsilon \sum_{n \mid R^\infty} (N/n)^\varepsilon = N^\varepsilon \prod_{p \mid R} (1 - p^{-\varepsilon})^{-1} \ll_\varepsilon N^\varepsilon R^\varepsilon$; so

$$\sum_{m_1,\ldots,m_r,n_1,\ldots,n_r \mid R^\infty} |\beta(\boldsymbol{m},\boldsymbol{n})| \ll_{r,\varepsilon} (\max|\beta|)^r (K_1 R)^{r\varepsilon} (K_2 R)^{r\varepsilon}. \tag{3.16}$$

By (3.16) and the bound $\sum_{R \leq (4K)^{r/2}} R^{2r\varepsilon - 1} \ll_{r,\varepsilon} K^{r^2\varepsilon}$, the right-hand side of (3.15) is

$$\ll_{r,\varepsilon} (\max|\beta|)^r \min(K_1, D^{3/2})^{r/2} K^{r/2+(1+r+r^2)\varepsilon}.$$

Plugging this into (3.14) gives (3.13). □

For integers $K \geq 1$ and reals $\eta > 0$, let

$$\mathscr{E}(K;\eta) := \{a \in \mathbb{Z} : a \not\equiv \pm 4 \bmod 9\} \cap \{a \in \mathbb{Z} : |s_a(K)| \leq \eta\}. \tag{3.17}$$

Theorem 3.9. *Let $A, K, j \geq 1$ be integers. Let $\varepsilon, \eta > 0$ be reals. If $A \geq K^{6j}$, then*

$$\frac{|\mathscr{E}(K;\eta) \cap [-A,A]|}{A} \ll_{j,\varepsilon} \eta^{1.8j} + A^\varepsilon K^{-1.5j} + \eta^{-0.2j} K^{-0.8j} + K^{-1/24}.$$

Proof. By Proposition 3.5, there exists a constant $C = C(\varepsilon) > 0$ such that every element of $\mathscr{E}(K;\eta) \cap [-A,A] \cap \mathcal{S}(D)$ lies in one of the following sets:

1. $\mathscr{E}_1 := \{a \in \mathcal{S}(D) : |M_a(K)| \geq \eta^{-9/10}\}$.
2. $\mathscr{E}_2 := \{a \in \mathcal{S}(D) : \prod_{p \mid a} (1 - p^{-1})^C \leq C \cdot (\eta^{1/10} + A^\varepsilon K^{\varepsilon - 1/6} D)\}$.
3. $\mathscr{E}_3 := \{a \in \mathcal{S}(D) : |s_a(K) M_a(K) - \sum_{n \leq K} c_a(n)| \geq \eta^{1/10}\}$.

Indeed, if C is sufficiently large and $a \in \mathscr{E}(K;\eta) \cap [-A,A] \cap \mathcal{S}(D) \setminus (\mathscr{E}_1 \cup \mathscr{E}_2)$, then $|s_a(K)| \leq \eta$ (since $a \in \mathscr{E}(K;\eta)$) and $|M_a(K)| \leq \eta^{-9/10}$ (since $n \notin \mathscr{E}_1$), and $\sum_{n \leq K} c_a(n) \geq 2\eta^{1/10}$ (by Proposition 3.5, since $n \notin \mathscr{E}_2$), so $a \in \mathscr{E}_3$.

We now bound $\mathscr{E}_1$, $\mathscr{E}_2$, $\mathscr{E}_3$. By (3.1) and Lemma 3.8 (with $K_1 = 1$ and $1 \leq K_2 \leq K$),

$$\eta^{-1.8j} \cdot |\mathscr{E}_1 \cap [-A,A]| \leq \sum_{a \in [-A,A] \cap \mathcal{S}(D)} |M_a(K)|^{2j} \ll_j A, \tag{3.18}$$

provided $A \geq K^{3j}$. By (3.6) and Lemma 3.8 (with $1 \leq K_2, K_2 \leq K$ and $(2K_1)(2K_2) \geq K$),

$$\eta^{0.2j} \cdot |\mathscr{E}_3 \cap [-A,A]| \leq \sum_{a \in [-A,A] \cap \mathcal{S}(D)} \left| s_a(K) M_a(K) - \sum_{n \leq K} c_a(n) \right|^{2j} \ll_{j,\varepsilon} A D^{3j/2} K^{-j+\varepsilon}, \tag{3.19}$$

provided $A \geq K^{6j}$. And $\prod_{p \mid a} (1 - p^{-1}) = \phi(|a|)/|a|$, so

$$C^{-18j} (\eta^{1/10} + A^\varepsilon K^{\varepsilon - 1/6} D)^{-18j} \cdot |\mathscr{E}_2 \cap [-A,A]| \leq \sum_{a \in [-A,A] \setminus \{0\}} \left(\frac{|a|}{\phi(|a|)} \right)^{18Cj} \ll_{Cj} A \tag{3.20}$$

by [MV07, p. 61, (2.32)]. Since $[-A,A]\setminus\mathcal{S}(D)$ has size $\ll D^{-1/2}A$, we conclude that

$$\frac{|\mathscr{E}(K;\eta)\cap[-A,A]|}{A}\ll_{j,\varepsilon}\eta^{1.8j}+(A^{\varepsilon}K^{\varepsilon-1/6}D)^{18j}+\eta^{-0.2j}D^{3j/2}K^{-j+\varepsilon}+D^{-1/2},$$

provided $A\geq K^{6j}$. Taking $D=\lfloor K^{1/12}\rfloor$ gives Theorem 3.9. □

4 Applying increasingly cuspidal weights

To prove Theorems 1.1 and 1.2, we will combine Theorems 2.13, 2.14, and 3.9. To apply Theorems 2.13 and 2.14, we need to choose a suitable weight ν. Fix a function $w_0\in C_c^\infty(\mathbb{R})$ with $w_0\geq 0$ everywhere, and $w_0\geq 1$ on $[-2,2]$. Fix a function $w_2\in C_c^\infty(\mathbb{R})$ with $w_2\geq 0$ everywhere, $w_2\geq 1$ on $[1,10]$, and $\operatorname{Supp} w_2\subseteq\mathbb{R}_{>0}$. Given a real $R\geq 2$, set

$$\nu^{\star}(\boldsymbol{y}):=w_0(F_0(\boldsymbol{y}))\int_{r\in[1,R]}d^{\times}r\prod_{1\leq l\leq 3}w_2(|y_l|/r)\prod_{1\leq i<j\leq 3}w_2(|y_i+y_j|/r),\tag{4.1}$$

where $d^{\times}r:=dr/r$. Clearly $\nu^{\star}\in C_c^\infty(\mathbb{R}^3)$, and $\nu^{\star}$ satisfies (2.28) and (2.29).

Let us consider what happens as we vary R. It is clear from (4.1) that

$$\operatorname{Supp}\nu^{\star}\subseteq\{\boldsymbol{y}\in\mathbb{R}^3:1\ll|y_1|,|y_2|,|y_3|\ll R\};\tag{4.2}$$

in particular, $B(\nu^{\star})\ll R$. On the other hand (reflecting the tentacled nature of $\operatorname{Supp}\nu^{\star}$),

$$\operatorname{vol}(\operatorname{Supp}\nu^{\star})\ll\int_{r\in[1,R]}d^{\times}r\int_{y_1,y_2\in\mathbb{R}}dy_1\,dy_2\,\mathbf{1}_{|y_1|,|y_2|\in r\cdot\operatorname{Supp}w_2}\cdot r^{-2}\ll\log R,\tag{4.3}$$

because for any $r\in[1,R]$ and $y_1,y_2\in\pm r\cdot\operatorname{Supp}w_2$, the set $\{y_3\in\pm r\cdot\operatorname{Supp}w_2:F_0(\boldsymbol{y})\in\operatorname{Supp}w_0\}$ has measure $\ll r^{-2}$. We also have the following key lemma.

Lemma 4.1. *Let $a\in\mathbb{R}$ with $|a|\leq X^3$. Then $\sigma_{\infty,a,\nu^{\star}}(X)\gg\log R$.*

Proof. Let $\tilde{a}:=a/X^3$. Given (y_2,y_3), let $y_1:=(\tilde{a}-y_2^3-y_3^3)^{1/3}$. For $r\geq 1$, let $D_r:=[3.99r,4r]^2$. Then for all $(y_2,y_3)\in D_r$, we have $F_0(\boldsymbol{y})=\tilde{a}\in[-1,1]$ and $y_1\in[-6r,-5r]$, and thus

$$w_0(F_0(\boldsymbol{y}))\prod_{1\leq l\leq 3}w_2(|y_l|/r)\prod_{1\leq i<j\leq 3}w_2(|y_i+y_j|/r)\geq 1.$$

By (2.24), (4.1), and the nonnegativity of w_0, w_2, it follows that

$$\sigma_{\infty,a,\nu^{\star}}(X)=\int_{\mathbb{R}^2}dy_2\,dy_3\,\nu^{\star}(\boldsymbol{y})\cdot(3y_1^2)^{-1}\gg\int_{r\in[1,R]}d^{\times}r\operatorname{vol}(D_r)\cdot r^{-2}\gg\log R,$$

since $\operatorname{vol}(D_r)\gg r^2$. □

We also need some control on the norms (1.13) of $\nu^\star$. Since w_0, w_2 are fixed, we have

$$\nu^\star(\boldsymbol{y}) \ll \int_{\mathbb{R}_{>0}} d^\times r\, w_2(|y_1|/r) = \mathbf{1}_{y_1\neq 0} \cdot \int_{\mathbb{R}_{>0}} d^\times r\, w_2(1/r) \ll 1, \tag{4.4}$$

uniformly over $\boldsymbol{y} \in \mathbb{R}^3$ and $R \geq 2$. In general, for integers $k \geq 0$, we have[3]

$$\|\nu^\star\|_{k,\infty} \ll_k B(\nu^\star)^{2k} \ll_k R^{2k}; \tag{4.5}$$

to see why, note that by the chain rule, any y_i-derivative of w_0 in (4.1) introduces a factor of $3y_i^2 \ll B(\nu^\star)^2$, while any y_i-derivative of w_2 in (4.1) only introduces a factor of $1/r \ll 1$.

We are finally prepared to prove our main theorems, by adapting Chebyshev's inequality to approximate variances like (2.5) (after removing a's for which $s_a(K)$ is small).

Proof of Theorems 1.1 and 1.2. We gradually increase our hypotheses. First assume that (1.9) for $d = 1$ holds for all clean functions $w \in C_c^\infty(\mathbb{R}^6)$. Plugging this and (2.30), (2.31) into Theorem 2.13 (with $d = 1$), we get that for integers $X, K \geq 1$ with $K \leq X^{9/10}$, we have

$$\frac{\mathrm{Var}(X,K;1)}{X^3} \ll \|\nu^\star\|^2_{L^2(\mathbb{R}^3)} + o_{\nu^\star;X\to\infty}(1) + K^{-1/2}\|\nu^\star\|^2_{100,\infty} B(\nu^\star)^{5000}, \tag{4.6}$$

where $o_{\nu^\star;X\to\infty}(1)$ denotes a quantity that tends to 0 as $X \to \infty$ (for any fixed $\nu^\star$). Here $\|\nu^\star\|^2_{L^2(\mathbb{R}^3)} \ll \log R$ by (4.3)–(4.4), and $\|\nu^\star\|^2_{100,\infty} B(\nu^\star)^{5000} \ll R^{5400}$ by (4.5). On the other hand, by Theorem 3.9 with $\eta = (\log R)^{-10/j}$ (for an integer $j \geq 1$), we have

$$\frac{|\mathscr{E}(K;\eta) \cap [-A,A]|}{A} \ll_{j,\varepsilon} (\log R)^{-18} + A^\varepsilon K^{-1.5j} + (\log R)^2 K^{-0.8j} + K^{-1/24} \tag{4.7}$$

for integers $A \geq K^{6j}$. By (2.5), (3.17), and Lemma 4.1, we conclude (by letting $A, X \to \infty$ with $A \in [X^3/2, X^3]$, taking $K = \lfloor A^{1/6j} \rfloor$, taking $\varepsilon = 1/6$, and taking $R \leq K^{1/20000}$) that

$$\begin{aligned} \frac{|\mathcal{E} \cap [-A,A]|}{A} &\ll \frac{|\mathscr{E}(K;\eta) \cap [-A,A]|}{A} + \frac{\mathrm{Var}(X,K;1)/X^3}{\eta^2 (\log R)^2} \\ &\ll_j (\log R)^{-18} + \frac{(\log R) + o_{R;A\to\infty}(1)}{(\log R)^{2-20/j}} \end{aligned} \tag{4.8}$$

as $A \to \infty$. Taking $j = 40$ and $R \to \infty$ proves the first part of Theorem 1.1.

What remains is similar. Fix $(\delta, k) \in \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 1}$, and assume (1.10) for $\xi = 0$. Note that (1.10) remains true if we decrease δ or increase k; so we may assume $\delta \in (0, 9/10)$ and $k \geq 5000$. In fact, it will be convenient to assume $\delta = 1/(12j)$ where $j \geq 1$ is a large integer. Theorem 2.14 for $\xi = 0$ now implies (assuming $X, K \geq 2$ and $K \leq X^{9/10-\delta}$)

$$\frac{\mathrm{Var}(X,K;1)}{X^3} \ll_j \|\nu^\star\|^2_{L^2(\mathbb{R}^3)} + \frac{\|\nu^\star\|^2_{k,\infty} B(\nu^\star)^k}{\min(X^{\delta/11}, K^{2/3} X^{-3\delta})}. \tag{4.9}$$

[3] In [Wan22, Remark 2.2.15] the derivatives of ν are incorrectly stated to be $O_k(1)$. This mistake does not affect the proof of [Wan22, Theorem 2.1.8]. In any case, our present work is independent of [Wan22].

Let $A, X \to \infty$ with $A \in [X^3/2, X^3]$; let $(K, \varepsilon, R, \eta) = (\lfloor A^{2\delta} \rfloor, 1/6, X^{\delta/300k}, (\log R)^{-10/j})$. Then $K = \lfloor A^{1/6j} \rfloor$, so $A \geq K^{6j}$. Applying (4.3)–(4.5), Theorem 3.9, and Lemma 4.1 as before, we get by the first line of (4.8) that

$$\frac{|\mathcal{E} \cap [-A, A]|}{A} \ll_j (\log R)^{-18} + \frac{(\log R) + R^{5k} X^{-\delta/11}}{(\log R)^{2-20/j}} \ll_j \frac{1}{(\log R)^{1-20/j}}, \tag{4.10}$$

since $R^{5k} = X^{\delta/60}$. Taking $j \to \infty$ proves the second part of Theorem 1.1.

Finally, fix $(\delta, k) \in \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 1}$, and assume (1.10) for $\xi = 1$. As in the previous paragraph, we assume $\delta = 1/(12j)$ and $k \geq 5000$, where $j \in \mathbb{Z}_{\geq 1}$, and let $(K, \varepsilon, R, \eta) = (\lfloor A^{2\delta} \rfloor, 1/6, X^{\delta/300k}, (\log R)^{-10/j})$, where $A \in [X^3/2, X^3]$ and $A, X \to \infty$. Theorem 2.14 for $\xi = 1$, when combined with (4.3)–(4.5), Theorem 3.9, and Lemma 4.1 as before, then gives

$$\frac{|\mathcal{E} \cap \{p \leq A\}|}{A} \ll_j (\log R)^{-18} + \frac{(\log R/\log X) + R^{5k} X^{-\delta/11}}{(\log R)^{2-20/j}} \ll_j \frac{1}{(\log R)^{2-20/j}}. \tag{4.11}$$

Taking $j \to \infty$ proves Theorem 1.2. □

5 Nonnegative cubes

Let $A \geq 2$. By the Selberg sieve, $\sum_{p \leq A} r_3(p) \ll A/\log A$. Assuming something like (1.10) (ideally for arbitrarily small $\delta > 0$), can one prove $\sum_{p \leq A} r_3(p) \gg A/\log A$? Something like (1.10) might let one handle certain "Type II" sums. The main difficulty might instead lie in "Type I_j" estimates (roughly corresponding to counting solutions to $dn_1 n_2 \cdots n_j = x^3 + y^3 + z^3$ for j large, where d is fixed). One may be able to handle "Type I_j" sums for $j \leq 2$ using the methods of [Hoo81], but it would be nice to treat larger j, even conditionally.

Or, assuming precise asymptotic second moments for $r_3(a)$ over $\{a \leq A : a \equiv 0 \bmod d\}$ for $d \leq A^\delta$, can one show that $\sum_{p \leq A} r_3(p)^2 \ll A/\log A$? Note the lack of exact multiplicative structure in d in the expected main term over $\{a \leq A : a \equiv 0 \bmod d\}$. (For $d = 1$, see [Hoo86a, Conjecture 2].) This may or may not be a serious obstacle.

Finally, in the conditional sense above, can one show that a positive proportion of primes p have $r_3(p) \neq 0$ (i.e. are sums of three nonnegative cubes)? Another direction, suggested by discussion with Christian Bernert and Damaris Schindler, would be to find a sequence of arithmetic progressions P_i along which $\{a \in P_i : r_3(a) \neq 0\}$ has relative density $\to 1$.

Acknowledgments

I thank Valeriya Kovaleva, Sarah Peluse, and Peter Sarnak for inspiring me to extend my thesis work on $x^3 + y^3 + z^3$ from integer values to prime values. I thank Manjul Bhargava, Tim Browning, Valeriya Kovaleva, Peter Sarnak, Katy Woo, Shuntaro Yamagishi, and Liyang Yang for conversations on closely related topics. I thank Simona Diaconu for sharing an early draft of her enlightening senior thesis. I thank the editors and referees for providing helpful feedback, corrections, and suggestions.

References

[BDPW23] J. Bergström, A. Diaconu, D. Petersen, and C. Westerland, *Hyperelliptic curves, the scanning map, and moments of families of quadratic L-functions*, arXiv:2302.07664 (2023). ↑3

[BGW24] T. Browning, J. Glas, and V. Y. Wang, *Sums of three cubes over a function field*, arXiv:2402.07146 (2024). ↑3

[Brü91] J. Brüdern, *Ternary additive problems of Waring's type*, Math. Scand. **68** (1991), no. 1, 27–45. MR1124817 ↑1

[Bro21] T. Browning, *Cubic forms and the circle method*, Progress in Mathematics, vol. 343, Birkhäuser/Springer, Cham, 2021. MR4387070 ↑6, 18

[BS21] A. R. Booker and A. V. Sutherland, *On a question of Mordell*, Proc. Natl. Acad. Sci. USA **118** (2021), no. 11, Paper No. 2022377118, 11. MR4279690 ↑4

[BW10] J. Brüdern and T. D. Wooley, *On Waring's problem: three cubes and a minicube*, Nagoya Math. J. **200** (2010), 59–91. MR2747878 ↑4

[CLT10] A. Chambert-Loir and Y. Tschinkel, *Igusa integrals and volume asymptotics in analytic and adelic geometry*, Confluentes Math. **2** (2010), no. 3, 351–429. MR2740045 ↑5, 11

[CTW12] J.-L. Colliot-Thélène and O. Wittenberg, *Groupe de Brauer et points entiers de deux familles de surfaces cubiques affines*, Amer. J. Math. **134** (2012), no. 5, 1303–1327. MR2975237 ↑1

[DHL06] J.-M. Deshouillers, F. Hennecart, and B. Landreau, *On the density of sums of three cubes*, Algorithmic number theory, 2006, pp. 141–155. MR2282921 ↑2

[Dia19] S. Diaconu, *On admissible integers of cubic forms*, Senior Thesis, Princeton University, 2019. URL: http://arks.princeton.edu/ark:/88435/dsp0112579w10h. ↑1, 2, 3, 4, 5, 12

[DK00] W. Duke and E. Kowalski, *A problem of Linnik for elliptic curves and mean-value estimates for automorphic representations*, Invent. Math. **139** (2000), no. 1, 1–39. With an appendix by Dinakar Ramakrishnan. MR1728875 ↑20

[DS19] K. Destagnol and E. Sofos, *Rational points and prime values of polynomials in moderately many variables*, Bull. Sci. Math. **156** (2019), 102794, 33. MR3996328 ↑4

[FMT89] J. Franke, Y. I. Manin, and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. **95** (1989), no. 2, 421–435. MR974910 ↑2

[GS03] A. Granville and K. Soundararajan, *The distribution of values of $L(1,\chi_d)$*, Geom. Funct. Anal. **13** (2003), no. 5, 992–1028. MR2024414 ↑15

[GS22] A. Ghosh and P. Sarnak, *Integral points on Markoff type cubic surfaces*, Invent. Math. **229** (2022), no. 2, 689–749. MR4448994 ↑1, 2, 4, 5

[HB92] D. R. Heath-Brown, *The density of zeros of forms for which weak approximation fails*, Math. Comp. **59** (1992), no. 200, 613–623. MR1146835 ↑2, 5

[HB98] ______, *The circle method and diagonal cubic forms*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. **356** (1998), no. 1738, 673–699. MR1620820 ↑3

[Hoo16] C. Hooley, *On the representation of numbers by quaternary and quinary cubic forms: I*, Acta Arith. **173** (2016), no. 1, 19–39. MR3494132 ↑1, 4

[Hoo81] ______, *On Waring's problem for two squares and three cubes*, J. Reine Angew. Math. **328** (1981), 161–207. MR636202 ↑24

[Hoo86a] ______, *On some topics connected with Waring's problem*, J. Reine Angew. Math. **369** (1986), 110–153. MR850631 ↑2, 24

[Hoo86b] ______, *On Waring's problem*, Acta Math. **157** (1986), no. 1-2, 49–97. MR857679 ↑3

[Hoo97] ______, *On Hypothesis K^* in Waring's problem*, Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995), 1997, pp. 175–185. MR1635754 ↑3

[Hua38] L.-K. Hua, *On Waring's problem*, Quart. J. Math. Oxford Ser. **9** (1938), no. 1, 199–202. ↑3

[IK04] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004. MR2061214 ↑15

[LMU23] J. Lyczak, V. Mitankin, and H. Uppal, *Cubic surfaces failing the integral Hasse principle*, `arXiv:2311.10008` (2023). ↑1

[MPPRW24] J. Miller, P. Patzt, D. Petersen, and O. Randal-Williams, *Uniform twisted homological stability*, `arXiv:2402.00354` (2024). ↑3

[MV07] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory. I. Classical theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, Cambridge, 2007. MR2378655 ↑22

[Pey95] E. Peyre, *Hauteurs et mesures de Tamagawa sur les variétés de Fano*, Duke Math. J. **79** (1995), no. 1, 101–218. MR1340296 ↑2, 11

[Vau20] R. C. Vaughan, *On some questions of partitio numerorum: Tres cubi*, Glasgow Mathematical Journal (2020), 1–22. ↑3

[Vau80] ———, *A ternary additive problem*, Proc. London Math. Soc. (3) **41** (1980), no. 3, 516–532. MR591653 ↑1

[Vau97] ———, *The Hardy-Littlewood method*, Second, Cambridge Tracts in Mathematics, vol. 125, Cambridge University Press, Cambridge, 1997. MR1435742 ↑6, 7, 16, 17

[VW95] R. C. Vaughan and T. D. Wooley, *On a certain nonary cubic form and related equations*, Duke Math. J. **80** (1995), no. 3, 669–735. MR1370112 ↑2

[Wan22] V. Y. Wang, *Families and dichotomies in the circle method*, Ph.D. Thesis, Princeton University, 2022. URL: `http://arks.princeton.edu/ark:/88435/dsp01rf55zb86g`. ↑3, 4, 5, 23

[Wan23] ———, *Sums of cubes and the Ratios Conjectures*, `arXiv:2108.03398v2` (April 19, 2023). ↑3

[Wan24] ———, *Notes on zeta ratio stabilization*, `arXiv:2402.01214` (2024). ↑3

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