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Isometric Immersions with Controlled Curvatures

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Abstract: We δ-approximate strictly short (e.g. constant) maps between Riemannian manifolds $f_0 : X^m \to Y^N$ for $N \gg m^2/2$ by C^∞-smooth isometric immersions $f_\delta : X^m \to Y^N$ with curvatures $curv(f_\delta) < \frac{\sqrt{3}}{\delta}$, for $\delta \to 0$.

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1 Compression and Approximation

A C^∞-immersion of a smooth manifold X to a smooth Riemannian $Y = (Y, h)$,

$$f_0 : X \to Y$$

is called $\mathcal{II}_h$ or just $\mathcal{II}$, if the Riemannian metric inducing operator from the space of smooth maps $f : X \to Y$ to the space $\mathcal{G}_+(X)$ of smooth semi definite positive quadratic forms on X,

$$\mathscr{I} = \mathscr{I}_h : \mathcal{F} = C^\infty(X,Y) \to \mathcal{G}_+(X) \text{ for } f \overset{\mathscr{I}}{\mapsto} g = f^*(h)$$

is ***i**nfinitesimally **i**nvertible in a C^∞-neighbourhood $\mathcal{F}_0 \subset \mathcal{F}$ of f_0.*

This means that the differential/linearization of $\mathscr{I}$,

$$\mathscr{L}_f : T_f(\mathcal{F}) \to T_{\mathscr{I}(f)}(\mathcal{G}),$$

of $\mathscr{I}$, $f \in \mathcal{F}_0$, is *right invertible* by a *differential operator*

$$\mathscr{M}_f : T_{\mathscr{I}(f)}(\mathcal{G}) \to \mathscr{T}_f(\mathcal{F}),\ \mathscr{L}_f \circ \mathscr{M}_f = Id : T_{\mathscr{I}(f)}(\mathcal{G}) \to T_{\mathscr{I}(f)}(\mathcal{G}),$$

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where $f \mapsto \mathscr{M}_f$ is a (possibly non-linear) differential operator defined on $\mathcal{F}_0$.

If $Y = \mathbb{R}^N$, (and in local coordinates for all Y, in general) the operator $\mathscr{L}$ can be written as an operator on maps $\overrightarrow{f}: X \to Y$,

$$\mathscr{L}_f(\overrightarrow{f}) = \mathscr{I}(f + \varepsilon \overrightarrow{f}) - \mathscr{I}(f) + o(\varepsilon),\ \varepsilon \to 0,$$

and where $\mathscr{M}_f(\overrightarrow{g})$ is a differential operator in $(f, \overrightarrow{g})$, which is linear in $\overrightarrow{g}$ and which satisfies

$$\mathscr{L}_f(\mathscr{M}_f(\overrightarrow{g})) = \overrightarrow{g}.$$

"Free" Example. Free immersions f, i.e. where the second osculating spaces $osc_2(f(x)) \in T_{f(x)}(Y)$ have *maximal possible* dimensions,

$$dim(osc_2(f(x)) = \frac{dim(X)(dim(X)+1)}{2} + dim(X)$$

at all points $x \in X$, are $\mathcal{II}$ by the *Janet-Burstin-Nash Lemma.*

Consequently,

generic f are $\mathcal{II}$ for $dim(Y) \geq \frac{dim(X)(dim(X)+1)}{2} + 2dim(X)$.[1]

1.A. Definitions of m-Free and of Flat $\mathcal{II}^{[m]}$ Maps. A smooth immersion $f: X = X^n \to Y$ is m-free, $m \leq n = dim(X)$, if the restrictions of f to all m-dimensional submanifolds in X are free.

For instance,

$$if\ \ N = dim(Y) \geq \frac{m(m+1)}{2} + m(n-m) + 2n,$$

then generic f are m-free.

An immersion $f: X = X^n \to Y$ is *flat* $\mathcal{II}^{[m]}$, $m \leq n$, if the induced metric in X is Riemannian flat, i.e. locally isometric to $\mathbb{R}^n$, and if the restrictions of f to all *flat*, (i.e. locally isometric to $\mathbb{R}^m \subset \mathbb{R}^n$) m-dimensional submanifolds $X^m \subset Y$ are $\mathcal{II}$.

For instance, isometric m-free immersions of flat tori are flat $\mathcal{II}^{[m]}$.

Such immersions, especially of the split tori $\mathbb{T}^n = \underbrace{\mathbb{T}^1 \times ... \times \mathbb{T}^1}_{n}$ to the Euclidean spaces, play a special role in our arguments,[2] where we use below the following.

1.B. Example. Let $\mathbb{T}^n$ be the torus with a flat Riemannian metric, i.e. the universal covering of this $\mathbb{T}^n$ is isometric to $\mathbb{R}^n$. Then:

$\mathbb{T}^n$ admits a free isometric C^∞-immersion to $\mathbb{R}^{\frac{n(n+1)}{2}+n+2}$

and

an isometric $\mathcal{II}$-immersion to $\mathbb{R}^{\frac{n(n+1)}{2}+n+1}$.

1.C. *Remarks on the Proof.* (a) The existence of free isometric immersions of flat split n-tori to N-dimensional Riemannian manifolds is proven for $N \geq \frac{n(n+1)}{2} + n + 2$ in section 3.1.8 in [4] and since non-split flat tori can be approximated by finite coverings of split ones, the general case follows from the Nash implicit function theorem.

[1] See [4], [5] and references therein.

[2] This is similar to how it is with non-isometric immersions with controlled curvature in [6].

(b) The existence of $\mathcal{JJ}$-immersion of split tori to $\mathbb{R}^{\frac{n(n+1)}{2}+n+1}$ is (implicitly) indicated in an exercise on p. 251 in [4][3], while the recent result by DeLeo [1] points toward a similar possibility for

$$dim(Y) \geq \frac{n(n+1)}{2} + n - \sqrt{\frac{n}{2}} + \frac{1}{2}.$$

In fact, it is not impossible that such immersions exist for

$$dim(Y) \geq \frac{n(n+1)}{2} + 1,$$

and it seems not hard to show these don't exist for $dim(Y) \leq \frac{n(n+1)}{2}$.

Definition of $\mathbf{curv(f) = curv(f(X))}$. This is (as in [6]) the curvature of a manifold X immersed by f to a Riemannian Y, that is *the supremum of the "Y-curvatures"* , i.e. curvatures measured in the Riemannian geometry of $Y \supset X$, of geodesics $\gamma \subset X$, for the induced Riemannian metric in X,

$$curv(f) = curv(f(X)) = \sup_{\gamma \in X} curv_Y(\gamma).$$

1.D. $\sqrt{3}$-Remark. One has only a limited control over the curvatures of the above immersions $\mathbb{T}^n \to Y$, even for $Y = \mathbb{R}^N$, but we shall prove in the next section the existence of

a free isometric immersion from the n-torus to the unit N-ball for $N \geq \frac{n(n+1)}{2} + 2n$ with the curvature bounded by a constant D independent of n, where, conceivably, $D < \sqrt{3}$.

In fact, $\sqrt{3}$ is *asymptotically optimal*, since, according to *Petrunin's inequality*,[4]

smooth isometric immersions $f : \mathbb{T}^n \to B^N(1)$ satisfy for all n and N:

$$curv(f) \geq \sqrt{3\frac{n}{n+2}}.$$

1.E. Compression Lemma. Let

$$F : \mathbb{T}^n \hookrightarrow B^N(1) \subset \mathbb{R}^N$$

be a flat $\mathcal{JJ}^{[m]}$-immersion.

Let $X^m = (X^m, g)$ be a compact Riemannian m-manifold, possibly with a boundary, which admits a smooth $(1+\varepsilon)$-bi-Lipschitz immersion $\phi_\varepsilon : X^m \to \mathbb{T}^n$. If $\varepsilon \leq \varepsilon_0(m) > 0$. [5]

Then there exist C^∞-smooth isometric immersions

$$f_i^\circ : X \hookrightarrow B^N\left(\frac{1}{i}\right), i = 1, 2, ...,$$

[3] We explain this in section 3. Also notice that there are no known obstructions to the existence of free isometric immersions of flat n-tori to $\mathbb{R}^{\frac{n(n+1)}{2}+n}$, but no example of a free (isometric or not) immersion from $\mathbb{T}^n$ to $\mathbb{R}^{\frac{n(n+1)}{2}+n}$ for $n \geq 2$ had been found either.

[4] See `https://anton-petrunin.github.io/twist/twisting.pdf` and [8].

[5] Possibly, this ε_0 doesn't depend on m.

such that

$$curv_{f_i^\circ}(X) \le i \cdot curv(F(\mathbb{T}^n)) + O(1).$$

Proof. Compose the maps F and ϕ_ε with the homothetic endomorphism $t \mapsto i \cdot t$ of the torus,

$$X \xrightarrow{\phi_\varepsilon} \mathbb{T}^n \xrightarrow{i\cdot} \mathbb{T}^n \xrightarrow{F} B^N(1)$$

and observe that the resulting composed maps, say

$$f_{i,\varepsilon} : X \to B^N(1)$$

satisfy the following conditions.

$\bullet_\varepsilon$ The map $f_{i,\varepsilon}$ is a smooth $(1+\varepsilon)$-bi-Lipschitz immersion with respect to the metrics $i^2 g$.

$\bullet_{1/i}$ The covariant derivatives of the metric $i^2 g$ and the covariant derivatives of the induced metric $f^*(h_{Eucl})$ with respect to $i^2 g$ converge to zero for $i \to \infty$,

$$\max(||\nabla^j_{ig}(i^2 g)||, ||\nabla^j_{ig}(f^*_{i,\varepsilon}(h_{Eucl}))||) \le const_j i^{-j}.$$

$\bullet_{\mathfrak{I}\mathfrak{I}}$ The immersions $f_{i,\varepsilon}$ are *uniformly* $\mathfrak{I}\mathfrak{I}_{h_{Eucl}}$:

the $j^2 g$-covariant derivatives of the (the coefficients of the) differential operators $\mathscr{M} = \mathscr{M}_i = \mathscr{M}_{f_{i,\varepsilon}}$, which invert the linearized operator $\mathscr{I} : f_{i,\varepsilon} \to\mapsto f^*_{i,\varepsilon}(h_{Eucl})$ on X are bounded, for all i independently of i,

$$||\nabla^j_{g_{i,\varepsilon}}(\mathscr{M})|| \le const = const_{m,j}.$$

It follows from the (generalized) Nash implicit function theorem (section 2.7.2 in [4]) that, for a sufficiently small $\varepsilon > 0$, depending only on m, the maps $f_{i,\varepsilon}$ for sufficiently large i admit C^∞-small, convergent to 0 with $\varepsilon \to 0$, perturbations to smooth isometric immersions $f_i^{\bigcirc} : (X, i^2 g) \to B^N(1)$.

Since the curvatures of these immersions are bounded by the curvature of F plus $O(1/i)$, the immersions

$$f_i^\circ = i^{-1} f_i^{\bigcirc} : (X, i^2 g) \to B^N(1/i)$$

are the required ones. QED.

1.F. Local Compression Corollary. Let X^m be a smooth Riemannian manifold with the sectional curvature bounded by

$$|sect.curv(X^m)| \le 1$$

and where the injectivity radius at a given point is bounded from below

$$inj.rad_{x_0}(X^m) \ge 1.$$

Then there exists a constant $\rho = \rho_m > 0$,[6] such that the ball $B(\rho) = B_{x_0}(\rho) \subset X$ admits a *smooth isometric immersion* to the Euclidean space

$$f : B(\rho) \to \mathbb{R}^{\frac{m(m+1)}{2}+m+1}.$$

[6] Probably, what we say here holds for $\rho \ge 10^{-10}$ and all m.

Moreover there exist immersions $f_\delta : B(\rho) \to \mathbb{R}^{\frac{m(m+1)}{2}+m+1}$ for all $0 < \delta \le 1$ with the *diameters of the images bounded by*

$$diam(f_\delta(B(\rho))) \le \delta$$

and the *curvatures of these images bounded by*

$$curv(f_\delta(B_{x_0}(\rho))) \le \frac{C_m}{\delta}.$$ [7]

Proof. The assumptions on the curvature and the injectivity radius of X imply that the $B_{x_0}(\rho)$ is $(1+3\rho)$-biLipschitz to the ρ-ball in the flat torus.

Remark. We shall prove in the next section the existence of smooth isometric $f_\delta : B_{x_0}(\rho) \to \mathbb{R}^{\frac{m(m+1)}{2}+2m+1}$ with $diam(f\delta_{x_0}) \le \delta$ and the curvatures of these images bounded by

$$curv(f_\delta(B_{x_0}(\rho))) \le \frac{C}{\delta}$$

for a universal constant C. [8]

1.G. Approximation Lemma. Let

$$X^m = (X^m, g_\varepsilon) \xrightarrow{\phi_\varepsilon} \mathbb{T}^n \xrightarrow{F} \mathbb{R}^N$$

be as in 1.E, let $Y = (Y, h)$ be a smooth Riemannian manifold and let $f_0 : X^m \to Y$ be a C^∞-smooth map.

Let the pullback to X of tangent bundle of Y,

$$f_0^*(T(Y)) \to X,$$

admit N *independent vector fields normal to the image of the differential of* f_0.

For instance, $dim(Y) \ge N$ and f_0 is a constant map, or f_0 is an immersion homotopic to a constant map and $dim(Y) \ge N + 2m - 1$.

If $0 < \varepsilon \le \varepsilon_0(m)$, then, for all $i = 1, 2, \dots$, there exist a δ_i-*approximation* of f_0 for $\delta_i \le \frac{1}{i}$ by C^∞-immersions $f_i : X^m \to Y$ with

$$curv_{f_i}(X) \le i \cdot curv(F(\mathbb{T}^n)) + o(i)$$

and where f_i increase the induced Riemannian metric $g_0 = f_0^*(h)$ in X by the above g_ε:

$$f_i^*(h) = g_0 + g_\varepsilon.$$ [9]

Proof. Let $E = E_{N,\delta_0} : X \times B^N(\delta) \to Y$ be the exponential map defined by the N vector fields, where this map is defined for all δ_0 if Y is complete Y and if Y is non-compete, then E_{N,δ_0} is defined if the δ_0-neighbourhood of $f_0(X) \subset Y$ is compact.[10]

[7] The proof of 1.B in [4] for $Y = \mathbb{R}^{\frac{m(m+1)}{2}+m+2}$ shows that $C_m < (100m)^{100m}$.

[8] Probably $C < 100$.

[9] δ_i-*Approximation* signifies that $dist_Y(f_0(x), f_i(x)) \le \delta_i$, $x \in X$.

[10] We assume here that Y has no boundary.

Let $\delta_i \leq \delta_0$ and let us restrict the map E to the graph of the above map $f_i^\circ : X \to B^N(\delta_i) \subset \mathbb{R}^N$,

$$\Gamma_i = \Gamma_{f_i^\circ} : X \hookrightarrow X \times B^N(\delta_i),$$

where our f_i° is now isometric for the metric g_ε on X.

Since the N fields are *normal* to $f_0(X) \subset Y$, the Riemannian metric in X induced by $\Gamma_i \circ E : X \to Y$ is $(1+\varepsilon+const \cdot \delta_i)$-bi-Lipschitz to $g_0 + g_\varepsilon$; hence, the (generalized) Nash implicit function theorem applied as in the proof of 1.E, now to the maps $\Gamma_i \circ E$ for small $\varepsilon > 0$ and large i, delivers C^∞-perturbations to these maps to the required immersions $f_i : X \to Y$ with $f_i^*(h) = g_0 + g_\varepsilon$ and with $curv_{f_i}(X) \leq i \cdot curv_f(\mathbb{T}^n) + o(i)$.

1.H. Global Approximation Corollary. Let $X^m = (X^m, g)$ and $Y^N = (Y^N, h)$ be smooth Riemannian manifolds and $f_0 : X \to Y = (Y, h)$ be a smooth *strictly short map*, i.e. the quadratic differential form $g - f^*(h)$ is positive definite.

If X^m is compact, if X admits a smooth immersion to $\mathbb{R}^n$ and if

$$N \geq \frac{n(n+1)}{2} + n + 1,$$

then there exists δ_i-*approximation* of f_0 for $\delta_i \leq \frac{1}{i}$, $i = 1, 2, \ldots$, by isometric C^∞-immersions $f_i : X^m \to Y$ with

$$curv(f_i(X)) \leq i \cdot C_m + o(i).$$

Proof. Apply the lemma to smooth $(1+\varepsilon)$-bi-Lipschitz immersions $\phi_\varepsilon : X^m \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, delivered by the Nash-Kuiper C^1-immersion theorem and to an isometric $\mathcal{JJ}$-immersion $F : \mathbb{T}^n \to \mathbb{R}^{N = \frac{n(n+1)}{2} + n + 1}$ from 1.B.

Remarks. (a) Since all X^m, $m \geq 2$, admit smooth immersions to $\mathbb{R}^{2m-1}$ by the Whitney theorem, the inequality

$$N \geq \frac{2m(2m-1)}{2} + 2m \approx 2m^2$$

suffices for all X^m. This is the worst case dimension-wise.

Our bound on N is much better for $n = m+1$, e.g. for compact hypersurfaces $X^m \subset \mathbb{R}^{m+1}$, where one needs

$$N \geq \frac{(m+1)(m+2)}{2} + m + 2 = \frac{m(m+1)}{2} + 2m + 3;$$

but this still seems far from optimal.

The best we can get for smaller N, namely for

$$N \geq \frac{m(m+1)}{2} + m + 1,$$

is the following special result.

1.I. Flat Torus Approximation Theorem. Let $\mathbb{T}_g^m$ the torus with an *invariant* (hence *flat*) Riemannian metric g and let

$$f_0 : \mathbb{T}_g^m \to Y$$

be a smooth map, where $Y = (Y,h)$ is a compact N-dimensional C^∞-smooth Riemannian manifold, possibly with a boundary, such that

$$dist(f_0(\mathbb{T}^m_g), \partial Y) > 0.$$

If $N = dim(Y) \geq \frac{m(m+1)}{2} + m + 1$, and if the bundle induced by f_0 from the tangent bundle of Y, that is $f_0^* : T(Y) \to \mathbb{T}^m_g$ is *trivial* (e.g. f_0 is contractible or Y is stably parallelizable), then, for all $\delta > 0$,[11] the map f_0 admits a δ-approximation by *metrically homothetic* maps

$$f_\delta : \mathbb{T}^m_g \to Y,$$

i.e. such that

$$f^*(h) = \lambda \cdot g \text{ for some } \lambda > 0$$

and where

$$curv(f_\delta(\mathbb{T}^m)) \leq \frac{const_m}{\delta} + o\left(\frac{1}{\delta}\right),$$

Proof. Let $E = E_{N,\delta_0} : X \times B^N(\delta) \to Y$ be the exponential map, similar to that in the proof of 1.F but now *not required to be normal* to $f_0(X)$.

Let $F : \mathbb{T}^m \to B^N(1)$, $N = \frac{m(m+1)}{2} + m + 1$ be as in 1.B and let

$$F_{ij} : \mathbb{T}^m \to B^N(1/i) \text{ for } t \mapsto \frac{1}{i} F(jt).$$

Let $\delta_i \leq \delta_0$ and let us restrict the map E to the graph of the map F_{ij}

$$\Gamma_{F_{ij}} = \Gamma_{i,j} : X \hookrightarrow X \times B^N(\delta_i).$$

Let the ratio $\lambda = j/i$ be very large. Then, in terms of the metric λg, the metric induced by $E \circ \Gamma_{ij} : \mathbb{T}^m \to Y$ becomes C^∞-close to λg and the proof follows as in 1.E and 1.G by the (generalized) Nash implicit. function theorem.

Remarks. The proof of Theorem (C) on p. 294 in [4] delivers (a stronger version of) the "local compression" for surfaces, $X^2 \to B^4(\delta)$, and this, seems to imply the torus approximation theorem for $\mathbb{T}^2 \to Y^N$ and all $N \geq 4$.

We don't know if one could comparably improve bounds on N in general, but in the next section we prove an approximation theorem for all X with the constant C *independent of m* and with a slightly improved dimension bounds in some cases.

2 Free Isometric Imbeddings of Tori to the Unit Balls with Small Curvatures

Let

$$\mathbb{T}^N_{\mathsf{Cl}} \subset S^{2N-1} \subset B^{2N} \subset \mathbb{R}^{2N}$$

[11] Since we are concerned with $\delta \to 0$, we assume here and below that $\delta \leq 1$.

be the Clifford torus, which observe, has the Euclidean curvature

$$curv(\mathbb{T}^N_{\mathsf{Cl}} \subset \mathbb{R}^{2N}) = \sqrt{N}.$$

2.A. $\Delta(n,N)$**-inequality.** If $1 \leq \frac{n^2}{2} \leq N$, then there exists a (flat invariant) subtorus $\mathbb{T}^n_0 \subset \mathbb{T}^N_{\mathsf{Cl}}$, the Euclidean curvature of which satisfies

$$curv(\mathbb{T}^n_0 \subset \mathbb{R}^{2N}) \leq \sqrt{3}\sqrt{\frac{n}{n+2}} + \Delta(n,N),$$ [12]

where $\Delta(n,N)$ is bounded by a universal constant, which, in fact, *vanishes* for $N \geq 8(n^2+n)$.[13]

This follows from the $D(m,N)$-inequalities 2.1.E and 3.B in [6].

2.B. m**-Freedom Corollary.** If $m \leq n$ and

$$\frac{m(m+1)}{2} + m(n-m) + 2n \leq 2N,$$

then, for all $\varepsilon > 0$, the n-torus $\mathbb{T}^n$ admits an immersion $\mathbb{T}^n \overset{F_\varepsilon}{\hookrightarrow} B^{2N} \subset \mathbb{R}^{2N}$ such that

$\bullet_{flat}$ the induced Riemannian metric $F^*_\varepsilon(h_{Eucl})$ in $\mathbb{T}^n$ is *Riemannian flat*;

$\bullet_{curv}$ the Euclidean *curvature* of this torus is *bounded* by

$$curv(F_\varepsilon(\mathbb{T}^n)) \leq \sqrt{\frac{3n}{n+2}} + \Delta(n,N) + \varepsilon;$$

$\bullet_{free}$ The restrictions of F_ε to all m-dimensional submanifolds in $\mathbb{T}^n$ are *free*.

Proof. The required F_ε is obtained by generically ε-perturbing the lift of the above $T^n_0 \subset \mathbb{T}^N_{\mathsf{Cl}}$ to a finite covering of $\mathbb{T}^N_{\mathsf{Cl}}$ as follows.

Firstly, replace $\mathbb{T}^n_0$ by a generic (flat invariant) subtorus, $\mathbb{T}^n_\varepsilon \subset \mathbb{T}^N_{\mathsf{Cl}}$ *tangentially* ε-close to $\mathbb{T}^n_0$, i.e. where the tangent spaces to $\mathbb{T}^n_\varepsilon$ are ε-parallel in $\mathbb{T}^N_{\mathsf{Cl}}$ to these of T^n_0,[14] where such an $\mathbb{T}^n_\varepsilon$ can be chosen split, i.e. being Riemannian product of 1-tori.

Moreover, we let

$$\tilde{\mathbb{T}}^N_{\mathsf{Cl}} = \underbrace{\mathbb{T}^1_1 \times \ldots \times \mathbb{T}^1_i \times \ldots \times \mathbb{T}^1_N}_{N},$$

(the circles $\mathbb{T}^1_i$ may have different lengths l_i), be a split finite covering of the Clifford torus such that

$$\tilde{\mathbb{T}}^n = \underbrace{\mathbb{T}^1_1 \times \ldots \times \mathbb{T}^1_n}_{N}$$

equals a covering of $\mathbb{T}^n_\varepsilon$.

[12] The summand $\sqrt{3}\sqrt{\frac{n}{n+2}}$ is optimal by Petrunin's inequality.

[13] Probably, $\Delta(n,N) < 10$ for all $N \geq n(n-1)/2$ and, possibly $\Delta(n,N) \leq 1/n$ for $N \geq \frac{n(n+2)}{2}$.

[14] This $\mathbb{T}^n_\varepsilon$ is far from T^n_0 as a subset in $\mathbb{T}^N_{\mathsf{Cl}}$.

Secondly, perturb the embedding $\tilde{\mathbb{T}}^n \subset \tilde{\mathbb{T}}^N_{\mathsf{Cl}}$ keeping the induced metric Riemannian flat in $N-n$ steps, where, at each step, we perturb a split K-torus, which contains $\tilde{\mathbb{T}}^n$, in a $(K+1)$-torus,

$$\tilde{\mathbb{T}}^n \subset \mathbb{T}^{K-1} \times \mathbb{T}^1_i \subset \mathbb{T}^{K-1} \times (\mathbb{T}^1_i \times \mathbb{T}^1_{i+1}) = \mathbb{T}^{K+1} \subset \tilde{\mathbb{T}}^N_{\mathsf{Cl}},\ K = n, ..., N-1, i = n+1, ...N,$$

by approximating the embedding $\mathbb{T}^1_i \subset \mathbb{T}^1_i \times \mathbb{T}^1_{i+1}$ by a generic isometric embedding $(1+\varepsilon)\cdot \mathbb{T}^1_i \overset{I_\varepsilon}{\to} \mathbb{T}_i \times \mathbb{T}^1_{i+1}$.

Here $(1+\varepsilon)\cdot \mathbb{T}^1_i$ is the same circle as $\mathbb{T}_i$ but with the metric of total length equal to $(1+\varepsilon)\cdot length(\mathbb{T}_i)$ and where, finally, we let

$$\mathbb{T}^{K-1} \times \mathbb{T}^1_i \ni (\theta, t) \to (\theta, I_\varepsilon(t)) \in \mathbb{T}^{K-1} \times \mathbb{T}^1_i \times \mathbb{T}^1_{i+1}.$$

If $\frac{m(m+1)}{2} + m(n-m) + 2n \leq 2N$, and granted all was done "sufficiently generically", then the resulting map $\tilde{\mathbb{T}}^n \to B^{2N}(1)$ via $\tilde{\mathbb{T}}^N_{\mathsf{Cl}} \subset B^{2N}(1)$ is free on all $X^m \subset \tilde{\mathbb{T}}^n$..

Checking this, which is similar to "*Making Non-free Maps Free*" on p. 259 in [4], is left to the reader.

2.C. Corollary. *Let X^m and Y^M be smooth Riemannian manifold and $f_0 : X \to Y$ be a smooth strictly short map. If X^m is a compact manifold, which admits a smooth immersion to $\mathbb{R}^n$, and if*

$$M > \frac{m(m+1)}{2} + m(n-m) + 2n,$$

then f_0 can be δ_i-approximated, by isometric C^∞-immersions $f_i : X^m \to Y^M$, for $\delta_i \leq \frac{1}{i}$, $i = 1,2,...,$ such that

$$curv(f_i(X)) \leq i\left(\sqrt{\frac{3n}{n+2}} + \Delta(n, \lfloor M/2 \rfloor)\right) + o(i).$$

For instance,

if X is a Euclidean hypersurface, then such an approximation is possible for

$$M > \frac{m(m+1)}{2} + 3m + 2$$

with

$$curv(f_i(X)) \leq i\left(\sqrt{\frac{3(m+1)}{m+3}} + \Delta(m+1, \lfloor M/2 \rfloor)\right) + o(i).$$

And – this is the worst case – the inequality

$$M > \frac{m(m+1)}{2} + m(m-1) + 4m - 2$$

is sufficient for all compact X^m, where the maps f_i satisfy:

$$curv(f_i(X)) \leq i\left(\sqrt{\frac{3(2m-1)}{2m+1}} + \Delta(2m-1, \lfloor M/2 \rfloor)\right) + o(i).$$

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2.D. Immersions with Prescribed Curvatures. The *symmetric normal curvature* Ψ_f of an immersion $f : X \to Y$ is the "square" of the second fundamental form, that is the symmetric differential 4-form, such that

$$\Psi_f(\partial,\partial,\partial,\partial) = ||\nabla_{\partial,\partial} f||^2_Y$$

for all tangent vectors $\partial \in T(X)$.

Observe that if f is free, then Ψ_f is *positive definite*. This means that

the 4-form Ψ_x on the tangent space $T = T_x(X)(=\mathbb{R}^m)$ is contained, for all $x \in X$, in the interior of the convex hull of the $GL(m)$-orbit of the fourth power of a non-zero 1-form on T.[16]

Also observe that

$$\sup_{||\partial||=1} \Psi_f(\partial,\partial,\partial,\partial) = (curv(f))^2.$$

2.C. C^2-Curvature Theorem.[17] *Let $X = X^m$ and $Y = Y^M$ be smooth Riemannian manifolds, let $f : X \to Y$ a free isometric C^∞-immersion and let Ψ be a symmetric positive definite differential 4-form on X.*

If

$$M = dim(Y) \geq \frac{m(m+1)}{2} + 3m + 5,$$

then f can be arbitrarily finely C^1-approximated by isometric C^2-immersions f' with the increase of their normal curvatures by Ψ.[18]

$$||\nabla_{\partial\partial} f'||^2 = ||\nabla_{\partial\partial} f||^2 + \Psi(\partial,\partial,\partial,\partial)$$

for all tangent vectors $\partial \in T(X)$.

2.E. Euclidean Example. *The standard embedding $f_0 : \mathbb{R}^n \to \mathbb{R}^{(n+2)(n+5)/2}$ can be C^1-approximated by isometric C^2-embeddings f with an arbitrary strictly positive definite normal curvature form Ψ.*

Proof. C^∞-approximate f_0 by *free* isometric embeddings (as in 1.B) and apply 2.C.

2.F. Toric Example. *Let Ψ be a smooth symmetric positive definite differential 4-form on the m torus $\mathbb{T}^m$.*

If $M \geq 16(m^2+m)$, then there exist a C^2-immersion $f : \mathbb{T}^M \to B^m(1)$, such that the induced metric is flat (split if you wish) and

$$\Psi_f(\partial,\partial,\partial,\partial) = \Psi(\partial,\partial,\partial,\partial) + \frac{3m}{m+2}||\partial||^4.$$

Proof. Observe that, according to 3.1 from [6] (this also follows from Petrunin's inequality), the isomeric immersion $f_0 : \mathbb{T}^m \to B^M$ with $(curv(f_0))^2 \leq \frac{3m}{m+2}$ has constant curvature,

$$\Psi_{f_0}(\partial,\partial,\partial,\partial) = \frac{3m}{m+2}||\partial||^4,$$

15 If the right-hand sides of the inequalities $M > ...$ are even, then these may be replaced by $M \geq ....$.

16 This interior makes the unique open $GL(m)$-invariant (non-empty!) minimal convex cone in the space $T^{\circledS 4}$ (of dimension $m(m+1)(m+2)(m+3)/24$, $m = dim(T)$ of symmetric 4-linear forms (4d-polynomials) on T.

(This cone is strictly smaller than the cone of the forms Φ, which are positive as polynomials, $\Phi(t,t,t,t) > 0$, $t \neq 0$.)

17 See 3.1.5.(A) in [4].

18 In general, such an f' can't be C^∞ for large m , but possibly, these C^2-smooth f' exist for all $m \geq 2$ and all M.

and apply 2.B and 2.D.

3 Perspective and Problems

The $\mathcal{JJ}$-property of free immersions was already implicitly present in the proof of the algebraic Janet lemma (1926), where this lemma brings the isometric immersion equations to the Cauchy–Kovalevskaya form and thus implies that real analytic Riemannian n-manifolds locally C^{an}-immerse to $\mathbb{R}^{\frac{n(n+1)}{2}}$.[19] [20]

The simplest non-free $\mathcal{JJ}$-immersions, $X \to Y$ are those, which are free on a totally geodesic hypersurface $X_0 \subset X$ and also on the complement $X \setminus X_0$ and where the Hessian of the second derivatives $\partial_{ij} f$ vanishes on X_0 with *finite order* (see 2.3.8, 3.1.8. 3.1.9 in [4]). Then one defines by induction k-*subfree* maps of n-manifolds X to Y, where

$$0\text{-}subfree = free,$$

and where

$$1\text{-}subfree\ maps\ f$$

generalize the above ones, namely, where there exist tangent hyperplanes $T_x^{n-1} \subset T_x(X)$ at all $x \in X$, such the *restrictions* of f to the *exponential images* $X_x = \exp(T_x) \subset X$ are free at x and also on the complements X_x near x, i.e. in the small balls:

$$B_x(\varepsilon) \cap (X \setminus X_x)$$

with “finite order of non-freedom” on (infinitesimal) neighbourhoods of X_x.

Finally, a map f is

$$k\text{-}subfree,\ k = 0, 1, ..., n,$$

if it is free on the above hypersurfaces $X_x \subset X$ near x for all $x \in X$ and it is $(k-1)$-subfree on $B_x(\varepsilon) \cap (X \setminus X_x)$ with “finite order of non-$(k-1)$-subfreedom” on (infinitesimal) neighbourhood of X_x.

Clarification. “Finite order of non-subfreedom” means that the coefficients of the relevant linear differential operator $\mathscr{M} = \mathscr{M}_f(g)$ on $X \setminus X_x$ are rational functions in partial derivatives of f, where these functions have their poles on X_x, (see 238 in [4] and [1]).

This seems to imply –I didn’t truly check this – that

generic immersions $X^n \to Y^N$ *for* $N \geq \frac{n(n+1)}{2} + n$ *are* $\mathcal{JJ}$.

It is also plausible, that

generic bendings of m-subtori in the Clifford torus, as in the proof of 2,B would make them $\mathcal{JJ}$ *for* $\frac{m(m+1)}{2} + m(n-m) + 2n - m \leq 2N$.

But this, even if true only slightly improve the lower bound on N in the above “worst dimension case”, where, in fact, we expect the following.

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[19]The Cauchy–Kovalevskaya theorem also yields a weak form of the Nash real analytic implicit function theorem. This, by Janet’s lemma combined with an analytic version of Nash twisting argument, delivers isometric C^{an}-immersions of compact C^{an}-manifolds to Euclidean spaces, see [2] and p 54 in Appendix 11 in [7].

[20]The existence of local isometric C^∞-immersions of C^∞-manifolds $X^n \to \mathbb{R}^N$ is known for $N = \frac{n(n+1)}{2} + n - 1$ [3] and it is easy to see that there is no local C^∞-immersions of generic smooth Riemannian n-manifolds to $\mathbb{R}^{\frac{n(n+1)}{2}-1}$.

But for *no* $n \geq 2$ one can prove the existence of such immersions to $\mathbb{R}^{N=\frac{n(n+1)}{2}+n-2}$ or to find a counterexample for $N = \frac{n(n+1)}{2}$.

It is also unclear for which $i = 2, 3, ...,$ (if any) all C^∞-smooth Riemannian manifolds $X = X^n$, admit (local or global) isometric C^i-immersion to $\mathbb{R}^{\frac{n(n+1)}{2}-1}$, where, for all we know, this may be possible, say for $i \leq 0.1\sqrt[4]{n}$, see [5] for more about it.

3.A. Conjecture. Let X^m and Y^N be smooth Riemannian manifolds, where X compact and Y is complete, and let $f_0 : X \to Y$ be a smooth strictly short map. If

$$N \geq \frac{m(m+1)}{2} + 1,$$

then f_0 admits a δ*-approximation* for all $\delta > 0$ by isometric C^∞-immersions $f_\delta : X^m \to Y$ with

$$curv(f(\delta(X))) \leq \frac{1}{\delta}\left(\sqrt{\frac{3m}{m+2}} + \Delta(m,N)\right) + o\left(\frac{1}{\delta}\right),$$

for the same Δ as in 2.A, where, if

$$N \geq \frac{m(m+1)}{2} + m,$$

these f_δ can be chosen *m-subfree*.

FAMILIES OF MAPS.

Given a locally defined class $\mathscr{C}$ of C^∞-maps $f : X \to Y$, where $Y = (Y,h)$ is a smooth Riemannian manifold, e.g. $\mathscr{C}$ consists of smooth immersions, of $\mathcal{II}$-isometric immersions etc, the δ-approximation problem is accompanied by a similar problem for families of maps.

More generally, such approximation makes sense for maps f from foliated leaf-wise Riemannian manifolds $\mathcal{X} = (\mathcal{X}, g)$ [21] to Y, where the restrictions of f to the leaves $X \subset \mathcal{X}$ are in $\mathscr{C}$.

3.B. $\mathscr{C}$-Homotopy Approximation Conjecture. Let $\mathcal{X} = (\mathcal{X}, g)$ be a compact manifold foliated into m-dimensional Riemannian leaves, let $Y = (Y,h)$ be complete Riemannian N-manifold.

Let $\mathscr{C}$ be a class of smooth leaf-wise isometric $\mathcal{II}$-maps and $\phi_0 : \mathcal{X} \to Y$ be a $\mathscr{C}$-map.

Let $f_0 : \mathcal{X} \to Y$ be a smooth leaf-wise strictly short $\mathscr{C}$-map, which is homotopic to ϕ.

Then, at least in in the following three cases, the map f_0 can be δ-approximated for all $\delta > 0$ by smooth leaf-wise $\mathscr{C}$-maps

$$f_\delta : \mathcal{X} \to Y,$$

where the maps f_δ can be joined with ϕ_0 by homotopies of leaf-wise $\mathscr{C}$-maps and where the leaf-wise curvatures of f_δ are bounded by

$$curv(f(\delta(X))) \leq \frac{\Xi}{\delta} + o\left(\frac{1}{\delta}\right),$$

where $\Xi = \Xi(m, N, dim(\mathcal{X})) \leq 100$.

Case 1. $\mathscr{C}$ is the class of leaf-wise *free* isometric maps.

Case 2. $\mathscr{C}$ is the class of leaf-wise *m-subfree* isometric maps.

Case 3. $\mathscr{C}$ is the class of *all* leaf-wise isometric $\mathcal{II}$-maps.

Remarks/Questions. (i) *How big is* Ξ? Possibly, Ξ is significantly smaller than 100, but it is unlikely to approach $\sqrt{3}$ for $N \gg m^2$.

Yet, it follows by the arguments in section 2 that

[21]This g – a smooth leaf-wise Riemannian metric on $\mathcal{X}$ – is a positive definite differential quadratic form on the tangent bundle $\mathcal{T} \subset T(\mathcal{X})$ to the leaves $X \subset \mathcal{X}$.

if $N \gg dim(\mathcal{X})^2$, then

$$\Xi \leq \sqrt{\frac{3(2dim(\mathcal{X})-1)}{2dim(\mathcal{X})+1}} < \sqrt{3}.$$

Conceivably, the correct bound on curvature needs $\Xi \sim 3(dim(\mathcal{X})) - m + 1)$.

(ii) *Why Integrable?* The above make sense for possibly non-integrable subbundles $\mathcal{T} \subset T(\mathcal{X})$, where the counterpart of the metric inducing operator sends maps f to the restrictions of the forms $f^*(h)$ to the subbundle $\mathcal{T} \subset \mathcal{X}$, where one can define the corresponding classes $\mathcal{C}_\mathcal{T}$ of maps $f : \mathcal{X} \to Y$ and where the $\mathcal{T}$-curvature of f is defined as follows.

Given a non-zero tangent vector $\tau \in T_x(\mathcal{X})$, $x \in \mathcal{X}$, let $curv_\tau(f)$ be the infimum of Y-curvatures at x of the f-images of the curves $C \subset \mathcal{X}$ which contain x and are tangent to τ,

$$curv_\tau(f) = \inf_C curv_x f(C)$$

and

$$curv_\mathcal{T}(f) = \sup_{\tau \in \mathcal{T}} curv_\tau(f).$$

Here is what one (may be unrealistically) expects in this regard.

3.C. Orthonormal Frame Conjecture. Let $\mathcal{X}$ be a compact smooth Riemannian manifold and Θ_i, $i = 1.,,,m \leq dim(\mathcal{X})$, be smooth orthonormal vector fields on $\mathcal{X}$.

Let $Y = (Y,h)$ be a complete Riemannian N-manifold and $f_0 : \mathcal{X} \to Y$, be a smooth strictly short map. If

$$N \geq \frac{m(m+1)}{2} + 1$$

and the induced (pullback) vector bundle $f^*(T(Y)) \to \mathcal{X}$ admits m linearly independent sections, e.g. f_0 is contractible, then the map f_0 can be δ-approximated for all $\delta > 0$ by smooth maps

$$f_\delta : \mathcal{X} \to Y,$$

such the differential images $d_f(\Theta_i) \in T(Y)$ are orthonormal with respect to h,

$$\langle d_f(\Theta_i), d_f(\Theta_j) \rangle_h = 0, \; ||d_f(\Theta_i)||_h = 1,$$

and such that

$$curv_\mathcal{T}(f) \leq \frac{\Xi}{\delta} + o\left(\frac{1}{\delta}\right),$$

for $\Xi \leq 100$, where $\mathcal{T} \subset T(\mathcal{X})$ is the subbundle spanned by the fields Θ_i.

Below we state without proof the only confirmation we have of the conjectures 3.A and 3.B.

3.D. Parametric $1D$-Approximation Theorem. Let $\mathcal{X}$ be compact smooth Riemannian manifold and $\mathcal{T}^1$ be a smooth line field on $\mathcal{X}$. Let $Y = (Y,h)$ be a complete Riemannian manifold of dimension $N \geq 2$ and let $f_0 : \mathcal{X} \to Y$ be a strictly short map.

If $\mathcal{T}^1$, regarded as a line bundle over $\mathcal{X}$, admits an injective homomorphism to the induced (pullback) bundle $f^*(T(Y)) \to \mathcal{X}$, e.g. $\mathcal{T}^1$ is orientable (defined by a vector field) and the map f_0 is contractible, then the map f_0 can be δ-approximated for all $\delta > 0$ by smooth maps

$$f_\delta : \mathcal{X} \to Y,$$

which are isometric on $\mathcal{T}^1$ and such that the $\mathcal{T}^1$-curvatures of f_δ, i.e. the Y-curvatures of the f_δ-images of the (1-dimensional) orbits/leaves of $\mathcal{T}^1$, are bounded by

$$curv_{\mathcal{T}^1}(f_\delta) \leq \frac{\Xi_1}{\delta} + o\left(\frac{1}{\delta}\right),$$

where $\Xi_1 \leq 4$.[22]

3.E. Example/Corollary. There exists a smooth map $f: S^{2n+1} \to B^2(1)$, for all $n = 1, 2, ...$, such that the f-images of all Hopf circles $S^1_p \subset S^{2n+1}$, $p \in \mathbb{C}P^n$, have equal length $l = l_n$ ($\leq 100n$) and curvatures

$$curv(f(S^1_p)) \leq 4 + \varepsilon$$

for a given $\varepsilon > 0$.

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[22] Probably, the optimal $\Xi_1 = 3$, where one proves that $\Xi_1 > 2$ by showing that:

if a $C^{1,1}$-immersion $f: S^1 \to B^2(2)$ with $curv(f(S^1)) \leq 1$ is regularly homotopic to the figure 8, then f factors through a map to an (8-shaped) union of two unit circles in $B^2(2)$, which touch each other at the center of the disc $B^2(2)$,

where "extremal members" in families of immersions $S^1 \to B^2(3-\varepsilon)$ with curvatures ≤ 1, are, probably, associated with certain patterns comprised of unit circles in $B^2(3-\varepsilon)$ tangent to the unit circle centered at zero. Such a pattern associated with a regular isotopy from the unit circle to an immersion with the image 8 is comprised of three mutually tangent unit circles inside $B^2(1+2/\sqrt{3})$ (which seems to imply that $\Xi_1 > 1 + 2/\sqrt{3} \approx 2.1547...$).

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